Poisson Process

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How to determine the expected number of failures

- Analyse known distributions of failures
  - The Poisson process

Poisson Distribution

- Definition
  - A random variable X has a Poisson Distribution if it is a discrete random variable that has **probability mass function** (pmf)

$$p(k) = P(X = k) = \frac{\varphi^{k} \exp(-\varphi)}{k!} \qquad k = 0, 1, 2, \ldots$$

  - If X has a Poisson distribution we write

$$X \sim POI(\varphi)$$

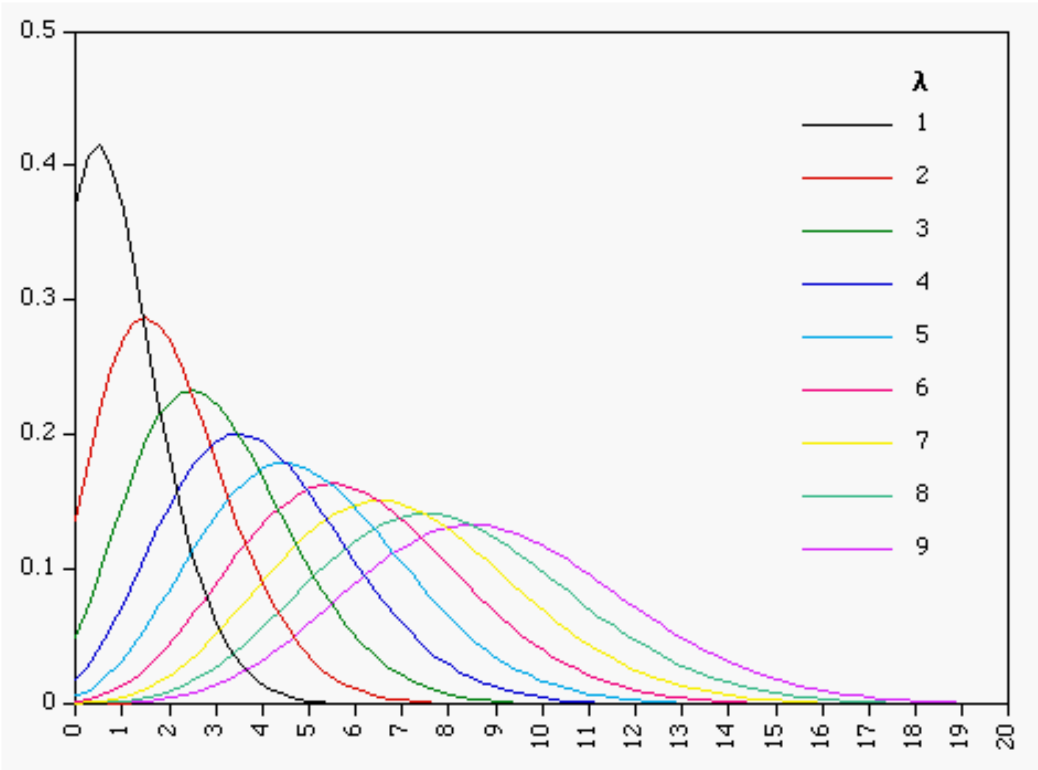
Poisson Distribution

- Theorem
  - If $X \sim POI(\varphi)$ then its expected value and its
  - variance are φ

  - $E[X]= e^{-\varphi} \sum(\varphi^k/k! *k) = e^{-\varphi} * e^{\varphi} * \varphi = \varphi$
- So the major unknown variable to identify is φ
- Once you know it you can draw the Poisson distribution
- Note: $d^k(e^x * x)=e^x*x + ke^x$. In 0 $d^n(e^x * x)=k$ and the Taylor series equals to the above sum

Graphs of Poisson distribution

- In the graph $\varphi = \lambda$



Poisson Process

- Definition of a Poisson Process
- A process defined by N(t) is said to be a Poisson process if
  1. N(0)=0
  2. For any $a < b \leq c < d$ the random variables N(a,b] and N(c,d] are independent.
  3. There is a function λ such that

$$\lambda(t) = \lim_{\Delta t --> 0} \frac{P(N(t,t+\Delta t] = 1)}{\Delta t}$$

DEF: Continues →

And the function λ is called the intensity function of the Poisson process

$$= \lim_{\Delta t --> 0} \frac{P(N(t, t + \Delta t] \geq 1)}{\Delta t}$$

4.

$$\lim_{\Delta t --> 0} \frac{P(N(t, t + \Delta t] \geq 2)}{\Delta t} = 0$$

Comments

- N(a,b] and N(c,d] are independent
- N(a,b]=N(b)-N(a) is a difference between two random variables
- Then independence of N(a,b] and N(c,d] means that the probability that a given number of failures falls in the interval (a,b] and the probability that the same number of failures falls in the interval (c,d] are independent

Comments

- The forth property precludes the possibility of simultaneous failures
- In fact it says that in a very small interval the probability that the the Number of failures jumps of two or more is zero

Major theorem for Poisson processes

- A point process defined by N(t) is Poisson process if and only if
  - N(0)=0
  - For any $a < b \le c < d$ the random variables N(a,b] and N(c,d] are independent. This is called increments property
  - For an a<b

$$N((a,b]) = POI(\int_a^b \lambda(t))$$

Comments

- The mass function of N(a,b] is

$$P(N(a,b]=k)=\frac{\left(\int_a^b \lambda(t)\right)^k * \exp(-\int_a^b \lambda(t))}{k!} \qquad k=0,1,2,\ldots$$

- and whose expected value
- E[N(a,b]) = $\int_a^b \lambda(t)dt$

Comments

- That is the probability that k failures fall in the interval (a,b] is described by a negative exponential
- The whole process is known once we know the rate of occurrence of failure (ROCOF)

$$\lambda(t)$$

- Poisson processes do not have multiple failures:
- ROCOF, intensity function of a process, and intensity function of the Poisson process coincide

Comments

- So to compute the mass function I have to compute the **intensity function of the Poisson process**

$$\lambda(t) = \lim_{\Delta t --> 0} \frac{P(N(t, t + \Delta t] = 1)}{\Delta t}$$

- Then integrate the intensity function between a and b (the area below the graph of the intensity function) getting a number C
- Compute the exponential of –C
- Multiply by C^k
- Divide by the factorial of k

Comments

- It is hard work!
- So how can I reduce some effort?
- We can define "nice" Poisson processes called Homogenous

Homogeneous Poisson Process

- Definition
  - The Homogeneous Poisson Process is a Poisson process that has constant failure intensity
- This means that

$$\lambda(t) = \lim_{\Delta t --> 0} \frac{P(N(t, t + \Delta t] = 1)}{\Delta t} = \lambda \qquad const$$

HPP

- A process is an HPP with intensity λ,

  if and only if

- The Times Between Failures are random variables
  - Independent and identically distributed
  - Exponential ($f_i(t) = \lambda_i e^{-\lambda_i t}$)
  - $E[X_i]=1/\lambda$ for all i

Intensity of one failure in the HPP

- Therefore the intensity of one failure λ defines
  - either the derivative of the number of failures expected an interval of time (E[N(t)])
  - The inverse of the Expected means of the Time Between Failures
  - and
  - the intensity function of a point process; the infinitesimal probability that a failure happens in a small interval

Comments

- When the X_i are independent and identically distributed the joint distribution becomes
  - $f(x_1, x_2, \ldots, x_n) = f(x_1) * f(x_2) * f(x_3) * \ldots * f(x_n)$
- Exponential means that these densities are all exp
  - In particular the Time between Failures have the same expected value $1/\lambda$

Comments

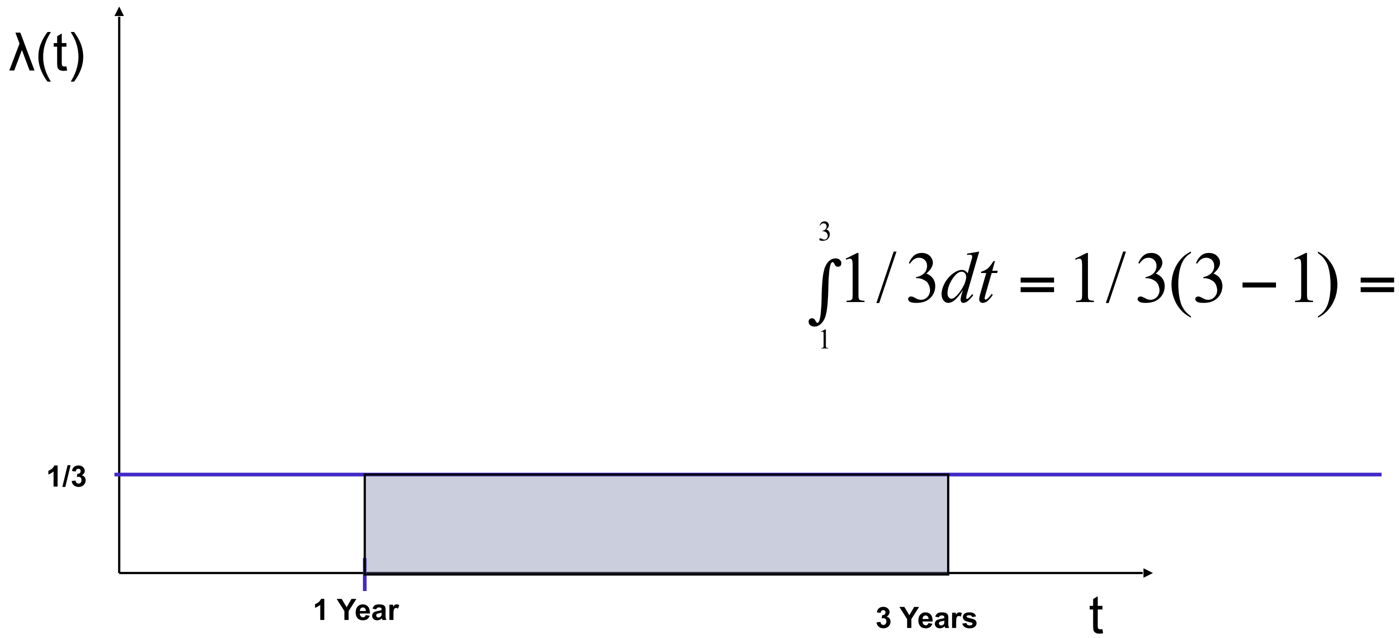
HPP model implies that

1. The system does not age in global time that is it does not deteriorate neither improve
    - Because Times between failures are exponential variables (random principle, the hazard function is constant)
    - and all their density functions look like the same (identically distributed same intensity function)
2. The system does not wear out in local time
    - The probability that in a small interval there are more than 2 failures is 0

Example

- Then

$$\int_a^b \lambda(t) = \lambda(b-a)$$



$$\int_1^3 1/3dt = 1/3(3-1) = 2/3$$

Comments

- $\lambda=1/3$ means that in each interval of time of the same length, I have the same probability to have 1 failure
- Every year I have always 1/3 of probability that one failure occurs
- In two years I have 2/3 that one failure occurs …

Exercise

$$P(N(a,b] = k) = \frac{(\lambda(b-a))^k * \exp(-\lambda(b-a))}{k!} \qquad k = 0,1,2,\ldots$$

- In the example

$$P(N(1,3] = 3) = \frac{(2/3)^3 * \exp(-2/3)}{3!} = \frac{4 * \exp(-2/3)}{81} = 0.025$$

- The expected value for N(1,3] is

$$\int_1^3 \lambda(\mathrm{t})dt = 1/3 * 2$$

- The expected number of failures in the interval (1,3] is less then 1. That is why the probability of having three failures is so small (2,5%)

Non-Homogenous Poisson Process

- A non homogeneous Poisson process is a Poisson Process with non – constant intensity function

$$\lambda(t)=3*t, =6*t+4, =t^2, =sqrt(t), =exp(t) \text{ etc.}$$

Example

- Suppose that a repairable system is modeled by a NHPP with intensity function $\lambda(t) = 0.02t^{0.8}$
- The mean is

$$: \int_0^t 0.02x^{0.8} = \frac{1}{90}t^{1.8}$$

Example

$$E(N(a,b]) = \int_a^b 0.02t^{0.8} = \frac{b^{1.8} - a^{1.8}}{90}$$

$$E(N(0,20]) = \int_0^{20} 0.02t^{0.8} = \frac{20^{1.8} - 10^{1.8}}{90} \approx 2.44$$

$$E(N(20,40]) = \int_{20}^{40} 0.02t^{0.8} = \frac{40^{1.8} - 20^{1.8}}{90} \approx 6.06$$

Comment

- Therefore the expected value varies with time intervals
- We will consider both cases homogeneous and non-homogeneous Poisson processes