The random counting variable

Barbara Russo

Repairable systems

- Altogether, we expect that the probability density function of T_i would be different than the one of T_{i-1}
  - For example, by **improving design** $E[T_{i-1}]$ tends to be **less** than the one of $E[T_i]$

Counting Random Variable

- Until now we have seen the point process through two sets of random variables
  - T_i
  - X_i
- We introduce a new random variable the Counting Random Variable N(t)

- Source: Software Reliability, Measurement Prediction and Apllication (Musa, Iannino, Okumoto. Pg. 253…)

Preamble

- Even if we have complete knowledge of the faults in the software **we are not able to state with certainty** when next the **system will fail**
- Namely, we do not know
  - what inputs will be supplied to the system and
  - in what order,
- So we cannot predict which fault will be triggered next and which failure will cause

Predict the time

- We may use the local expected mean to evaluate the next time to failure

$$\mathbf{E[T_i]=\int t f_i(t) dt}$$

- Does it really answer our problem?

System of prediction

- Theoretically yes, but the **pdf and cdf** are in general **not known**
- Remember? if we consider each single state separately we can consider the system non-repairable and the local description is given by the hazard rate
- In the simplest case when hazards are constant (operational region for all the states), the $\{\lambda_i\}$ are unknown, they depend on the system
- … but we can use history to estimate them

Example

- Assume that all the failures occur randomly. Then the model is negative exponential and

$$F_i(t_i)=1-e^{-\lambda_i t_i}$$

- and

$$E[T_i]=1/\lambda_i$$

- Inference Procedure: for each i there is an unknown parameter λ_i

Remember that …

- $E[T]= \int_0^{+\infty} \lambda t e^{-\lambda t} =(-te^{-\lambda t})|_0^{+\infty} - \int_0^{+\infty} -e^{-\lambda t} =$
- $=0+ (1/\lambda) * (e^{-\lambda t})|_0^{+\infty} = 1/\lambda$

Example

- If we want to determine λ_i we can use the average of the two previously observed values:
- Since it is

$$E[T_i] = \frac{1}{\lambda_i}$$

- and

$$E[T_i] = \frac{\bar{t}_{i-2} + \bar{t}_{i-1}}{2}$$

$$\frac{1}{\lambda_i} = \frac{\bar{t}_{i-2} + \bar{t}_{i-1}}{2} \qquad \lambda_i = \frac{2}{\bar{t}_{i-2} + \bar{t}_{i-1}}$$

Open questions

- Which arithmetic mean?
- Are instants of time t_i all the same?
- Until when should we sum?

- We introduce a new random variable that "counts" …

Counting processes

- A stochastic process is counting if the X(t) is the number of items counted by time t
- The **random counting variable** defines a counting process

The random counting variable

- N(t) = # of failures in the interval (0,t]

- N((a,b])= # of failures in the interval (a,b]

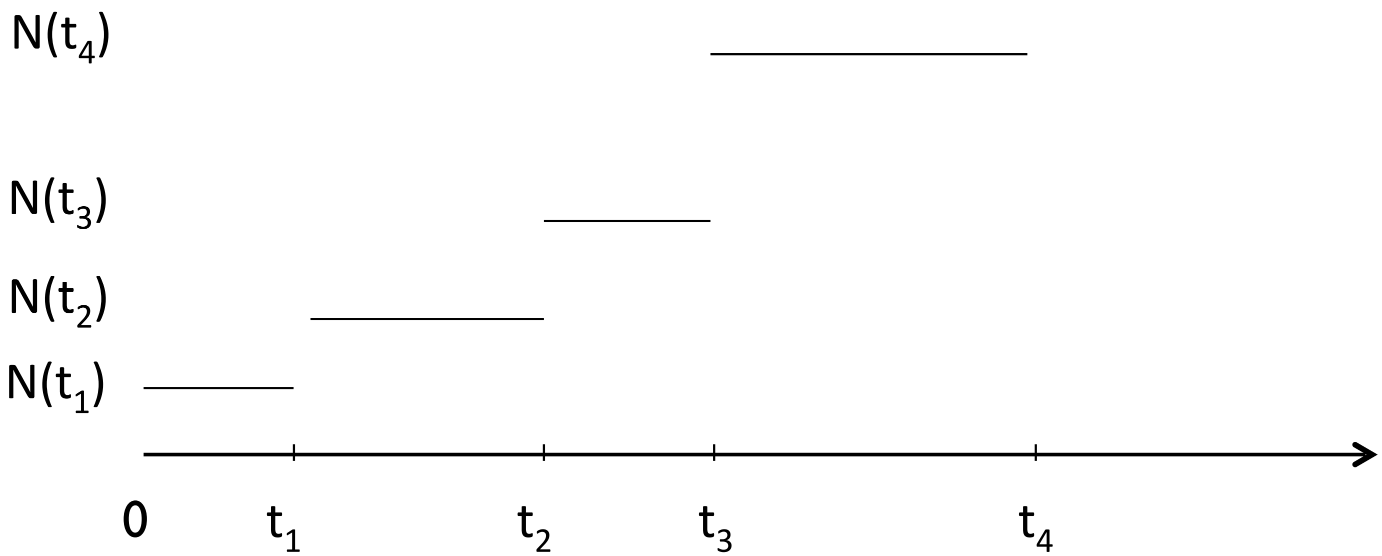
- N((a,b])=N(b)-N(a)

Counting Function N(t)

- N(t) is a counting function that keeps track of the cumulative number of failures a given system is experiencing from 0 to time t
- N(t) is a step function that jumps one up every time a failure occurs and stays at the new level until the next failure

N(t)

- Note: if you have multiple failures N(t) jumps more than one



Probabilities

- $N(t)$ number of failures in the interval $(0,t]$
- $p(k)=P\{N(t)=k\}$ probability that in the interval $(0,t]$ there are k failures
- $F(k)=P\{N(t)\leq k\}$ probability that in the interval $(0,t]$ there are less than k failures
- $f(k)= (P(N(t)\leq k) - P(N(t)\leq k-1))/ 1 = p(k)$ – because it is a discrete variable

Probabilities

- $N(a,b]=N(b)-N(a)$ difference of # of failures

- $p(k)=P\{N(a,b]=k\}$ probability that in the interval (a,b] there are k failures

- $F(k)= P\{N(a,b]\leq k\}$ probability that in the interval (a,b] there are less than k failures

- $f(k) =p(k)$ – bc. discrete variable

Expected value for N(t)

$$E[N(t)] = \sum_i i p(i)$$

- Problem: **until when should I sum up?**
- Deciding this is equivalent to expecting that a software has a known number of failures in the interval (0,t] – the expected value of N(t)
- So we need to find instruments to estimate the expected value – this is the hard work!
- With the counting random variable we can define a global process!

N(t) as point process

- A point process is also defined by the joint distribution of the random variables
  - $N(t_1),\ldots,N(t_n)$ for any n and for any $t_1,\ldots,t_n$
  - In our case t_i are the ith failure times

Comparison

- $N(t_1),\ldots,N(t_n)$; $t_1,\ldots,t_n$
- $T_1,\ldots,T_n$; $1\ldots n$
- $X_1,\ldots,X_n$; $1\ldots n$

- What is the relation among them?

Mean function

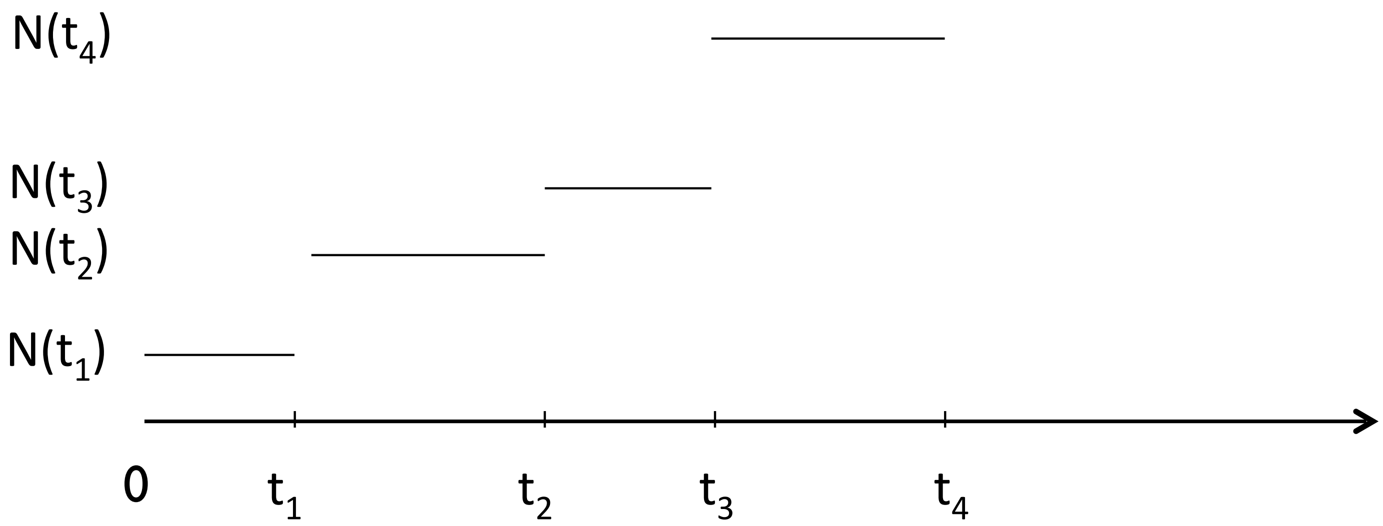
- The mean function is the expected value

$$\mu(t)=E(N(t))$$

- Note that for N, t is the global time!
- $\mu(t)$ is the expected number of failures at time t

μ(t)

- μ(t) must be a **non-**decreasing function as N(t) it is a non decreasing step function



ROCOF

- When $\mu(t)$ is differentiable then the **Rate of Occurence of Failure** is defined as
- $$\lambda(t) = \frac{d}{dt}\mu(t)$$
- It is the instantaneous rate of change of the expected number of failures

Failure intensity function

- The intensity function of a process is

$$\lambda\_(t) = \lim_{\Delta t \to o} \frac{P(N(t,t+\Delta t] \geq 1)}{\Delta t}$$

- We shall see that the intensity function will be the key function describing the most known software reliability models

Intensity function

- The intensity function is the probability of failure in a small interval divided by the length of the interval (a speed)
- Therefore there will be many failures over intervals in which λ _(t) is large and fewer failures over intervals on which λ _(t) is small

Comparing f(t), h(t) and λ(t)

$$f_i(t) = \lim_{\Delta t \to o} \frac{P(T_i \in (t, t+\Delta t])}{\Delta t} = \lim \frac{F(t+\Delta t) - F(t)}{\Delta t}$$

$$\lambda\_(t) = \lim_{\Delta t \to o} \frac{P(N(t, t+\Delta t] \geq 1)}{\Delta t}$$

$$h_i(t) = \lim_{\Delta t \to o} \frac{P(T_i \in (t, t+\Delta t] \mid T_i \geq t)}{\Delta t}$$

$$\lambda(t) = \frac{d}{dt} \mu(t) = \lim_{\Delta t \to o} \frac{E[N(t+\Delta t)] - E[N(t)]}{\Delta t}$$

Comparing f(t), h(t) and λ(t)

- f(t) is the velocity in t of the cumulative distribution function
- h(t) is the velocity in t of the conditional probability that one and only one failure will occur in a small interval
- λ _(t) is a kind of velocity. It is the probability that a "change of # of failures around t is greater than 1" over the length of the interval

Theorem

- If simultaneous failures do not occur, the intensity function and ROCOF are equal

$$\lambda\_(t)=\lambda(t)$$

- Note that in case of multiple failures, E[N(t)] is discontinuous

Meaning of the theorem

- The uncertainty of the ROCOF function (until when should I sum up in the expected mean and which probability function p for individual output "i" should I consider) is reduced to the computation of

$$\frac{P(N(t,t+\Delta t] \geq 1)}{\Delta t}$$

- The intensity function does not care of how many failures in total a process should have; it only considers the probability of having at least one failure

For the most curious

- The Theorem proof follows below
- The most curious can read it
- The major fact that you need to know and understand is

- $$P\{N(t) \geq k\} = P\{T_k \leq t\}$$

ROCOF of a point process

- Lemma. Let $g_k(t)$ denote the density of T_k, the time to the k^{th} failure. Then the ROCOF function is

$$\boldsymbol{\lambda(t)=\sum(g_k(t))}$$

Proof.

$P\{N(t)\geq k\}= P\{T_k\leq t\}$ and $P\{N(t)\geq k+1\}= P\{T_{k+1}\leq t\}$

$P\{N(t)=k\} = P\{N(t)\geq k\} - P\{N(t)\geq k+1\} =$
$P\{T_k\leq t\} - P\{T_{k+1}\leq t\}$

Proof

$\mu(t) = E(N(t)) = \sum_k (k * P\{N(t)=k\}) =$

$\sum_k (k * (P\{T_k \le t\} - P\{T_{k+1} \le t\})) = \sum_k (P\{T_k \le t\})$

- Then we derive $\frac{d}{dt}\mu(t)$

$\lambda(t)$ vs. $\lambda\_(t)$

- If the probability of simultaneous failures is zero then

$$\lambda(t)=\lambda\_(t)$$

- Proof.
- When simultaneous failures are impossible then
- $N(t,t+\Delta t] \geq 1$ for small Δt, is equivalent to either the first failure occurs in $(t,t+\Delta t]$ or the second failure occurs in $(t,t+\Delta t]$ or the third failure …

Proof

- $\lambda\_(t) = \lim_{\Delta t \to o} P(N(t,t+\Delta t] \geq 1)/\Delta t =$
- $\lim_{\Delta t \to o} P(T_1 \text{ in } (t,t+\Delta t] \text{ or } T_2 \text{ in } (t,t+\Delta t] \text{ or } \ldots)/\Delta t =$
- $\sum (\lim_{\Delta t \to o} P\{T_k \text{ in } (t,t+\Delta t]\}/\Delta t) = \sum(g_k(t))$
- $= \lambda(t)$ by previous lemma