Chapter 12
Recursion

Java Software Solutions
Foundations of Program Design
Seventh Edition

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Recursion

• Recursion is a fundamental programming technique that can provide an elegant solution to certain kinds of problems.

• Chapter 12 focuses on:
  – thinking in a recursive manner
  – programming in a recursive manner
  – the correct use of recursion
  – recursion examples
Recursive Thinking

- A *recursive definition* is one which uses the word or concept being defined in the definition *itself*

- When defining an English word, a recursive definition is often not helpful

- But in other situations, a recursive definition can be an appropriate way to express a concept

- Before applying recursion to programming, it is best to practice thinking recursively
Recursive Definitions

• Consider a list of numbers:

    24, 88, 40, 37

• A list can be defined as follows:

    A List is a: number
    or a: number comma List

• That is, a List is defined to be a single number, or a number followed by a comma followed by a List

• The concept of a List is used to define itself
Recursive Definitions

• The recursive part of the LIST definition is used several times, terminating with the non-recursive part:

```
LIST: number comma LIST
   24   ,   88, 40, 37
    number comma LIST
       88   ,   40, 37
        number comma LIST
           40   ,   37
            number
               37
```
Peano's def. of Natural Numbers

• The following two **axioms** define the natural numbers
  – 0 is a natural number
  – For every natural number \( n \), \( S(n) \) – the successor of \( n \) - is a natural number

• The number 1 can be defined as \( S(0) \), 2 as \( S(S(0)) \) (which is also \( S(1) \)), and, in general, any natural number \( n \) as \( S^n(0) \)

• The next two axioms define their properties:
  – For every natural number \( n \), \( S(n) = 0 \) is false. That is, **there is no natural number whose successor is 0**
  – For all natural numbers \( m \) and \( n \), if \( S(m) = S(n) \), then \( m = n \). That is, **\( S \) is an injection**.
Infinite Recursion

• All recursive definitions **have to have** a non-recursive part called the **base case**

• If they didn't, there would be **no way to terminate** the recursive path

• For instance:
  
  A List is a: number comma List

• Such a definition would cause **infinite recursion**

• This problem is similar to an infinite loop
Quiz

What is printing the program described in this flowchart?
public int itFactorial (int n) {
    int m = 1, f = 1;
    while (m < n) {
        ++m;
        f = f * m;
    }
    return f;
}
Recursive Definition: Factorial

• N!, for any positive integer N, is defined to be the product of all integers between 1 and N inclusive

\[ N! = N \times (N-1) \times (N-2) \times \ldots \times 2 \times 1 \]

• This definition can be expressed recursively as:

\[ 1! = 1 \]
\[ N! = N \times (N-1)! \]

\[ f(1) = 1 \]
\[ f(n) = n \times f(n-1) \]

• A factorial is defined in terms of another factorial

• Eventually, the base case of 1! is reached
Recursive Factorial

5!
5 * 4!
4 * 3!
3 * 2!
2 * 1!
1

120
24
6
2
1
Quick Check

Write a recursive definition of \( f(n) = e^n \), where \( n \geq 0 \).
Quick Check

Write a recursive definition of \( f(n) = e^n \), where \( n \geq 0 \).

\[
\begin{align*}
e^0 &= 1 & f(0) &= 1 \\
e^n &= e \times e^{n-1} & f(n) &= e \times f(n-1)
\end{align*}
\]

In this way you can compute \( e^n \) by just using multiplications.
Quick Check

Write a recursive definition of

\[ f(n) = 5 \times n \]

where \( n > 0 \).
Quick Check

Write a recursive definition of

\[ f(n) = 5 \times n \]

where \( n > 0 \).

\[
5 \times 1 = 5 \\
5 \times n = 5 + (5 \times (n-1))
\]

\[ f(1) = 5 \]
\[ f(n) = 5 + f(n-1) \]
Quick Check

Write a recursive definition of $f(n) = \frac{(n+1)n}{2}$, where $n > 0$. 
Quick Check

Write a recursive definition of \( f(n) = \frac{(n + 1)n}{2} \), where \( n > 0 \).

\[
f(1) = 1
\]

\[
f(n+1) = \frac{(n + 2)(n + 1)}{2} = \frac{(n + 1)n}{2} + \frac{2(n+1)}{2} = f(n) + (n + 1)
\]
Outline

Recursive Thinking
Recursive Programming
Using Recursion
Recursion in Graphics
A (non recursive) Method

```java
private class SliderListener implements ChangeListener {
    private int red, green, blue;

    // gets the value of each slider, then updates the labels and the color panel.

    public void stateChanged (ChangeEvent event) {
        red = rSlider.getValue();
        green = gSlider.getValue();
        blue = bSlider.getValue();

        rLabel.setText ("Red: "+ red);
        gLabel.setText ("Green: "+ green);
        bLabel.setText ("Blue: " + blue);

        colorPanel.setBackground (new Color (red, green, blue));
    }
}
```
Recursive Programming

• A **recursive method** is a method that **invokes** itself

• A recursive method must be structured to handle both the **base case** and the **recursive case**

• **Each call** to the method sets up a **new execution environment**, with new parameters and local variables

• As with any method call, when the method completes, control returns to the method that invoked it (which may be an earlier invocation of itself).
Sum of 1 to N

- Consider the problem of computing the sum of all the numbers between 1 and any positive integer N.

- This problem can be recursively defined as:

\[
\sum_{i=1}^{N} i = N + \sum_{i=1}^{N-1} i = N + (N-1) + \sum_{i=1}^{N-2} i \\
= N + (N-1) + (N-2) + \sum_{i=1}^{N-3} i \\
= N + (N-1) + (N-2) + \cdots + 2 + 1
\]
Sum of 1 to N

- The summation could be implemented recursively as follows:

```java
// This method returns the sum of 1 to num
public int sum (int num) {
    int result;

    if (num == 1)
        result = 1;
    else
        result = num + sum (num-1);

    return result;
}
```
public int sum (int num) {
    int result;
    if (num == 1)
        result = 1;
    else
        result = num + sum (num-1);
    return result;
}
Recursive Programming

• Note that just because we can use recursion to solve a problem, doesn't mean we should

• We usually would not use recursion to solve the summation problem, because the iterative version is easier to understand

• However, for some problems, recursion provides an elegant solution, often cleaner than an iterative version (e.g. Fibonacci)

• You must carefully decide whether recursion is the correct technique for any problem
Quiz

• Write a recursive method that computes the factorial of a non-negative int number n:
  factorial(0)=1, factorial(n) = n * factorial(n-1)
public int factorial(int n) {
    if (n == 0)
        return 1;
    return n * factorial(n-1);
}

factorial(4)
    factorial(3)
        factorial(2)
            factorial(1)
                factorial(0)
                    return 1
                    return 1*1 = 1
                    return 2*1 = 2
                    return 3*2 = 6
                    return 4*6 = 24
Indirect Recursion

• A method invoking itself is considered to be *direct recursion*

• A method could invoke another method, which invokes another, etc., until eventually the original method is invoked again

• For example, method m1 could invoke m2, which invokes m3, which in turn invokes m1 again

• This is called *indirect recursion*, and requires all the same care as direct recursion

• It is often more difficult to trace and debug
Indirect Recursion
Quiz

L is the left propagation and R is the right propagation (they are two functions that map a non negative integer into a non negative integer):

- \( L(n) = L(R(n-1)) \)
- \( R(n) = R(L(n-1)) \)
- \( L(1) = R(1) = k \)

- For which values of \( k > 0 \), \( L(n) \) and \( R(n) \) computation terminates for all \( n > 0 \)?
Quiz

L is the left propagation and R is the right propagation:

• \( L(n) = L(R(n-1)) \)
• \( R(n) = R(L(n-1)) \)
• \( L(1) = R(1) = k \)

For which values of \( k > 0 \), \( L(n) \) and \( R(n) \) computation terminates for all \( n > 0 \)?

• **Only \( k = 1 \)** (and \( L \) and \( R \) are constant functions = 1)

In fact, if \( k = 2 \) then:
  – \( L(2) = L(R(1)) = L(2) \) infinite loop

If \( k = 3 \) then:
  – \( L(3) = L(R(2)) = L(R(L(1))) = L(R(3)) = L(R(2)) \) infinite loop

If \( k = 4 \) …
Exercise

• Implement a class Chap12 that contains the right and left (static) methods described before and test that there is an infinite loop if \( k > 1 \)
public class Chap12 {

    private final static int K = 1;

    public static int right(int n) {
        System.out.println("right");
        if (n == 1)
            return K;
        return right(left(n-1));
    }

    public static int left(int n) {
        System.out.println("left");
        if (n == 1)
            return K;
        return left(right(n-1));
    }

    public static void main(String[] args) {
        System.out.println(right(2));
    }
}
Quiz

• What does the following recursive function return? Try it when the parameter \( s \) is your name.

```java
public String mystery(String s) {
    int N = s.length();
    if (N <= 1)
        return s;
    String a = s.substring(0, N/2);
    String b = s.substring(N/2, N);
    return mystery(b) + mystery(a);
}
```
Quiz

• What does the following recursive function return?

```java
public String mystery(String s) {
    int N = s.length();
    if (N <= 1)
        return s;
    String a = s.substring(0, N/2);
    String b = s.substring(N/2, N);
    return mystery(b) + mystery(a);
}
```

The reverse of the input string.
Trace

```
return "nhoj"

mistery("john")

return "oj"

mistery("jo")

return "j"

mistery("j")

return "o"

mistery("o")

return "h"

mistery("h")

return "n"

mistery("n")
```

(hopefully the tree or chart is displayed correctly)
Mathematical Induction

- Recursive programming is directly related to mathematical induction, a technique for proving facts about discrete functions.
- Proving by mathematical induction that a statement involving an integer \( N \) is true for all \( N \) involves two steps:
  - **The base case:** to prove the statement true for some specific value or values of \( N \) (usually 0 or 1).
  - **The induction step:** assume that a statement is true for all positive integers less than \( N \), then use that fact to prove it true for \( N \).
Proof by Induction Example

• Prove that:
  – \( 1 + 2 + 3 + 4 + \ldots + N = \frac{(N+1)N}{2} \)

• Base case:
  – \( 1 = \frac{(1+1)1}{2} \) TRUE

• Induction step:
  – Assume that it is true for \( N-1 \)
    • \( 1+ \ldots+ (N-1) = \frac{N(N-1)}{2} \)
  – Then:
    • \( 1+ \ldots+ (N-1) + N = \frac{N(N-1)}{2} + N \)
    • = \( \frac{N^2 - N + 2N}{2} \)
    • = \( \frac{N^2 + N}{2} = \frac{(N + 1)N}{2} \) Q.E.D.
Without using Induction

• Prove that:
  – $1 + 2 + 3 + 4 + \ldots + N = S_n = (N+1)N/2$

• $(1 + 2 + 3 + 4 + \ldots + N) + (1 + 2 + 3 + 4 + \ldots + N) = 2S_n$

• $(1 + 2 + 3 + 4 + \ldots + N) + (N + (N-1) + \ldots + 1) = 2S_n$

• $(1 + N) + (2 + N-1) + (3 + N-2) + \ldots + (N + 1) = 2S_n$

• $N(N+1) = 2S_n$

• $S_n = (N+1)N/2$
Quiz

• Consider the fibonacci sequence:
  – \( f(0) = 0, f(1) = 1, f(n) = f(n-1) + f(n-2) \)
  – \( f(0) = 0, f(1) = 1, f(2) = f(1) + f(0) = 1, f(3) = f(2) + f(1) = 2, f(4) = 3, f(5) = 5, f(6) = 8, f(7) = 13, \ldots \)

• Prove by induction that:
  – For all \( n > 0 \), \( f(3n) \) is even
Quiz

• Consider the fibonacci sequence:
  – \( f(0) = 0, f(1) = 1; f(n) = f(n-1) + f(n-2) \)
  – 0, 1, 1, 2, 3, 5, 8, 13, ..

• Prove by induction that:
  – For all \( n > 0 \), \( f(3n) \) is even

• Base case \( n=1 \)
  – \( f(3) = 2 \) TRUE

• Induction step
  – if \( f(3n) \) is even we must prove that \( f(3(n+1)) \) is even
  – \( f(3(n+1)) = f(3n+2) + f(3n+1) = f(3n+1) + f(3n) + f(3n+1) = 2f(3n+1) + f(3n) \) THIS is EVEN
Recursion can be inefficient

```c
int fibonacci(int n) {
    if (n < 2)
        return n;
    return fibonacci(n - 1) + fibonacci(n - 2);
}
```

How many calls to `fibonacci` for computing `fibonacci(5)`?
Recursion can be inefficient

```c
int fibonacci(int n) {
    if (n < 2)
        return n;
    return fibonacci(n - 1) + fibonacci(n - 2);
}
```

How many calls to `fibonacci` for computing `fibonacci(5)`?

```
f(5)
f(4)    f(3)    f(3)
f(3)    f(2)    f(2)    f(1)
f(2)    f(1)    f(1)    f(0)    f(1)    f(0)
f(1)    f(0)    f(1)    f(0)
```

15 calls
Trace

```
return 5
  return 3
    return 2
      return 1
        return 1
      return 1
    return 1
    return 1
  return 1
    return 0
      return 1
    return 1
  return 1
    return 0
      return 1
    return 0
```
Iterative version of Fibonacci

```c
int itFibonacci(int n) {
    int result = 0, prec = 1;
    for (int i = 1; i <= n; i++) {
        result += prec; // f(i) = f(i-1) + f(i-2)
        prec = result - prec; // f(i-1) = f(i) - f(i-2)
    }
    return result;
}
```

\[ f(0) = 0, \ f(1) = 1, \ f(2) = 1, \ldots, \ f(n) = f(n-1) + f(n-2) \]
\[ 0, 1, 1, 2, 3, 5, 8, 13, \ldots \]
Outline

Recursive Thinking
Recursive Programming
Using Recursion
Recursion in Graphics
Maze Traversal

• We can use recursion to find a path through a maze
• From each location, we can search in each direction
• The recursive calls keep track of the path through the maze
• The base case is an invalid move or reaching the final destination

• See MazeSearch.java
• See Maze.java
public class MazeSearch
{
    public static void main (String[] args)
    {
        Maze labyrinth = new Maze();

        System.out.println (labyrinth);

        if (labyrinth.traverse (0, 0))
            System.out.println ("The maze was successfully traversed!");
        else
            System.out.println ("There is no possible path.");

        System.out.println (labyrinth);
    }
}
public class MazeSearch {
    // Creates a new maze, prints its original form, attempts to solve it, and prints out its final form.
    public static void main (String[] args) {
        Maze labyrinth = new Maze();
        System.out.println(labyrinth);
        if (labyrinth.traverse(0, 0)) System.out.println("The maze was successfully traversed!");
        else System.out.println("There is no possible path.");
        System.out.println(labyrinth);
    }
}
public class Maze
{
    private final int TRIED = 3;
    private final int PATH = 7;

    private int[][] grid = {
        {1,1,1,0,1,0,0,1,1,1,1,1},
        {1,0,1,1,0,1,1,1,1,0,0,1},
        {0,0,0,1,0,1,0,1,0,1,0,0},
        {1,1,1,0,1,1,1,0,1,0,1,1},
        {1,0,1,0,0,0,1,1,1,0,0,1},
        {1,0,1,1,1,1,1,1,1,1,1,1},
        {1,0,1,0,0,0,0,0,0,0,0,0},
        {1,1,1,1,1,1,1,1,1,1,1,1}
    };
}

//********************************************************************************
// Maze.java       Author: Lewis/Loftus
//
// Represents a maze of characters. The goal is to get from the
// top left corner to the bottom right, following a path of 1s.
//********************************************************************************

continued
public boolean traverse (int row, int column) {
    boolean done = false;

    if (valid (row, column)) {
        grid[row][column] = TRIED; // this cell has been tried

        if (row == grid.length - 1 && column == grid[0].length - 1)
            done = true; // the maze is solved — base case
        else {
            done = traverse (row+1, column); // down
            if (!done)
                done = traverse (row, column+1); // right
            if (!done)
                done = traverse (row-1, column); // up
            if (!done)
                done = traverse (row, column-1); // left
        }

        if (done) // this location is part of the final path
            grid[row][column] = PATH;
    }

    return done;
}
// Determines if a specific location is valid.
// -----------------------------------------------------------------
private boolean valid (int row, int column)
{
    boolean result = false;

    // check if cell is in the bounds of the matrix
    if (row >= 0 && row < grid.length &&
        column >= 0 && column < grid[row].length)
    // check if cell is not blocked and not previously tried
    if (grid[row][column] == 1)
        result = true;

    return result;
}
public String toString ()
{
    String result = "\n";
    for (int row=0; row < grid.length; row++)
    {
        for (int column=0; column < grid[row].length; column++)
            result += grid[row][column] + "";
        result += "\n";
    }
    return result;
}
Quiz

• Trace the calls to `traverse()` for the maze row0=11, row1=01
Trace

```
traverse(0,0)
traverse(0,1)
traverse(1,0)
traverse(1,1)
return true
return false
return true
return true
```
Quiz

• More elaborated trace the calls to traverse() and valid() for the maze row0=11, row1=01
Towers of Hanoi

• The *Towers of Hanoi* is a puzzle made up of three vertical pegs and several disks that slide onto the pegs.

• The disks are of varying size, initially placed on one peg with the largest disk on the bottom with increasingly smaller ones on top.
Towers of Hanoi

• The goal is to move all of the disks from one peg to another

• Under the following rules:
  – Move only one disk at a time
  – A larger disk cannot be put on top of a smaller one
Towers of Hanoi

Original Configuration

Move 1

Move 2

Move 3
Towers of Hanoi

Move 4

Move 5

Move 6

Move 7 (done)
Recursive Solution

• To solve a N-tower

  1. Solve the (N-1)-tower: move the (N-1)-tower in the middle peg
  2. Move the largest disc to target peg
  3. Solve the (N-1)-tower: move the (N-1)-tower from the middle peg to the target peg
Towers of Hanoi

- An iterative solution to the Towers of Hanoi is quite complex
- A recursive solution is much shorter and more elegant
- See `SolveTowers.java`
- See `TowersOfHanoi.java`
public class SolveTowers
{
    //---------------------------------------------
    // Creates a TowersOfHanoi puzzle and solves it.
    //---------------------------------------------
    public static void main (String[] args)
    {
        TowersOfHanoi towers = new TowersOfHanoi (4);

        towers.solve();
    }
}
public class SolveTowers {

//-----------
// Creates a TowersOfHanoi puzzle and solves it.
//-----------
public static void main (String[] args) {
    TowersOfHanoi towers = new TowersOfHanoi (4);
    towers.solve();
}

Output
Move one disk from 1 to 2
Move one disk from 1 to 3
Move one disk from 2 to 3
Move one disk from 1 to 2
Move one disk from 3 to 1
Move one disk from 3 to 2
Move one disk from 1 to 2
Move one disk from 1 to 3
Move one disk from 2 to 3
Move one disk from 2 to 1
Move one disk from 3 to 1
Move one disk from 2 to 3
Move one disk from 1 to 2
Move one disk from 1 to 3
Move one disk from 2 to 3
public class TowersOfHanoi
{
    private int totalDisks;

    public TowersOfHanoi (int disks)
    {
        totalDisks = disks;
    }

    public void solve ()
    {
        moveTower (totalDisks, 1, 3, 2);
    }
}
private void moveTower (int numDisks, int start, int end, int temp) {
    if (numDisks == 1)
        moveOneDisk (start, end);
    else
    {
        moveTower (numDisks-1, start, temp, end);
        moveOneDisk (start, end);
        moveTower (numDisks-1, temp, end, start);
    }
}

private void moveOneDisk (int start, int end) {
    System.out.println ("Move one disk from " + start + " to " + end);
}
Quiz

public int mystery(int x, int y) {
    if (x == y) return 0;
    else return mystery(x - 1, y) + 1;
}

If the method is called as mystery(8, 3), what is returned?

A) 11  
B) 8    
C) 5    
D) 3    
E) 24

public int mystery(int x, int y) {
    if (x == y) return 0;
    else return mystery(x - 1, y) + 1;
}

If the method is called as mystery(8, 3), what is returned?

C) 5

The method computes x - y if x > y. The method works as follows: each time the method is called recursively, it subtracts 1 from x until (x == y) is becomes true, and adds 1 to the return value. So, 1 is added each time the method is called, and the method is called once for each int value between x and y.
Quiz

Calling the previous method will result in infinite recursion if which condition below is initially true?

A) \((x == y)\)
B) \((x != y)\)
C) \((x > y)\)
D) \((x < y)\)
E) \((x == 0 && y != 0)\)
Quiz

Calling the previous method will result in infinite recursion if which condition below is initially true?

A) \((x == y)\)
B) \((x != y)\)
C) \((x > y)\)
D) \((x < y)\)
E) \((x == 0 && y != 0)\)

If \((x < y)\) is true initially, then the else clause is executed resulting in the method being recursively invoked with a value of \(x - 1\), or a smaller value of \(x\), so that \((x < y)\) will be true again, and so for each successive recursive call, \((x < y)\) will be true and the base case, \(x == y\), will never be true.
Quiz
What does the following method compute? Assume the method is called initially with a string a, a char b and i = 0

public int mystery(String a, char b, int i) {
    if (i == a.length()) return 0;
    else if (b == a.charAt(i))
        return mystery(a, b, i+1) + 1;
    else return mystery(a, b, i+1);
}
Quiz

What does the following method compute? Assume the method is called initially with $i = 0$

```java
public int mystery(String a, char b, int i) {
    if (i == a.length()) return 0;
    else if (b == a.charAt(i))
        return mystery(a, b, i+1) + 1;
    else return mystery(a, b, i+1);
}
```

The number of times char $b$ appears in String $a$. The method compares each character in String $a$ with char $b$ until $i$ reaches the length of String $a$. 1 is added to the return value for each match.
Outline

Recursive Thinking
Recursive Programming
Using Recursion
Recursion in Graphics
Tiled Pictures

• Consider the task of repeatedly displaying a set of images in a mosaic
  – Three quadrants contain individual images
  – Upper-left quadrant repeats pattern

• The base case is reached when the area for the images shrinks to a certain size

• See TiledPictures.java
import java.awt.*;
import javax.swing.JApplet;

public class TiledPictures extends JApplet {

    private final int APPLET_WIDTH = 320;
    private final int APPLET_HEIGHT = 320;
    private final int MIN = 20;  // smallest picture size

    private Image world, everest, goat;

    // Demonstrates the use of recursion.

    // TiledPictures.java    Author: Lewis/Loftus
    //
   //********************************************************************
   //********************************************************************
    

continue
// Loads the images.

public void init()
{
    world = getImage (getDocumentBase(), "world.gif");
    everest = getImage (getDocumentBase(), "everest.gif");
    goat = getImage (getDocumentBase(), "goat.gif");

    setSize (APPLET_WIDTH, APPLET_HEIGHT);
}

// Draws the three images, then calls itself recursively.

public void drawPictures (int size, Graphics page)
{
    page.drawImage (everest, 0, size/2, size/2, size/2, this);
    page.drawImage (goat, size/2, 0, size/2, size/2, this);
    page.drawImage (world, size/2, size/2, size/2, size/2, this);

    if (size > MIN)
        drawPictures (size/2, page);
}
continue

    //  Performs the initial call to the drawPictures method.
    //-----------------------------------------------
    public void paint (Graphics page)
    {
        drawPictures (APPLET_WIDTH, page);
    }
}
continue

// ------------------
// Performs the initial call to the drawPictures method.
// ------------------
public void
paint (Graphics page)
{
    drawPictures (APPLET_WIDTH, page);
}
}
import javax.swing.JFrame;

public class TiledPicturesApp {

    public static void main(String[] args) {
        JFrame frame = new JFrame("Tiled Pictures");
        frame.setDefaultCloseOperation(JFrame.EXIT_ON_CLOSE);
        frame.getContentPane().add(new TiledPicturesPanel());
        frame.pack();
        frame.setVisible(true);
    }
}

Application version of the previous applet
import java.awt.*;
import java.awt.image.BufferedImage;
import java.io.File;
import java.io.IOException;
import javax.imageio.ImageIO;
import javax.swing.JPanel;

public class TiledPicturesPanel extends JPanel {

    private final int PANEL_WIDTH = 320;
    private final int PANEL_HEIGHT = 320;
    private final int MIN = 20; // smallest picture size

    private BufferedImage world, everest, goat;

    continue
// Loads the images.
public TiledPicturesPanel() {
    try {
        world = ImageIO.read(new File("world.gif"));
        everest = ImageIO.read(new File("everest.gif"));
        goat = ImageIO.read(new File("goat.gif"));
    } catch (IOException e) {
    }
    setPreferredSize(new Dimension(PANEL_WIDTH, PANEL_HEIGHT));
}

continue
public void drawPictures(int size, Graphics page) {
    page.drawImage(everest, 0, size / 2, size / 2, size / 2, this);
    page.drawImage(goat, size / 2, 0, size / 2, size / 2, this);
    page.drawImage(world, size / 2, size / 2, size / 2, size / 2, this);

    if (size > MIN) {
        drawPictures(size / 2, page);
    }
}

public void paintComponent(Graphics page) {
    super.paintComponent(page);
    drawPictures(PANEL_WIDTH, page);
}

Fractals

• A fractal is a geometric shape made up of the same pattern repeated in different sizes and orientations

• The Koch Snowflake is a particular fractal that begins with an equilateral triangle

• To get a higher order of the fractal, the sides of the triangle are replaced with angled line segments

• See KochSnowflake.java
• See KochPanel.java
This is an application; in the book you find an applet.
// Demonstrates the use of recursion in graphics.

import java.awt.*;
import java.awt.event.*;
import javax.swing.*;

public class KochMainPanel extends JPanel implements ActionListener {
    private final int PANEL_WIDTH = 400;
    private final int PANEL_HEIGHT = 440;

    private final int MIN = 1, MAX = 9;

    private JButton increase, decrease;
    private JLabel titleLabel, orderLabel;
    private KochPanel drawing;
    private JPanel tools;

    continue
// Sets up the components for the panel.

public KochMainPanel()
{
    tools = new JPanel();
    tools.setLayout (new BoxLayout(tools, BoxLayout.X_AXIS));
    tools.setPreferredSize (new Dimension (PANEL_WIDTH, 40));
    tools.setBackground (Color.yellow);
    tools.setOpaque (true);

    titleLabel = new JLabel ("The Koch Snowflake");
    titleLabel.setForeground (Color.black);

    increase = new JButton (new ImageIcon ("increase.gif");
    increase.setPressedIcon (new ImageIcon ("increasePressed.gif");
    increase.addActionListener (this);

    decrease = new JButton (new ImageIcon ("decrease.gif");
    decrease.setPressedIcon (new ImageIcon ("decreasePressed.gif");
    decrease.addActionListener (this);
}
orderLabel = new JLabel("Order: 1");
orderLabel.setForeground(Color.black);

tools.add (titleLabel);
tools.add (Box.createHorizontalStrut (40));
tools.add (decrease);
tools.add (increase);
tools.add (Box.createHorizontalStrut (20));
tools.add (orderLabel);

drawing = new KochPanel (1);

add (tools);
add (drawing);

setPreferredSize (PANEL_WIDTH, PANEL_HEIGHT);
}
public void actionPerformed (ActionEvent event) {
    int order = drawing.getOrder();

    if (event.getSource() == increase)
        order++;
    else
        order--;

    if (order >= MIN && order <= MAX)
    {
        orderLabel.setText("Order: " + order);
        drawing.setOrder(order);
        repaint();
    }
}
```java
public void actionPerformed(ActionEvent event) {
    int order = drawing.getOrder();
    if (event.getSource() == increase)
        order++;
    else
        order--;
    if (order >= MIN && order <= MAX)
        orderLabel.setText("Order: " + order);
    drawing.setOrder(order);
    repaint();
}
```
Koch Snowflakes

\[ \langle x_5, y_5 \rangle \]

Becomes

\[ \langle x_5, y_5 \rangle \]

\[ \langle x_4, y_4 \rangle \]

\[ \langle x_3, y_3 \rangle \]

\[ \langle x_2, y_2 \rangle \]

\[ \langle x_1, y_1 \rangle \]

\[ \langle x_1, y_1 \rangle \]

\[ = \sqrt{3/6} \cdot |P_5 - P_1| \]
public class KochPanel extends JPanel {
    private final int PANEL_WIDTH = 400;
    private final int PANEL_HEIGHT = 400;
    private final double SQ = Math.sqrt(3.0) / 6;
    private final int TOPX = 200, TOPY = 20;
    private final int LEFTX = 60, LEFTY = 300;
    private final int RIGHTX = 340, RIGHTY = 300;
    private int current; // current order
    continue
public void drawFractal (int order, int x1, int y1, int x5, int y5, Graphics page) {
    int deltaX, deltaY, x2, y2, x3, y3, x4, y4;

    if (order == 1)
        page.drawLine (x1, y1, x5, y5);
    else
        {  
            deltaX = x5 - x1;  // distance between end points
            deltaY = y5 - y1;

            x2 = x1 + deltaX / 3;  // one third
            y2 = y1 + deltaY / 3;

            x3 = (int) ((x1+x5)/2 + SQ * (y1-y5));  // tip of projection
            y3 = (int) ((y1+y5)/2 + SQ * (x5-x1));

            continue
\[ x_4 = x_1 + \text{deltaX} \times \frac{2}{3}; \quad \text{// two thirds} \]
\[ y_4 = y_1 + \text{deltaY} \times \frac{2}{3}; \]

drawFractal (order-1, x1, y1, x2, y2, page);
drawFractal (order-1, x2, y2, x3, y3, page);
drawFractal (order-1, x3, y3, x4, y4, page);
drawFractal (order-1, x4, y4, x5, y5, page);
}
}

//---

// Performs the initial calls to the drawFractal method.
//---

public void paintComponent (Graphics page) {
  super.paintComponent (page);

  page.setColor (Color.green);

  drawFractal (current, TOPX, TOPY, LEFTX, LEFTY, page);
drawFractal (current, LEFTX, LEFTY, RIGHTX, RIGHTY, page);
drawFractal (current, RIGHTX, RIGHTY, TOPX, TOPY, page);

  continue
```java
continue

// Sets the fractal order to the value specified.
public void setOrder (int order)
{
    current = order;
}

// Returns the current order.
public int getOrder ()
{
    return current;
}
```
Summary

• Chapter 12 has focused on:
  – thinking in a recursive manner
  – programming in a recursive manner
  – the correct use of recursion
  – recursion examples
Exercise

- Write a recursive definition of a valid Java identifier (see Chapter 1). Imagine that a Java Letter is either a letter of the English alphabet or $ or _
Exercise

• Write a recursive definition of a valid Java identifier (see Chapter 1). Imagine that a Java Letter is either a letter of the English alphabet or $ or _

A Java-Identifier is a: Letter
or a: Java-Identifier followed by a Letter
or a: Java-Identifier followed by a digit
Exercise

• Write a recursive definition of \( f(n) = n \times k \) (integer multiplication), where \( n > 0 \).

• Define the multiplication process in terms of integer addition.
  – For example, \( 4 \times 7 \) is equal to 7 added to itself 4 times.
Exercise

• Write a recursive definition of \( f(n) = n \times k \) (integer multiplication), where \( n > 0 \).

• Define the multiplication process in terms of integer addition.
  – For example, \( 4 \times 7 \) is equal to 7 added to itself 4 times.

  • \( 1 \times k = k \), if \( n=1 \)
  • \( n \times k = k + (n-1) \times k \) for \( n > 1 \)
Exercise

• Modify the method that calculates the sum of the integers between 1 and N shown in this lecture. Have the new version match the following recursive definition of the sum of the integers between n and m (n\leq m):
  – Sum(n, m) = n, if n=m
  – Sum(n, m) = Sum(n, (n+m)/2) + Sum((n+m)/2 +1, m), if n<m

• Trace your solution using for N = 7 (n=1, m=7).
public int sum(int n1, int n2) {
    int result;
    if (n2 - n1 == 0) {
        result = n1;
    } else {
        int mid = (n1 + n2) / 2;
        result = sum(n1, mid) + sum(mid + 1, n2);
    }
    return result;
}
Trace
Exercise

• Write (another) recursive method to reverse a string.
• Use the following String methods
  – charAt(int n) : char
  – substring(int beginIndex, int endIndex) : String
• Implement a procedure that concatenates the last character of the input string with the (recursive) reverse of the string composed by the first N-1 characters of the string (N is the length of the string).
public String reverse(String text) {
    String result = text;
    if (text.length() > 1) {
        result = text.charAt(text.length() - 1) +
                 reverse(text.substring(0, text.length() - 1));
    }
    return result;
}
Tower of Hanoi

- Produce a chart showing the number of moves required to solve the Towers of Hanoi puzzle using the following number of disks: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10
- Write a recursive definition of the formula giving Moves(n), the number of moves required to solve the Hanoi tower of n disks
### Solution

<table>
<thead>
<tr>
<th>Disks</th>
<th>Moves</th>
<th>Moves(n) = Moves(n-1) + 1 + Moves(n-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Moves(1) = 1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>31</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>63</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>127</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>255</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>511</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1023</td>
<td></td>
</tr>
</tbody>
</table>
Kock Snowflake

• How many line segments are used to construct a Koch snowflake of order N? Produce a chart showing the number of line segments that make up a Koch snowflake for orders 1 through 7.

• Give the general formula Segments(n) = ?
## Solution

<table>
<thead>
<tr>
<th>Order</th>
<th>Segments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4*3 = 12</td>
</tr>
<tr>
<td>3</td>
<td>4<em>4</em>3 = 48</td>
</tr>
<tr>
<td>4</td>
<td>4<em>4</em>4*3 = 192</td>
</tr>
<tr>
<td>5</td>
<td>4<em>4</em>4<em>4</em>3 = 768</td>
</tr>
<tr>
<td>6</td>
<td>4<em>4</em>4<em>4</em>4*3 = 3072</td>
</tr>
<tr>
<td>7</td>
<td>4<em>4</em>4<em>4</em>4<em>4</em>3 = 12288</td>
</tr>
</tbody>
</table>

Segments(1) = 3  
Segments(n) = Segments (n-1) * 4  
Segments(n) = 3*4^{n-1}
Exercise

• Give the value of mistery2(3):

```java
public static String mistery2(int n) {
    if (n <= 0) {
        return "";
    }
    return mistery2(n - 3) + n + mistery2(n - 2) + n;
}
```
Trace

misery2(3)

return "" + 3 + "11" + 3 = "3113"

misery2(0)

return ""

misery2(1)

return "" + 1 + "" + 1

misery2(-2)

return ""

misery2(-1)

return ""
Exercise

• Criticize the following recursive function:

```java
public static String mistery3(int n) {
    String s = mistery3(n - 3) + n + mistery3(n - 2) + n;
    if (n <= 0)
        return "";
    return s;
}
```