Will my Ontologies Fit Together?

Technical Report

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1 Motivation

In realistic applications, it is often desirable to integrate different ontologies[1] into a single, reconciled ontology. Ideally, one would expect the individual ontologies to be developed as independently as possible from the rest, and the final reconciliation to be seamless and free from unexpected results. This would allow for the modular design of large ontologies and would facilitate knowledge reuse tasks. Few ontology development tools, however, provide any support for integration, and there has been relatively little study of the problem at a fundamental level.

So far, ontology integration problems have been mostly tackled in an ad-hoc manner, with no clear notion of what to expect from the integrated ontology. The result is the adoption of unpredictable techniques as a common practice, which partially ignore the semantics of the ontologies, and may lead to undesired results, even if the ontologies to be integrated are widely tested and understood [6].

We distinguish three basic ontology integration scenarios which capture some of the common practices in the Ontology Engineering community:

1. *Foundational integration*: an ontology is integrated with a foundational (or "upper") ontology. The foundational ontology describes more general terms, and may be domain independent.
2. *Broadening integration*: two ontologies describing different (and largely independent) domains are integrated in order to cover a broader subject matter.

[1]Throughout this paper, we do not distinguish between ontologies and TBoxes.

3. *Deepening integration*: two ontologies describing different (and largely independent) aspects of the same domain are integrated in order to provide more detailed coverage.

In this paper we define, for each scenario, *semantic* properties that should (we believe) be satisfied by the integrated ontology; that is, we specify how the consequences of the integrated ontology relate to those from its parts. Next, we specify *syntactic* constraints on the ontologies to be integrated which ensure the satisfaction of these properties. These constraints clearly depend on the DL used in the ontologies, and mainly concern the way the symbols occurring in the different ontologies (their *signatures*) are used. Finally, we discuss whether these constraints are realistic for the scenario, i.e., whether users could be expected to stick happily to these constraints in order to ensure that the integrated ontology will satisfy the desired semantic properties.

These issues depend on the integration scenario under consideration, and are tightly related: if one does not maintain a certain syntactic discipline, then it is unlikely that the integrated ontology will behave as expected; conversely, we need to define precisely what it means for the integrated ontology to be well-behaved in a given scenario, in order to be able to define a suitable discipline.

In what follows we will discuss, for each of our scenarios, which are the desirable semantic properties of the integration; define, for some cases, appropriate syntactic constrains, and prove that integrations satisfying these constraints enjoy the desired semantic properties. This is a first step towards our goal to gain a deeper understanding of the issues involved in ontology integration tasks and to provide a well-founded methodology that can be easily supported in existing ontology development tools.

We assume the reader to be familiar with the basics of description logics and use, throughout this paper, *axiom* for any kind of TBox, RBox, or ABox assertion, and $\mathsf{Sig}(\mathcal{T})$ for the set of concept and roles names in $\mathcal{T}$.

2 Integration Scenarios

Suppose that two ontologies $\mathcal{T}_1, \mathcal{T}_2$ shall be integrated in some application. The ontologies may be the result of a collaborative ontology development process and may have been designed in a coordinated way by different groups of experts, or they may have been simply "borrowed" from the Web. In any case, we assume that they have both been tested and debugged individually prior to the integration and, hence, are consistent and do not contain unsatisfiable concept names. More precisely, given an ontology $\mathcal{T}$, we call $\mathcal{T}$ *instantiable* if $\mathcal{T}$ is consistent and s.t. there is a model $\mathcal{I} \models \mathcal{T}$ where $A^I \neq \emptyset$ and

$R^{\mathcal{I}} \neq \varnothing$ for all concept names (role names) A (R) in the signature of $\mathcal{T}$.[2]

In the simplest case, one would construct a reconciled ontology $\mathcal{T}$ by simply taking the *union* of both. In general, the ontologies $\mathcal{T}_1$ and $\mathcal{T}_2$ may be related and share symbols in their signatures $\mathsf{Sig}(\mathcal{T}_1)$ and $\mathsf{Sig}(\mathcal{T}_2)$.[3] We will first identify the semantic properties that $\mathcal{T}$ should satisfy in order to capture the modeling intuitions of each scenario, and then to see which "acceptable" syntactic restrictions on the $\mathcal{T}_i$ make sure that $\mathcal{T}$ will behave as expected. The intuition is simple: the more liberal the syntactic constraints including the use of shared symbols, the more freedom is given to the modeler, but the less likely it is that the integrated ontology will behave as expected.

In this Section, we describe and formalize our integration scenarios and argue, in each case, which properties should be expected to hold in the merged ontology $\mathcal{T}$. Then we specify, for all but the deepening integration, syntactic restrictions that guarantee these properties and discuss their usefulness. The proofs for our initial results are provided in the Appendix.

2.1 Foundational Integration

Often, interoperability between different *domain* ontologies $\mathcal{T}_{dom}$ and their data is achieved through the use of a *foundational* (or "upper") ontology $\mathcal{T}_{up}$. A well designed foundational ontology should provide a carefully conceived high level axiomatization of general purpose concepts. Foundational ontologies, thus, provide a structure upon which ontologies for specific subject matters can be constructed.

A prominent example of an ontology conceived as the integration of a foundational ontology and a set of domain ontologies is GALEN [7], a large medical ontology designed for supporting clinical information systems. The foundational ontology contains generic concepts, such as *Process* or *Substance*. The domain ontologies contain concepts such as *Gene* or *Research Institution*, which are specific to a certain subject matter. The domain ontologies in GALEN are connected to the foundational ontology through subsumption relations between concept and role names. For example, Microorganism in the domain ontology is a subconcept of Organism in the foundational ontology:

$$\mathsf{Microorganism} \sqsubseteq \mathsf{Organism}$$

Some prominent ontologies, such as CYC, SUMO and DOLCE have been designed specifically to be used in applications as foundational ontologies. For

[2] For a logic that is closed under disjoint unions, such as $\mathcal{SHIQ}$, in order to ensure instantiability of $\mathcal{T}$ it suffices to check that all the concept and role names in $\mathcal{T}$ are satisfiable.

[3] There may be some previous reconciliation w.r.t. symbols, e.g., to identify different symbols in the two ontologies that have the same intended meaning [6]. This is a separate problem, often referred to as *ontology alignment*, which we do not address here.

example, given a large dataset about chemicals annotated with NCI concepts, one may want to annotate it semi-automatically with concepts of a different biomedical ontology. For such a purpose, one may align organic chemicals in NCI to substances in SUMO using the axiom:

$$\mathsf{Organic\_Chemical} \sqsubseteq \mathsf{Substance}.$$

Similarly, one may want to use a foundational ontology to generalize the roles of a given domain ontology. For example, a University ontology may use SUMO to generalize the role writes as follows:

$$\mathsf{writes} \sqsubseteq \mathsf{authors}$$

where authors, is defined in SUMO and does not occur in the University ontology.

Foundational ontologies, such as, for example, CYC, DOLCE and SUMO, are well-established ontologies that one does not control and, typically, does not fully understand. When one of these ontologies is borrowed from the Web and integrated in an application, it is especially important to make sure that the merge preserves their semantics. In particular, we shall not allow the classification tree in $\mathcal{T}_{up}$ to change as a consequence of the merge. This property can be conveniently formalized by the notion of a *conservative extension* [3].

Definition 1 ***(Conservative Extensions)*** *The TBox $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ if, for every axiom α in the signature of $\mathcal{T}_1$, if $\mathcal{T} \models \alpha$, then $\mathcal{T}_1 \models \alpha$.*

Clearly, if $\mathcal{T}$ is a conservative extension of $\mathcal{T}_1$ and $\mathcal{T}_1, \mathcal{T}_2$ are consistent, then so is $\mathcal{T}$. However, conservativeness is indeed a much stronger condition than instantiability: even if $\mathcal{T}$ is instantiable, new (and probably unintended) subsumptions between (possibly complex) concepts in $\mathcal{T}_1$ may still occur as a consequence of the merge.

In general, it may still be tolerable, and even desirable, to allow new subsumptions to occur in the domain ontology as a consequence of the integration, and in such a case, $\mathcal{T}$ will not be a conservative extension of $\mathcal{T}_{dom}$.

Also, the notion of a conservative extension is not sufficient to capture all the intended and unintended consequences. In particular, one would not expect concept names originally in $\mathcal{T}_{up}$ to be subsumed by concepts originally in $\mathcal{T}_{dom}$. In other words, the rôles of the foundational and domain ontologies should not be inverted after the merge. In contrast, new subsumptions may and should be entailed between concepts (respectively roles) in $\mathcal{T}_{dom}$ and concepts (roles) in $\mathcal{T}_{up}$. For example, since the shared concept Substance is subsumed by SelfConnectedObject in SUMO, it is expected that $\mathcal{T} = \mathcal{T}_{NCI} \cup \mathcal{T}_{SUMO}$ will entail the subsumption $\mathsf{Organic\_Chemical} \sqsubseteq \mathsf{SelfConnectedObject}$, where

Organic_Chemical occurs in NCI, but not in SUMO, whereas SelfConnectedObject occurs in SUMO, yet not in NCI.

Next, we specify the syntactic restrictions that will ensure these "nice" properties of $\mathcal{T} = \mathcal{T}_{up} \cup \mathcal{T}_{dom}$. Given the examples, it seems reasonable to limit the coupling between $\mathcal{T}_{up}$ and $\mathcal{T}_{dom}$ to subsumptions relating concept (role) names in $\mathcal{T}_{dom}$ and concept (role) names occurring in $\mathcal{T}_{up}$.

Definition 2 *The pair $\Im = \langle \mathcal{T}_{up}, \mathcal{T}_{dom} \rangle$ is* f-*compliant*[4] *if, given the shared signature $\mathbf{S} = \mathsf{Sig}(\mathcal{T}_{up}) \cap \mathsf{Sig}(\mathcal{T}_{dom})$, concept and role names $A, R \in \mathbf{S}$ occur in $\mathcal{T}_{dom}$ only in axioms of the form $B \sqsubseteq A$ and $S \sqsubseteq R$ respectively, where $B, S \in \mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S}$.*

f-compliance suffices for capturing the coupling between the foundational and the domain ontologies in GALEN. However, is f-compliance enough to guarantee our "nice" properties for $\mathcal{T} = \mathcal{T}_{up} \cup \mathcal{T}_{dom}$? A simple example will provide a negative answer: just assume that $\mathcal{T}_{dom}$ contains a GCI of the form $\top \sqsubseteq A$; after the merge, every concept in $\mathcal{T}_{up}$ will be subsumed by $A \in \mathsf{Sig}(\mathcal{T}_{dom})$ and, thus, the foundational ontology does not act as such anymore.

In order to guarantee our nice properties, we impose an additional *safety* condition on $\mathcal{T}_{up}$ and $\mathcal{T}_{dom}$ which is identical to the one introduced in [2] and which can be checked syntactically, we call this condition *localness* .

Intuitively, local ontologies contain only GCIs with a limited "global" effect. Examples of non-local axioms are GCIs that fix the size of the domain in every model of the ontology (e.g. $\top \sqsubseteq$ *bob*), or GCIs that establish the existence of a "universal" named concept (e.g. $\top \sqsubseteq Car$). Examples of local GCIs are role domain and range, and concept disjointness. And indeed, f-compliance and localness suffice:

Theorem 1 *Let $\Im = \langle \mathcal{T}_{up}, \mathcal{T}_{dom} \rangle$ be* f-*compliant. If $\mathcal{T}_{dom}$ is a local $\mathcal{SHOIQ}$ TBox, $\mathcal{T}_{up}$ is a $\mathcal{SHIQ}$ TBox (not necessarily local), and $\mathcal{T} = \mathcal{T}_{up} \cup \mathcal{T}_{dom}$ is instantiable:*

1. *$\mathcal{T} = \mathcal{T}_{up} \cup \mathcal{T}_{dom}$ is a conservative extension of $\mathcal{T}_{up}$.*
2. *There are no concept names $A \in \mathsf{Sig}(\mathcal{T}_{up})$ and $B \in \mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S}$ such that $\mathcal{T} \models A \sqsubseteq B$.*
3. *There are no role names $R \in \mathsf{Sig}(\mathcal{T}_{up})$ and $S \in \mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S}$ such that $\mathcal{T} \models R \sqsubseteq S$.*

As desired, the merge is a conservative extension of the foundational ontology, and the rôles of the foundational and domain ontologies are preserved after the merge (Items 2 and 3). Note, however, that f-compliance does not suffice for ensuring the instantiability of the merge: only if $\mathcal{T}$ is consistent and free from unsatisfiable names the guarantees provided by the theorem

[4] "f" stands for "foundational".

apply. Although instantiability, as opposed to conservative extensions, can be easily checked using a reasoner, it would indeed be desirable to strengthen the theorem to ensure the instantiability of $\mathcal{T}$ as well. Also, note that localness certainly is a too restrictive safety condition since it rules out "harmless" GCIs as well. The investigation of new f-compliance conditions that ensure the instantiability of the integrated ontology and of less strict safety conditions is the focus of our ongoing work.

2.2 Broadening Integration

In this scenario, an ontology $\mathcal{T}_1$ is to be integrated with another $\mathcal{T}_2$ that describes in more detail one or more of the domains that are only touched on in $\mathcal{T}_1$. For example, we may wish to integrate the Wine Ontology [8] with an ontology describing in more detail the regions in which wines are produced or the kinds of grapes they contain.

The Wine Ontology illustrates a common pattern: although ontologies usually refer to a *core* application domain, they also refer to other *secondary* domains that deal with different objects. This modeling paradigm is not only characteristic of small and medium sized ontologies, but also occurs in large, high-quality knowledge bases, written by groups of experts. A prominent example is the NCI Thesaurus [4], a huge ontology covering areas of basic and clinical science. The core of NCI is focused on genes; other subject matters described in the ontology include diseases, drugs, chemicals, diagnoses treatments, professional organizations, anatomy, organisms, and proteins.

In this scenario, concepts in the core application domain can be defined in terms of concepts in the secondary domains. For example, in the Wine Ontology, a Bordeaux is described as a Wine produced in France, where France is defined in the Regions ontology:

$$\mathsf{Bordeaux} \sqsubseteq \mathsf{Wine} \sqcap \exists \mathsf{producedIn}.\mathsf{France}$$

In NCI, the gene ErbB2 is an Oncogene that is found in humans and is associated with a disease called Adrenocarcinoma.

$$\mathsf{ErbB2} \sqsubseteq \mathsf{Oncogene} \sqcap \exists \mathsf{foundIn}.\mathsf{Human} \sqcap \exists \mathsf{associatedWith}.\mathsf{Adrenocarcinoma}$$

Concepts in the secondary ontologies, however, do not use the core concepts in their definitions, i.e. regions are not defined in terms of wines or diseases in terms of genes. Note, in this connection, that a 'broadening scenario' in this interpretation is closely related to the way ontologies would be integrated using the framework of $\mathcal{E}$-connections, but is rather mimicking than directly adopting the syntax and semantics of $\mathcal{E}$-connections [5].

Ontologies following this pattern can evolve by expanding their domain of discourse with knowledge about new subject matters. For example, we may extend the Wine ontology by representing the kinds of dishes each wine is most appropriate for, or NCI by adding information about useful publications on cancer research. This evolution process will typically consist of adding a new "secondary" ontology, either developed by a group of experts, or borrowed directly from the Web. As a consequence, this ontology should be "good" as it is, and thus we want to make sure that it will not be affected by the integration, i.e., we should require $\mathcal{T} = \mathcal{T}_{core} \cup \mathcal{T}_{side}$ to be a conservative extension of $\mathcal{T}_{side}$.

Furthermore, since we assume $\mathcal{T}_{core}$ and $\mathcal{T}_{side}$ to cover different aspects of the world, we require that the merged ontology $\mathcal{T}$ does not entail subsumptions in any directions between non-shared concept names $A \in \mathsf{Sig}(\mathcal{T}_{core})$ and $B \in \mathsf{Sig}(\mathcal{T}_{side})$. This condition ensures that the ontologies actually describe different objects.

Let $\mathcal{T}_{core}$ and $\mathcal{T}_{side}$ be ontologies with signatures $\mathbf{S}_{core} = \mathbf{C}_{core} \cup \mathbf{R}_{core}$ and $\mathbf{S}_{side} = \mathbf{C}_{side} \cup \mathbf{R}_{side}$, let the shared signature $\mathbf{S} = \mathbf{S}_{core} \cap \mathbf{S}_{side}$ contain only concept names, and let $\mathbf{R}_{out} \subseteq \mathbf{R}_{core}$ be a distinguished subset of roles. Intuitively, the roles in $\mathbf{R}_{out}$ connect objects in different ontologies. Some concepts in $\mathcal{T}_{core}$ are defined in terms of restrictions on these roles; for example, the Bordeaux wines are related to France via the role $\mathsf{producedIn}$ and the ErbB2 oncogenes with organisms and diseases through the roles $\mathsf{foundIn}$ and $\mathsf{associatedWith}$, respectively.

Definition 3 *The pair $\mathfrak{I} = \langle \mathcal{T}_{core}, \mathcal{T}_{side} \rangle$ is* b*-compliant if:* ***1)*** $\mathbf{S} = \mathbf{S}_{core} \cap \mathbf{S}_{side} = \mathbf{C}_{core} \cap \mathbf{C}_{side}$, $\varnothing \neq \mathbf{R}_{out} \subseteq \mathbf{R}_{core}$; ***2)*** *for every role inclusion axiom $R \sqsubseteq S \in \mathcal{T}_{core}$, either both $R, S \in \mathbf{R}_{out}$ or both $R, S \notin \mathbf{R}_{out}$;* ***3)*** *for every GCI $C_1 \sqsubseteq C_2 \in \mathcal{T}_{core}$, C_1, C_2 can be generated using the following grammar:*

$$C_i \leftarrow A | C \sqcap D | \neg C_i | \exists R.C_i | \exists P.A' | \geq nR.C_i | \geq nP.A'$$

where $A \in \mathbf{C}_{core} \setminus \mathbf{C}_{side}$, C, D and C_i are concepts generated using the grammar, $A' \in \mathbf{C}_{side}$, $R \notin \mathsf{Rol}(\mathbf{R}_{out})$, and $P \in \mathbf{R}_{out}$.

As a consequence, concept names in $\mathcal{T}_{side}$ can *only* be used in $\mathcal{T}_{core}$ through restrictions on the "outgoing" relations. Condition **2)** makes sure that the hierarchies for the two kinds of roles are disconnected from each other. The Wine Ontology and the "modules" that can be extracted from NCI [2] are local and b-compliant.

Theorem 2 *Let $\mathfrak{I} = \langle \mathcal{T}_{core}, \mathcal{T}_{side} \rangle$ be* b*-compliant. If $\mathcal{T}_{core}$ is a local $\mathcal{SHOIQ}$ TBox, $\mathcal{T}_{side}$ is a local $\mathcal{SHIQ}$ TBox, and $\mathcal{T} = \mathcal{T}_{core} \cup \mathcal{T}_{side}$ is instantiable, then:*

1. *$\mathcal{T} = \mathcal{T}_{core} \cup \mathcal{T}_{side}$ is a conservative extension of $\mathcal{T}_{side}$.*
2. *There are no concept names $A \in \mathsf{Sig}(\mathcal{T}_{core}) \setminus \mathbf{S}$ and $B \in \mathsf{Sig}(\mathcal{T}_{side})$ such that either $\mathcal{T} \models A \sqsubseteq B$, or $\mathcal{T} \models B \sqsubseteq A$.*

3. *There are no role names* $R \in \mathsf{Sig}(\mathcal{T}_{side})$ *and* $S \in \mathsf{Sig}(\mathcal{T}_{core})$ *such that* $\mathcal{T} \models R \sqsubseteq S$ *or* $\mathcal{T} \models S \sqsubseteq R$*.*
4. *There are no role names* $R \in \mathbf{R}_{out}$ *and* $S \in \mathbf{R}_{core} \setminus \mathbf{R}_{out}$ *such that* $\mathcal{T} \models R \sqsubseteq S$ *or* $\mathcal{T} \models S \sqsubseteq R$*.*

As in the foundational scenario, the theorem requires the instantiability (and thus the consistency) of the merged ontology $\mathcal{T}$.

2.3 Deepening Integration

In this scenario, an ontology $\mathcal{T}_1$ is to be integrated with another $\mathcal{T}_2$ that describes a different aspect of the same domain. For example, we may wish to integrate the Wine Ontology with an ontology that describes wines from the point of view of a wine producer rather than a wine consumer, or we may describe cathedrals from the point of view of an architect, a priest or a tourist:

$$
\begin{aligned}
\mathsf{Cathedral} &\sqsubseteq \mathsf{Building} \\
\mathsf{Cathedral} &\sqsubseteq \exists \mathsf{holdsEvent}.\mathsf{Mess} \sqcap \exists \mathsf{hasFunction}.\mathsf{Prayer} \\
\mathsf{Cathedral} &\sqsubseteq \mathsf{SightSeeingAttraction} \sqcap \exists \mathsf{hasDressCode}.\mathsf{Proper}
\end{aligned}
$$

Intuitively, each ontology provides a different *view* of the shared objects. If we assume that the different views are largely independent, then it seems reasonable to expect that the integrated ontology will not entail subsumptions between concepts from different ontologies that are in *not* in the shared signature (e.g., Prayer should not become a sub-concept of $\mathsf{Building}$ after the merge). More formally, given $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ and concept names A, B such that $A \in \mathsf{Sig}(\mathcal{T}_1)$, $B \in \mathsf{Sig}(\mathcal{T}_2)$ and $A, B \notin \mathsf{Sig}(\mathcal{T}_1) \cap \mathsf{Sig}(\mathcal{T}_2)$, we shall expect that $\mathcal{T} \not\models A \sqsubseteq B$ and $\mathcal{T} \not\models B \sqsubseteq A$. Also, if we want the integration to be seamless, the merged ontology $\mathcal{T}$ should be, at least, consistent and should not contain unsatisfiable concept names, although, in general, it may not necessarily be a conservative extension of both $\mathcal{T}_1$ and $\mathcal{T}_2$.

In some applications, one may also assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ present a well-established body of knowledge about the concepts they share, and hence no new subsumptions between concept names in the common signature should be entailed in $\mathcal{T}$. For example, if $\mathcal{T}_1, \mathcal{T}_2 \not\models \mathsf{Cathedral} \sqsubseteq \mathsf{Church}$, then $\mathcal{T} \not\models \mathsf{Cathedral} \sqsubseteq \mathsf{Church}$. However, in other applications, one may assume that the relationships about shared concepts in both ontologies are largely underspecified, and these new subsumptions constitute precisely one of the goals of the merge.

Hence this scenario might need to be split into several sub-scenarios. We are currently identifying new requirements, based on application needs, and exploring how to provide sensible syntactic counterparts for each case.

3 Related work and Outlook

So far, the problem of predicting and controlling the consequences of ontology integration has been largely overlooked by the Ontology Engineering and Semantic Web communities. To the best of our knowledge, the problem has been tackled only, very recently, in [3]. The authors propose a set of reasoning services based on the notion of conservative extensions and provide decidability and complexity results for the proposed services and for ontologies in $\mathcal{ALC}$. This work is focused on deciding conservative extensions whereas the work presented here is focused on providing suitable syntactic restrictions that guarantee conservative extensions and similar properties.

Complementary to the problem of Ontology Integration is the problem of Ontology Segmentation, often referred to in the literature as Ontology Modularization and Ontology Partitioning. In [2], the authors presented a method for automatically identifying and extracting relevant fragments of ontologies with precise semantic guarantees. The method has been designed for ontologies that contain information about disjoint subject matters, just as described in Section 2.2. In particular, the proof of Theorem 2 is implicit in the results presented in [2].

In this paper, we have formalized three basic scenarios for ontology integration. In each case, we have identified a set of *semantic* properties that the integrated ontology should satisfy and, under certain simplifying assumptions, we have shown how these properties can be guaranteed by imposing certain *syntactic* constraints on the ontologies to be integrated. However, we have been very conservative in both the (syntactic) compliance and safety conditions (localness) in the scenarios.

In the future, we aim at investigating how these can be relaxed in each case without losing the nice properties of the integrated ontology. We expect that our results will constitute the basis for a new methodology for ontology integration that is both well-founded and understandable to modelers, and that can be supported by ontology editors: so far, all notions of compliance and safety are decidable and can be checked syntactically.

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A Preliminaries and Notation

In the following, by a (background) logic $\mathcal{L}$ we shall mean just one of the description logics $\mathcal{SHIQ}$ or $\mathcal{SHOIQ}$. A logic $\mathcal{L}$ comes with a **signature** $\mathbf{S}_\mathcal{L} = \mathbf{C}_{\mathbf{S}_\mathcal{L}} \cup \mathbf{R}_{\mathbf{S}_\mathcal{L}}$, where $\mathbf{C}_{\mathbf{S}_\mathcal{L}}$ is a set of concept names, and $\mathbf{R}_{\mathbf{S}_\mathcal{L}}$ is a set of role names. Generally, by a signature $\mathbf{S} = \mathbf{C}_\mathbf{S} \cup \mathbf{R}_\mathbf{S}$ we mean any (mostly finite) set of concept and role names.

The set $\mathbf{C}_{\mathbf{S}_\mathcal{L}}$ may have a subset $\mathbf{I}_{\mathbf{S}_\mathcal{L}} \subseteq \mathbf{C}_{\mathbf{S}_\mathcal{L}}$ of nominals (when $\mathcal{L}$ is $\mathcal{SHOIQ}$). Usually, when the logic $\mathcal{L}$ is clear from the context, we will refer to such sets just as $\mathbf{S}$, $\mathbf{C}_\mathbf{S}$, etc. Thus, the **signature** $\mathsf{Sig}(\alpha)$ (respectively $\mathsf{Sig}(\mathcal{T})$) **of** an axiom α (respectively a TBox $\mathcal{T}$) denotes the set of concept and role names occurring in it. Given a background logic $\mathcal{L}$ and a signature $\mathbf{S}$, we use $\mathsf{Con}_\mathcal{L}(\mathbf{S})$ and $\mathsf{Rol}_\mathcal{L}(\mathbf{S})$ to denote the set of concepts and roles respectively that can be constructed in the logic $\mathcal{L}$ using only concept and role names in $\mathbf{S}$. Again, if the logic $\mathcal{L}$ is clear from the context, we usually use simply $\mathsf{Con}(\mathbf{S})$ and $\mathsf{Rol}(\mathbf{S})$. Moreover, let $\mathsf{Sub}(\mathcal{T})$ ($\mathsf{Sub}(C)$) denote the set of sub-concepts (defined in the usual way) occurring in a TBox $\mathcal{T}$(concept C).

B Local Ontologies

In order to assess the "globality" of a GCI, we introduce the notion of a *domain expansion*.

Definition 4 *(**Domain Expansion**) Let $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, .^{\mathcal{J}})$ be interpretations such that:*

1. *$\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}} \cup \nabla$, where ∇ is a non-empty set disjoint with $\Delta^{\mathcal{I}}$;*
2. *$A^{\mathcal{J}} = A^{\mathcal{I}}$ for each concept name;*
3. *$R^{\mathcal{J}} = R^{\mathcal{I}}$ for each role name.*

*We say that $\mathcal{J}$ is the **expansion** of $\mathcal{I}$ with ∇.*

Intuitively, the interpretation $\mathcal{J}$ is identical to $\mathcal{I}$ except for the fact that it contains some additional elements in the interpretation domain. These elements do not participate in the interpretation of concepts or roles. The following question naturally arises: if $\mathcal{I}$ is a model of $\mathcal{T}$, is $\mathcal{J}$ also a model of $\mathcal{T}$? *Local* ontologies are precisely those whose models are closed under domain expansions.

Definition 5 *(**Localness**) A concept C is **local** if, for every interpretation $\mathcal{I}$ for C and every non-empty set ∇ disjoint with $\Delta^{\mathcal{I}}$, the expansion $\mathcal{J}$ of $\mathcal{I}$ with ∇ verifies:*

$$C^{\mathcal{J}} = C^{\mathcal{I}}.$$

*Otherwise, we say that C is **non-local**. For a logic $\mathcal{L}$ and a signature $\mathbf{S}$, we denote by $\mathsf{Local}_{\mathcal{L}}(\mathbf{S})$ the set of local concepts that can be constructed in $\mathcal{L}$ using only concept names and roles in $\mathbf{S}$. Again, we usually abbreviate this to $\mathsf{Local}(\mathbf{S})$.*

*Let $\mathcal{T}$ be a TBox. We say that $\mathcal{T}$ is **local** if, for every $\mathcal{I} \models \mathcal{T}$ and every set ∇ disjoint with $\Delta^{\mathcal{I}}$, the expansion $\mathcal{J}$ of $\mathcal{I}$ with ∇ is a model of $\mathcal{T}$.*

Thus, local concepts are those whose interpretation remains invariant under domain expansions, and local TBoxes are those whose class of models is closed under domain expansions. The following theorem establishes the syntactic counterpart to the notion of localness of a concept:

Theorem 3 *Let $\mathcal{L}$ be a logic, $\mathbf{S}$ its signature, and C a concept in $\mathsf{Con}(\mathbf{S})$, then:*

- *If C is a concept name (including nominals) then $C \in \mathsf{Local}(\mathbf{S})$.*
- *If C is of the form $\exists R.D$ or $\geq nR.D$ then $C \in \mathsf{Local}(\mathbf{S})$.*
- *If C of the form $D \sqcap E$ then: $C \in \mathsf{Local}(\mathbf{S})$ iff $D \in \mathsf{Local}(\mathbf{S})$ or $E \in \mathsf{Local}(\mathbf{S})$.*
- *If C of the form $\neg D$ then: $C \in \mathsf{Local}(\mathbf{S})$ iff $D \notin \mathsf{Local}(\mathbf{S})$.*

Furthermore, for every pair of interpretations $\mathcal{I}, \mathcal{J}$ s.t. $\mathcal{J}$ is an expansion of $\mathcal{I}$ with ∇, if $C \notin \mathsf{Local}(\mathbf{S})$ then $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \nabla$.

Using Theorem 3, we can easily find an effective procedure for deciding localness of a Tbox:

Theorem 4 *Let $\mathcal{T}$ be consistent. $\mathcal{T}$ is local iff it does not contain a GCI $C \sqsubseteq D$ such that C is non-local and D is local.*

Clearly, the problem of deciding whether $C \in \mathsf{Local}(\mathbf{S})$ for some logic $\mathcal{L}$ and its signature $\mathbf{S}$ is polynomial in the length $|C|$ of the concept C. Thus, as a direct consequence of Theorem 4, the problem of deciding whether a consistent ontology $\mathcal{T}$ is local is polynomial w.r.t the size $|\mathcal{T}|$ of $\mathcal{T}$.

Proofs of these results are provided in [2].

C Proofs

C.1 Proof of Theorem 1

Let $\mathcal{T}_{dom}$ be a local $\mathcal{SHOIQ}$ TBox, $\mathcal{T}_{up}$ a (not necessarily local) $\mathcal{SHIQ}$ TBox, and assume the pair $\Im = \langle \mathcal{T}_{up}, \mathcal{T}_{dom} \rangle$ is f-compliant and that $\mathcal{T} = \mathcal{T}_{up} \cup \mathcal{T}_{dom}$ is instantiable. We have to show that:

1. $\mathcal{T} = \mathcal{T}_{up} \cup \mathcal{T}_{dom}$ is a conservative extension of $\mathcal{T}_{up}$.
2. There are no concept names $A \in \mathsf{Sig}(\mathcal{T}_{up})$ and $B \in \mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S}$ such that $\mathcal{T} \models A \sqsubseteq B$.
3. There are no role names $R \in \mathsf{Sig}(\mathcal{T}_{up})$ and $S \in \mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S}$ such that $\mathcal{T} \models R \sqsubseteq S$.

We begin by constructing a special model for $\mathcal{T}$ given models for $\mathcal{T}_{up}$ and $\mathcal{T}$. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ be an interpretation for $\mathsf{Sig}(\mathcal{T}_{up})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, .^{\mathcal{J}})$ an interpretation for $\mathsf{Sig}(\mathcal{T}_{up}) \cup \mathsf{Sig}(\mathcal{T}_{dom})$. Since $\mathcal{T}_{up}$ and $\mathcal{T}_{dom}$ are instantiable, and $\mathcal{T}$ is consistent, we can assume that $\mathcal{I} \models \mathcal{T}_{up}$ and $\mathcal{J} \models \mathcal{T}$, and, w.l.o.g., that $\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}} = \varnothing$. Furthermore, since $\mathcal{T}_{up}$ is a $\mathcal{SHIQ}$ TBox, the shared signature $\mathbf{S} = \mathsf{Sig}(\mathcal{T}_{up}) \cap \mathsf{Sig}(\mathcal{T}_{dom})$ can not contain shared nominals, and so the following construction of the interpretation $\mathcal{M} = (\Delta^{\mathcal{M}}, .^{\mathcal{M}})$ for the signature $\mathsf{Sig}(\mathcal{T}_{up}) \cup \mathsf{Sig}(\mathcal{T}_{dom})$ is well-defined:

$$\Delta^{\mathcal{M}} := \Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}}$$

$$A^{\mathcal{M}} := \begin{cases} A^{\mathcal{J}} \cup A^{\mathcal{I}} & A \in \mathsf{Sig}(\mathcal{T}_{up}) \\ A^{\mathcal{J}} & A \in \mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S} \end{cases}$$

$$R^{\mathcal{M}} := \begin{cases} R^{\mathcal{J}} \cup R^{\mathcal{I}} & R \in \mathsf{Sig}(\mathcal{T}_{up}) \\ R^{\mathcal{J}} & R \in \mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S} \end{cases}$$

First note the following. Relative to the signature $\mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S}$, the model $\mathcal{M}$ is a domain expansion of $\mathcal{J}$ with $\Delta^{\mathcal{I}}$. Thus, by Theorem 3, we immediately have the following:

(♠) For every concept $C \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S})$: if C is not local (i.e., $C^{\mathcal{M}} \neq C^{\mathcal{J}}$), then $C^{\mathcal{M}} = C^{\mathcal{J}} \cup \Delta^{\mathcal{I}}$.

Next, relative to the signature $\mathsf{Sig}(\mathcal{T}_{up})$, the model $\mathcal{M}$ is the disjoint union of the two models $\mathcal{I}$ and $\mathcal{J}$ for $\mathsf{Sig}(\mathcal{T}_{up})$. Therefore, since $\mathcal{T}_{up}$ is a $\mathcal{SHIQ}$ ontology, $\mathcal{I}, \mathcal{J} \models \mathcal{T}_{up}$, and $\mathcal{SHIQ}$ is 'closed' under the formation of disjoint unions, we obtain the following:

(♣) For every (complex) concept $C \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{up}))$: $C^{\mathcal{M}} = C^{\mathcal{J}} \cup C^{\mathcal{I}}$ (independently of whether C is local or not). Moreover, $\mathcal{M} \models \mathcal{T}_{up}$.

For a proof, compare [1]. We now show that $\mathcal{M}$ is also a model of $\mathcal{T}_{dom}$, and therefore of $\mathcal{T}$:

($\diamondsuit$) $\mathcal{M} \models \mathcal{T}_{dom}$.

Proof of ($\diamondsuit$): Since $\mathcal{T}_{dom}$ is an f-compliant $\mathcal{SHOIQ}$ ontology, we have axioms of the following forms:

- $C \sqsubseteq D$ with $C, D \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S})$: by (♠), $C^{\mathcal{M}} = C^{\mathcal{J}}$ if C is local, and $C^{\mathcal{M}} = C^{\mathcal{J}} \cup \Delta^{\mathcal{I}}$ if C is non-local, and analogously for D. Since $\mathcal{T}_{dom}$ is local, a GCI α can only be of one of the following three forms: **1)** C, D local; **2)** C local and D non-local; **3)** Both C and D are non-local.

  In all these cases, since $\mathcal{J} \models \alpha$, it is immediate to verify that $\mathcal{M} \models \alpha$ as well.

- $A \sqsubseteq B$, where A, B are concept names s.t. $A \notin \mathbf{S}$ and $B \in \mathbf{S}$: in this case, $A^{\mathcal{M}} = A^{\mathcal{J}}$ and $B^{\mathcal{M}} = B^{\mathcal{I}} \cup B^{\mathcal{J}}$; since $A^{\mathcal{J}} \subseteq B^{\mathcal{J}}$, we have that $A^{\mathcal{M}} \subseteq B^{\mathcal{M}}$.

- $R \sqsubseteq S$, where $R, S \in \mathsf{Rol}(\mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S})$: in this case, $R^{\mathcal{M}} = R^{\mathcal{J}}$ and $S^{\mathcal{M}} = S^{\mathcal{J}}$, and since $R^{\mathcal{J}} \subseteq S^{\mathcal{J}}$, we also have $R^{\mathcal{M}} \subseteq S^{\mathcal{M}}$.

- $P \sqsubseteq Q$, where P, Q are role names s.t. $P \notin \mathbf{S}$ and $Q \in \mathbf{S}$: in this case, $P^{\mathcal{M}} = P^{\mathcal{J}}$ and $Q^{\mathcal{M}} = Q^{\mathcal{I}} \cup Q^{\mathcal{J}}$, and since $P^{\mathcal{J}} \subseteq Q^{\mathcal{J}}$, we also have $P^{\mathcal{M}} \subseteq Q^{\mathcal{M}}$.

- $\mathsf{Trans}(R)$, where $R \notin \mathbf{R_S}$: then $R^{\mathcal{M}} = R^{\mathcal{J}}$, and so $R^{\mathcal{M}}$ is transitive since $R^{\mathcal{J}}$ is. $\square$ ($\diamondsuit$)

Now, to prove (1), suppose there are concepts $C_0, D_0 \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{up}))$ such that $\mathcal{T} \models C_0 \sqsubseteq D_0$, but $\mathcal{T}_{up} \not\models C_0 \sqsubseteq D_0$. Let $\mathcal{I}_0$ be an interpretation for $\mathsf{Sig}(\mathcal{T}_{up})$ s.t. $\mathcal{I}_0 \not\models C_0 \sqsubseteq D_0$, $\mathcal{J}_0$ a model of $\mathcal{T}$ with $\Delta^{\mathcal{I}_0} \cap \Delta^{\mathcal{J}_0} = \varnothing$, and $\mathcal{M}_0$ constructed from $\mathcal{I}_0, J_0$ as above. By ($\diamondsuit$) and ($\clubsuit$), $\mathcal{M}_0 \models \mathcal{T}$, and thus $\mathcal{M}_0 \models C_0 \sqsubseteq D_0$. Moreover, by ($\clubsuit$), we have $C^{\mathcal{M}_0} = C^{\mathcal{J}_0} \cup C^{\mathcal{I}_0}$ and $D^{\mathcal{M}_0} = D^{\mathcal{J}_0} \cup D^{\mathcal{I}_0}$ for all $C, D \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{up}))$. But since $\Delta^{\mathcal{I}_0} \cap \Delta^{\mathcal{J}_0} = \varnothing$ by construction, $C_0^{\mathcal{I}_0} \not\subseteq D_0^{\mathcal{I}_0}$ implies $C_0^{\mathcal{M}_0} \not\subseteq D_0^{\mathcal{M}_0}$. Thus, we encounter a contradiction.

It remains to prove (2) and (3). For (2), suppose that $A \in \mathsf{Sig}(\mathcal{T}_{up})$ and $B \in \mathsf{Sig}(\mathcal{T}_{dom}) \setminus \mathbf{S}$. Pick a model $\mathcal{I}_1$ that instantiates $\mathcal{T}_{up}$, i.e., where $A^{I_1} \neq \varnothing$, a model $\mathcal{J}_1$ of $\mathcal{T}$ with $\Delta^{\mathcal{I}_1} \cap \Delta^{\mathcal{J}_1} = \varnothing$, and construct a model $\mathcal{M}_1$ for $\mathcal{T}$ as above. Then $A^{\mathcal{M}_1} = A^{J_1} \cup A^{I_1}$ and $B^{\mathcal{M}_1} = B^{J_1}$. Since $\Delta^{I_1} \cap \Delta^{J_1} = \varnothing$, we have $A^{\mathcal{M}_1} \not\subseteq B^{\mathcal{M}_1}$, i.e., $\mathcal{T} \not\models A \sqsubseteq B$. (3) is shown analogously. $\square$

C.2 Proof of Theorem 2

Let $\mathcal{T}_{core}$ be a local $\mathcal{SHOIQ}$ TBox, $\mathcal{T}_{side}$ a local $\mathcal{SHIQ}$ TBox, and assume the merged Tbox $\mathcal{T} = \mathcal{T}_{side} \cup \mathcal{T}_{core}$ is instantiable (and therefore also $\mathcal{T}_{core}$ and $\mathcal{T}_{side}$). Suppose the pair $\Im = \langle \mathcal{T}_{side}, \mathcal{T}_{core} \rangle$ is b-compliant. We have to show that:

1. $\mathcal{T} = \mathcal{T}_{side} \cup \mathcal{T}_{core}$ is a conservative extension of $\mathcal{T}_{side}$.
2. There are no concept names $A \in \mathsf{Sig}(\mathcal{T}_{side})$ and $B \in \mathsf{Sig}(\mathcal{T}_{core}) \setminus \mathbf{S}$ such that $\mathcal{T} \models A \sqsubseteq B$ or $\mathcal{T} \models B \sqsubseteq A$.
3. There are no role names $R \in \mathsf{Sig}(\mathcal{T}_{side})$ and $S \in \mathsf{Sig}(\mathcal{T}_{core})$ such that $\mathcal{T} \models R \sqsubseteq S$ or $\mathcal{T} \models S \sqsubseteq R$.
4. There are no role names $R \in \mathbf{R}_{out}$ and $S \in \mathbf{R}_{core} \setminus \mathbf{R}_{out}$ such that $\mathcal{T} \models R \sqsubseteq S$ or $\mathcal{T} \models S \sqsubseteq R$.

Analogously to the proof of Theorem 1, we can construct an interpretation $\mathcal{M} = (\Delta^{\mathcal{M}}, .^{\mathcal{M}})$ for the signature $\mathsf{Sig}(\mathcal{T}_{core}) \cup \mathsf{Sig}(\mathcal{T}_{side})$ from interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, .^{\mathcal{J}})$ for, respectively, $\mathsf{Sig}(\mathcal{T}_{side})$ and $\mathsf{Sig}(\mathcal{T}_{core}) \cup \mathsf{Sig}(\mathcal{T}_{side})$, where $\mathcal{I} \models \mathcal{T}_{side}$, $\mathcal{J} \models \mathcal{T}$, and such that $\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}} = \varnothing$. Also, recall that, by b-compliance, $\mathsf{Sig}(\mathcal{T}_{core})$ and $\mathsf{Sig}(\mathcal{T}_{side})$ do not share role names, i.e., $\mathbf{S} = \mathbf{C}_{\mathbf{S}}$ and $\mathbf{R}_{\mathbf{S}} = \varnothing$. Define the interpretation $\mathcal{M}$ as follows:

$$\Delta^{\mathcal{M}} := \Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}}$$

$$A^{\mathcal{M}} := \begin{cases} A^{\mathcal{J}} \cup A^{\mathcal{I}} & A \in \mathsf{Sig}(\mathcal{T}_{side}) \\ A^{\mathcal{J}} & A \in \mathsf{Sig}(\mathcal{T}_{core}) \setminus \mathbf{S} \end{cases}$$

$$R^{\mathcal{M}} := \begin{cases} R^{\mathcal{J}} \cup R^{\mathcal{I}} & R \in \mathsf{Sig}(\mathcal{T}_{side}) \\ R^{\mathcal{J}} & R \in \mathsf{Sig}(\mathcal{T}_{core}) \end{cases}$$

Since $\mathcal{T}_{side}$ is a $\mathcal{SHIQ}$ Tbox and $\mathcal{SHIQ}$ models are closed under the formation of disjoint unions, we have, analogously to Theorem 1:

(♣) For every concept $C \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{side}))$: $C^{\mathcal{M}} = C^{\mathcal{J}} \cup C^{\mathcal{I}}$ (independently of whether C is local or not). Moreover, $\mathcal{M} \models \mathcal{T}_{side}$.

Call a concept $D \in \mathsf{Sub}(\mathcal{T}_{core})$ b-**admissible** if it can be constructed in $\mathsf{Sig}(\mathcal{T}_{core})$ according to the rules specified by b-compliance. We claim the following:

(♠) Let $C \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{core}))$ be b-admissible. If C is local then $C^{\mathcal{M}} = C^{\mathcal{J}}$, and if C is not local then $C^{\mathcal{M}} = C^{\mathcal{J}} \cup \Delta^{\mathcal{I}}$.

PROOF OF (♠): The proof is by induction on the structure of C as induced by b-compliance. Note first that, by definition of $\mathcal{M}$, we have, for concept names A occuring in $\mathcal{T}_{core}$: $A^{\mathcal{M}} = A^{J}$ if $A \notin \mathbf{S}$, and $A^{\mathcal{M}} = A^{\mathcal{J}} \cup A^{I}$ if $A \in \mathbf{S}$. Furthermore, for roles $R \in \mathsf{Rol}(\mathsf{Sig}(\mathcal{T}_{core}))$ we have $R^{\mathcal{M}} = R^{\mathcal{J}}$ since $\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}} = \varnothing$.

A concept name A is b-admissible if $A \notin \mathbf{S}$. They are local and $A^{\mathcal{M}} = A^{J}$ holds by definition.

For the induction step, note that the case of Booleans, negation and conjuntion, can be proved just as in Theorem 1. Thus we give only the case of existential quantification:

Let C be of the form $\exists R.D$ and b-admissible. Then C is local. By the b-compliance conditions we have to distinguish two cases:

- $R \notin \mathbf{R}_{out}$, and $D \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{core}))$ is b-admissible.
  - if D is local then, by induction, $D^{\mathcal{M}} = D^{\mathcal{J}}$, and so $C^{\mathcal{M}} = C^{\mathcal{J}}$;
  - if D is non-local then, by induction, $D^{\mathcal{M}} = D^{\mathcal{J}} \cup \Delta^{\mathcal{I}}$. Since $R^{\mathcal{M}} = R^{\mathcal{J}}$ and $\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}} = \varnothing$, there is no $y \in \Delta^{\mathcal{I}}$ s.t. $\langle x, y \rangle \in R^{\mathcal{M}}$ and so $C^{\mathcal{M}} = C^{\mathcal{J}}$.
- $R \in \mathbf{R}_{out}$, and $D \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{core}))$ is b-admissible. In this case, D is a shared concept name $A \in \mathbf{S}$, and hence $D^{\mathcal{M}} = D^{\mathcal{J}} \cup D^{\mathcal{I}}$. But since $R^{\mathcal{M}} \subseteq \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}}$, we obtain $C^{\mathcal{M}} = C^{\mathcal{J}}$.

□ (♠)

We now show that $\mathcal{M}$ is also a model of $\mathcal{T}_{core}$, and therefore of $\mathcal{T}$.

(♢) $\mathcal{M} \models \mathcal{T}_{core}$.

PROOF OF ($\diamondsuit$): If $C \sqsubseteq D \in \mathcal{T}_{core}$ then C and D are b-admissible, and by ($\spadesuit$), $C^{\mathcal{M}} = C^{\mathcal{J}}$, if C is local, and $C^{\mathcal{M}} = C^{\mathcal{J}} \cup \nabla$, if C is non-local; analogously for D. Since $\mathcal{T}$ is local, we can only have local GCIs in $\mathcal{T}_{core}$, and so the proof is as in Theorem 1. Finally, $\mathcal{M}$ obviously verifies the role inclusion and transitivity axioms in $\mathcal{T}_{core}$ since there are no shared roles.

$\square$ ($\diamondsuit$)

Now, to prove (1), we can proceed analogously to Theorem 1 using ($\diamondsuit$) and ($\clubsuit$). In order to prove (2)–(4), we will first show the following claim:

($\heartsuit$) There exists a model $\mathcal{N} = (\Delta^{\mathcal{N}}, .^{\mathcal{N}})$ of $\mathcal{T}$ such that:

$$\Delta^{\mathcal{N}} := \Delta_1^{\mathcal{N}} \cup \Delta_2^{\mathcal{N}}; \quad \Delta_1^{\mathcal{N}} \cap \Delta_2^{\mathcal{N}} = \varnothing; \quad \Delta_i^{\mathcal{N}} \neq \varnothing$$

$$A^{\mathcal{N}} \subseteq \begin{cases} \Delta_1^{\mathcal{N}} & A \in \mathsf{Sig}(\mathcal{T}_{core}) \setminus \mathbf{S} \\ \Delta_2^{\mathcal{N}} & A \in \mathsf{Sig}(\mathcal{T}_{side}) \end{cases}$$

$$R^{\mathcal{N}} \subseteq \begin{cases} \Delta_1^{\mathcal{N}} \times \Delta_2^{\mathcal{N}} & R \in \mathbf{R}_{out} \\ \Delta_1^{\mathcal{N}} \times \Delta_1^{\mathcal{N}} & R \in \mathbf{R}_{core} \setminus \mathbf{R}_{out} \\ \Delta_2^{\mathcal{N}} \times \Delta_2^{\mathcal{N}} & R \in \mathsf{Sig}(\mathcal{T}_{side}) \end{cases}$$

PROOF OF ($\heartsuit$): Since $\mathcal{T}$ is consistent, there exists an interpretation $\mathcal{J} = (\Delta^{\mathcal{J}}, .^{\mathcal{J}})$ for $\mathsf{Sig}(\mathcal{T})$ s.t. $\mathcal{J} \models \mathcal{T}$. We show that we can construct from $\mathcal{J}$ an interpretation $\mathcal{N}$ of the desired form such that $\mathcal{N} \models \mathcal{T}$. First, take an isomorphic disjoint copy $\mathcal{J}' = (\Delta^{\mathcal{J}'}, .^{\mathcal{J}'})$ of $\mathcal{J}$, i.e., such that $\lambda : \Delta^{J} \longrightarrow \Delta^{J'}$ is a bijective map, $\Delta^{J} \cap \Delta^{J'} = \varnothing$, and

$$\begin{aligned} x \in A^{\mathcal{J}} &\iff \lambda(x) \in A^{\mathcal{J}'} \\ \langle x, y \rangle \in R^{\mathcal{J}} &\iff \langle \lambda(x), \lambda(y) \rangle \in R^{\mathcal{J}'} \end{aligned}$$

for all concept names A and role names R in $\mathsf{Sig}(\mathcal{T})$. Now, define the interpretation $\mathcal{N}$ as follows:

$$\Delta^{\mathcal{N}} := \Delta^{\mathcal{J}} \cup \Delta^{\mathcal{J}'};$$

$$A^{\mathcal{N}} := \begin{cases} A^{\mathcal{J}} \subseteq \Delta^{\mathcal{J}} & A \in \mathsf{Sig}(\mathcal{T}_{core}) \setminus \mathbf{S} \\ \lambda(A^{\mathcal{J}}) \subseteq \Delta^{\mathcal{J}'} & A \in \mathsf{Sig}(\mathcal{T}_{side}) \end{cases}$$

$$R^{\mathcal{N}} := \begin{cases} \{\langle x, \lambda(y) \rangle \mid \langle x, y \rangle \in R^{\mathcal{J}}\} & R \in \mathbf{R}_{out} \\ R^{\mathcal{J}} & R \in \mathbf{R}_{core} \setminus \mathbf{R}_{out} \\ \{\langle \lambda(x), \lambda(y) \rangle \mid \langle x, y \rangle \in R^{\mathcal{J}}\} & R \in \mathsf{Sig}(\mathcal{T}_{side}) \end{cases}$$

By construction, $\mathcal{N}$ is of the desired form. It remains to be shown that $\mathcal{N} \models \mathcal{T}$. First, since $\mathcal{J}' \models \mathcal{T}_{side}$, and the fact that $\mathcal{T}_{side}$ is local and that $\mathcal{N}$ is a domain expansion of $\mathcal{J}'$ with $\Delta^{\mathcal{J}}$ relative to the signature $\mathsf{Sig}(\mathcal{T}_{side})$, it follows that $\mathcal{N} \models \mathcal{T}_{side}$.

We next show that $\mathcal{N} \models \mathcal{T}_{core}$. We claim the following:

($\star$) Let $C \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{core}))$ be b-admissible. If C is local then $C^{\mathcal{N}} = C^{\mathcal{J}}$, and if C is non-local then $C^{\mathcal{N}} = \Delta^{\mathcal{J}'} \cup C^{\mathcal{J}}$.

PROOF OF ($\star$): We proceed by induction on the structure of b-admissible concepts $C \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{core}))$, using the definition of $\mathcal{N}$ and the notion of b-compliance.

Note first that, by definition of $\mathcal{N}$, we have, for b-admissible concept names A, $A \notin \mathbf{S}$, and hence $A^{\mathcal{N}} = A^{J}$; also note that A is local and thus the claim holds. For roles $R \in \mathsf{Rol}(\mathsf{Sig}(\mathcal{T}_{core}))$ occuring in b-admissible concepts, we have $R^{\mathcal{N}} = R^{\mathcal{J}}$ if $R \in \mathsf{Rol}(\mathbf{R}_{core} \setminus \mathbf{R}_{out})$ (including inverse roles) and $R^{\mathcal{N}} = \{\langle x, \lambda(y)\rangle \mid \langle x, y\rangle \in R^{\mathcal{J}}\}$ if $R \in \mathbf{R}_{out}$. Note that, if $R \in \mathbf{R}_{out}$, the inverse of R does not occur in $\mathcal{T}$ due to b-compliance.

For the induction step, note that the case of Booleans, negation and conjuntion, can be proved just as in Theorem 1. Thus we give only the case of existential quantification: let C be of the form $\exists R.D$ and b-admissible. Then C is local. By the b-compliance conditions we have to distinguish two cases:

- $R \notin \mathbf{R}_{out}$, and $D \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{core}))$ is b-admissible.
  - if D is local then, by induction, $D^{\mathcal{N}} = D^{\mathcal{J}}$, and so $C^{\mathcal{N}} = C^{\mathcal{J}}$;
  - if D is non-local then, by induction, $D^{\mathcal{N}} = D^{\mathcal{J}} \cup \Delta^{\mathcal{J}'}$. Since $R^{\mathcal{N}} = R^{\mathcal{J}}$ and $\Delta^{\mathcal{J}} \cap \Delta^{\mathcal{J}'} = \emptyset$, there is no $y \in \Delta^{\mathcal{J}'}$ s.t. $\langle x, y\rangle \in R^{\mathcal{N}}$ and so $C^{\mathcal{N}} = C^{\mathcal{J}}$.
- $R \in \mathbf{R}_{out}$, and $D \in \mathsf{Con}(\mathsf{Sig}(\mathcal{T}_{core}))$ is b-admissible. In this case, D is a shared concept name, and hence $D^{\mathcal{N}} = \lambda(D^{\mathcal{J}})$ with $\lambda(D^{\mathcal{J}}) \subseteq \Delta^{\mathcal{J}'}$. But since $R^{\mathcal{N}} = \{\langle x, \lambda(y)\rangle \mid \langle x, y\rangle \in R^{\mathcal{J}}\}$ and λ is an isomorphism, it follows that $C^{\mathcal{N}} = C^{\mathcal{J}}$.

$\square$ ($\star$)

Since $\mathcal{T}_{core}$ is local and contains only GCIs $C \sqsubseteq D$ such that C, D are b-admissible, the proof of $\mathcal{N} \models \mathcal{T}_{core}$ using ($\star$) is almost identical to the proof of ($\diamondsuit$) using ($\spadesuit$) above. Notice also that $\mathcal{N}$ obviously satisfies the role inclusion axioms in $\mathcal{T}_{core}$. Concerning the transitivity axioms $\mathsf{Trans}(R)$, $\mathcal{N}$ straigtforwardly satisfies them in case $R \notin \mathbf{R}_{out}$, and, if $R \in \mathbf{R}_{out}$, we have that $R^{\mathcal{N}} = \{\langle x, \lambda(y)\rangle \mid \langle x, y\rangle \in R^{\mathcal{J}}\}$ and $\Delta^{\mathcal{J}} \cap \Delta^{\mathcal{J}'} = \emptyset$. Thus, if $\langle x, \lambda(y)\rangle \in R^{\mathcal{N}}$, there is no $z \in \Delta^{\mathcal{N}}$ such that $\langle \lambda(y), z\rangle \in R^{\mathcal{N}}$ by definition of $R^{\mathcal{N}}$, and consequently $\mathcal{N} \models \mathsf{Trans}(R)$.

$\square$ ($\heartsuit$)

Properties (2)–(4) of Theorem 2 are now a straightforward consequence of ($\heartsuit$). $\square$