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Notes on Logics of Metric Spaces

Abstract. In [14], we studied the computational behaviour of various first-order and modal languages interpreted in metric or weaker distance spaces. [13] gave an axiomatisation of an expressive and decidable metric logic. The main result of this paper is in showing that the technique of representing metric spaces by means of Kripke frames can be extended to cover the modal (hybrid) language that is expressively complete over metric spaces for the (undecidable) two-variable fragment of first-order logic with binary predicates interpreting the metric. The frame conditions needed correspond rather directly with a Boolean modal logic that is, again, of the same expressivity as the two-variable fragment. We use this representation to derive an axiomatisation of the modal hybrid variant of the two-variable fragment, discuss the compactness property in distance logics, and derive some results on (the failure of) interpolation in distance logics of various expressive power.

Keywords: Metric spaces; expressive completeness; Boolean modal logic; hybrid logic; axiomatisation; interpolation.

1. Introduction

Logics of distance spaces were conceived as knowledge representation formalisms aimed to bring a numerical, quantitative concept of distance into the conventional qualitative representation and reasoning [14]. The main application area of these formalisms envisaged was spatial reasoning. However, the notion of 'distance' allows a wide variety of interpretations.

Distances can be induced by different measures. We may be interested in the physical distance between two cities a and b, the length of the railroad connecting a and b, or the time it takes to go from a to b by plane. But we can also define the distance as the number of cities on the way from a to b, as the difference in altitude between a and b, and so forth. A more abstract notion of distance is obtained by assuming the distance between two points to be induced by a **similarity measure**: we may say that two points have distance 1 if they share a certain number of properties, distance 2 if they share a certain smaller number of properties, etc.

The standard mathematical models, capturing common features of various notions of distance, are known as metric spaces. A **metric space** is a

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pair $\langle W, d\rangle$, where W is a set (of points) and d a function from $W \times W$ into the set $\mathbb{R}^+$ (of non-negative real numbers) satisfying, for all $x, y, z \in W$, the following axioms:

$$d(x, y) = 0 \iff x = y; \tag{i}$$

$$d(x, y) = d(y, x); \tag{ii}$$

$$d(x, z) \leq d(x, y) + d(y, z). \tag{iii}$$

We denote the class of all metric spaces by $\mathcal{MS}$, refer to (ii) as **symmetry** of the metric, to (iii) as **triangularity**, and call the value $d(x, y)$ the **distance** from the point x to the point y. Axiom (i) is related to the Leibnizian principle of the *indiscernibility of identicals* and is assumed throughout. Clearly, the distance from a point to itself should be zero in any sensible interpretation of 'distance'.

Note, however, that the axiom also implies the converse, namely the *identity of indiscernibles*: if we assume the distance function to measure similarity, perfect similarity, i.e. distance zero, implies identity.[1]

Although acceptable in many cases, the concept of metric space is not universally applicable to all interesting measures of distance between points, especially those used in everyday life. Consider, for instance, the following two examples:

(i) $d(x, y)$ measures the flight-time from location x to location y;

(ii) $d(x, y)$ measures the similarity of scientific topics x and y identified with some subset of key words from a list K, i.e., computes the ratio of non-shared to shared key words.

In (i), d is clearly not necessarily symmetric. As concerns (ii), assume K is some list of key words, and topics x, y are identified with some non-empty subset of K. If $x \cap y = \varnothing$, we may want to define $d(x, y) := \big|K\big|$. Otherwise, we may set

$$d(x, y) := \frac{\big|(x \cup y) \setminus (x \cap y)\big|}{\big|x \cap y\big|},$$

using the set-theoretic symmetric difference to count the number of key words on which x and y disagree, and intersection to count the number of

[1] On the other hand, it does make sense to allow for the situation where the distance between points x and y is zero (because, e.g., they share all properties in question) and where x and y still denote distinct points. The investigation of such kinds of models is left for future work.

key words on which they agree. This measure clearly satisfies (i) and (ii), but it does not satisfy the triangular inequality (iii). For instance, assume K consists of all words appearing in this article and let

$$\begin{array}{lcl} \mathrm{x} & = & \{\textit{Euclidean spaces}, \textit{metric spaces}\}; \\ \mathrm{y} & = & \{\textit{metric spaces}, \textit{frame-companions}\}; \\ \mathrm{z} & = & \{\textit{frame-companions}, \textit{compactness}\}. \end{array}$$

Then $d(x, y) = \frac{2}{1}$, $d(y, z) = \frac{2}{1}$, but $d(x, z) = 8003$, since x and z share no key words at all and $\left|K\right| = 8003$.

For this reason, more general **distance spaces** $\langle W, d\rangle$ satisfying at least axiom (i) were also considered, and it was shown in [14] that the modal languages $\mathcal{L}^{\mathcal{O}}[M]$ (introduced below) are expressively complete over distance spaces satisfying (i) and (ii) for the (undecidable) two-variable fragments $\mathcal{LF}_2[M]$ of first-order logic having, besides unary predicates, binary predicates $\boldsymbol{\delta}(x, y) < a$ and $\boldsymbol{\delta}(x, y) = a$, for each $a \in M \subseteq \mathbb{R}^+$, interpreted by the distance function d in the obvious way.

In [14], we investigated in-depth the computational behaviour of various sublanguages of $\mathcal{L}^{\mathcal{O}}[M]$ and pointed at promising applications in knowledge representation. [13] gave an axiomatisation of an expressive and decidable sublanguage $\mathcal{MS}^\sharp$ of $\mathcal{L}^{\mathcal{O}}[M]$, introduced below as $\mathcal{L}_D[M]$, by using a relational representation of metric spaces.

The main result of the present paper is in showing that the technique of representing metric spaces by means of Kripke frames can be extended to cover the modal (hybrid) languages $\mathcal{L}^{\mathcal{O}}[M]$ that extend $\mathcal{L}_D[M]$ by more expressive distance operators as well as nominals.

After introducing the first-order and modal languages in Section 2, we prove in Section 3 that $\mathcal{LF}_2[M]$ can be characterised in another interesting way by showing that the modal languages $\mathcal{L}^{\mathcal{O}}[M]$ are expressively complete for natural Boolean modal languages $\mathcal{LB}[M]$ over models based on arbitrary distance spaces.

In Section 4, we prove the main result (Theorem 4.4), a 'finitary' and elementary relational representation of metric spaces that captures theoremhood for $\mathcal{L}^{\mathcal{O}}[M]$ in metric spaces and whose frame conditions correspond closely with the Boolean modal languages discussed in Section 3.

Section 5 provides an axiomatisation of the validities of the language $\mathcal{L}^{\mathcal{O}}[M]$ in metric spaces by using results from hybrid completeness theory, thus axiomatising the two-variable fragments (via translation).

Finally, in Sections 6 and 7, we discuss, respectively, the failure of compactness in metric spaces, and some results on (the failure of) interpolation in distance logics of various expressive power.

2. First-order and modal languages

Call a set $M \subseteq \mathbb{R}^+$ a **parameter set** if $0 \in M$ and whenever $a + b < c$ for some $a, b, c \in M$ then also $a + b \in M$. Typical sets M of parameters are $\mathbb{Q}^+$ (the non-negative rational numbers) or $\mathbb{N}$ (the natural numbers including 0), but note that parameter sets are not necessarily infinite, for instance, $M = \{0, 1, \ldots, n\}$, for $n \in \mathbb{N}$. The parameter sets specify to which distances a language can explicitly refer. In some situations, the particular choice of M is not important. For instance, axiomatisability does not depend on the choice of M. Compactness, however, does depend on whether or not M is infinite and unbounded, and to prove decidability, we obviously require that M is a recursive set.

Consider the first-order languages $\mathcal{LF}[M]$ (of first-order distance logic) containing a countably infinite set $c_1, c_2, \ldots$ of **constant symbols**, a countably infinite set $x_1, x_2, \ldots$ of **individual variables**, a countably infinite set $P_1, P_2, \ldots$ of **unary predicate symbols**, the **equality symbol** $\doteq$, two (possibly infinite) sets of **binary predicates**

$$\boldsymbol{\delta}(\_,\_) < a \quad \text{and} \quad \boldsymbol{\delta}(\_,\_) = a \quad (a \in M),$$

the **Booleans** (including the **propositional constants** $\top$ for **verum** and $\bot$ for **falsum**), and the quantifier $\exists x_i$ for every variable x_i. Thus, the **atomic formulae** of $\mathcal{LF}[M]$ are of the form

$$\top, \quad \bot, \quad \boldsymbol{\delta}(t, t') < a, \quad \boldsymbol{\delta}(t, t') = a, \quad t \doteq t,' \quad \text{and} \quad P_i(t),$$

where t and t' are **terms**, i.e., variables or constants, and $a \in M$. **Compound $\mathcal{LF}[M]$-formulae** are obtained from atomic ones by applying the Booleans and quantifiers in the usual way:

$$\varphi ::= \text{atom} \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \exists x_i \varphi.$$

The formula $\boldsymbol{\delta}(t_1, t_2) > a$ can be used as an abbreviation for $\neg\,\boldsymbol{\delta}(t_1, t_2) < a \wedge \neg\,\boldsymbol{\delta}(t_1, t_2) = a$.

$\mathcal{LF}[M]$-formulae are interpreted in structures of the form

$$\mathfrak{A} = \left\langle W, d, P_1^{\mathfrak{A}}, \ldots, c_1^{\mathfrak{A}}, \ldots \right\rangle,$$

where $\langle W, d\rangle$ is a distance space, the $P_i^{\mathfrak{A}}$ are subsets of W interpreting the unary predicates P_i, and the $c_i^{\mathfrak{A}}$ are elements of W interpreting the constants c_i. Moreover, the binary predicates are interpreted as

$$\mathfrak{M} \models \boldsymbol{\delta}(t_1, t_2) < a \iff d(t_1^{\mathfrak{M}}, t_2^{\mathfrak{M}}) < a, etc.$$

The **truth-relation** $\mathfrak{M} \models \varphi$, for an $\mathcal{LF}[M]$-formula φ, is defined inductively in the standard way. Also, the notions of **validity** and **satisfiability** are defined as in standard first-order logic, but respect the class of intended models. The **first-order logic** $\mathcal{FM}$ of metric spaces, then, is the set of all $\mathcal{LF}$-formulae valid in the class $\mathcal{MS}$ of all metric spaces, and similarly for the **two-variable first-order logic** $\mathcal{FM}_2$ that is defined just like $\mathcal{FM}$, but over the sublanguage $\mathcal{LF}_2 \subset \mathcal{LF}$ containing only two variables, say, x and y.

Whereas the two-variable fragment of standard first-order logic is decidable, the satisfiability problem for the sublanguage $\mathcal{LF}_2[\mathbb{N}]$ in two variables is undecidable in the class of metric spaces, which is shown by a reduction to the undecidable $\mathbb{N} \times \mathbb{N}$ tiling problem [14].[2]

Consider now the modal languages $\mathcal{L}_O^{\mathcal{O}}[M]$, which, instead of first-order quantifiers, use a set $O[M] \subset \mathsf{Op}[M]$ of **distance operators** from the set

$$\mathsf{Op}[M] = \{\mathsf{A}^{=a}, \mathsf{A}^{>a}, \mathsf{A}^{\geq a}, \mathsf{A}^{<a}, \mathsf{A}^{\leq a}, \mathsf{A}^{>a}_{<b}, \mathsf{A}^{\geq a}_{<b}, \mathsf{A}^{>a}_{\leq b}, \mathsf{A}^{\geq a}_{\leq b} \mid a, b \in M\}$$

(usually abbreviated to Op and O when M is clear from the context), the **universal modality** $\blacksquare$, and, additionally, **Boolean operators** $\wedge$ and $\neg$, **propositional constants** $\top$ and $\bot$, some denumerably infinite set $\mathcal{P} = \{p_l \mid l < \omega\}$ of **propositional variables**, and a denumerably infinite set $\mathsf{Nom} = \{i_l \mid l < \omega\}$ of **nominals**—formulae are constructed in the usual way. Also, we denote by $\mathcal{L}_O[M]$ the languages that are just like $\mathcal{L}_O^{\mathcal{O}}[M]$ with the exception that neither nominals nor the universal modality are present.

Models for these languages are structures of the form

$$\mathfrak{B} = \left\langle W, d, p_0^{\mathfrak{B}}, \dots, i_0^{\mathfrak{B}}, \dots \right\rangle,$$

where, again, $\langle W, d\rangle$ is a distance space, the $p_l^{\mathfrak{B}}$ are subsets of W, the $i_l^{\mathfrak{B}}$ are singleton subsets of W, and where the distance operators are interpreted similarly to 'boxes' from modal logic, e.g., if $\langle \mathfrak{B}, v\rangle$ is a pointed model based on a metric space $\langle W, d\rangle$, then

$$\langle \mathfrak{B}, v\rangle \models \mathsf{A}^{>a}_{<b}\varphi \iff \text{for all } w \in W \text{ such that } a < d(v, w) < b : \langle \mathfrak{B}, w\rangle \models \varphi,$$

and similarly for the other operators from the list above.[3]

[2] The undecidability of $\mathcal{LF}_2[\mathbb{N}]$ does not hinge on the fact that $\mathbb{N}$ is infinite. $\mathcal{LF}_2[\mathbb{N}]$ satisfiability, however, is decidable in classes of distance spaces satisfying only (i) and (ii), which can be shown by a simple reduction to the two-variable fragment of standard first-order logic [14].

[3] It should be clear that the universal modality $\blacksquare$ can be defined in some of the languages thus defined, e.g. as $\mathsf{A}^{\leq a}\_ \vee \mathsf{A}^{>a}\_$ in the full language $\mathcal{L}_{\mathsf{Op}}^{\mathcal{O}}[M]$, whence it can be regarded as 'syntactic sugar' in these cases. However, in weaker fragments of $\mathcal{L}_{\mathsf{Op}}^{\mathcal{O}}[M]$ it is needed as a 'binding device' for nominals and is thus added generally for uniformity.

Other Booleans as well as the dual distance operators $\mathsf{E}^{\leq a}$, $\mathsf{E}^{<a}$ etc. and the universal diamond $\blacklozenge$ are defined as abbreviations, e.g., $\mathsf{E}^{\leq a} = \neg\mathsf{A}^{\leq a}\neg$, etc. We use lower case Latin letters $p, q, r, \ldots$ to denote propositional variables, i, j, k to denote nominals, lower case Greek letters $\chi, \varphi, \psi, \ldots$ to denote formulae, and upper case Greek letters $\Gamma, \Delta, \Theta, \ldots$ to denote sets of formulae.

As usual, a formula φ is said to be **valid in a model** if it is true at every point of the model, φ is **valid in a metric space** $\langle W, d\rangle$ if it is valid in every model based on $\langle W, d\rangle$, and φ is **valid in a class** $\boldsymbol{K}$ of models (or metric spaces) if it is valid in every model (respectively, metric space) of $\boldsymbol{K}$.

By replacing quantifiers with distance operators we do not lose expressive power as compared with $\mathcal{LF}_2[M]$: the language $\mathcal{L}^{\mathcal{O}}_{\mathsf{Op}}[M]$ ($\mathcal{L}^{\mathcal{O}}[M]$ for short) is expressively complete for $\mathcal{LF}_2[M]$ in the class of all models based on metric spaces [14].[4] Note, however, that while the languages $\mathcal{L}^{\mathcal{O}}[M]$ and $\mathcal{LF}_2[M]$ are equally expressive over metric spaces, $\mathcal{LF}_2[M]$-formulae can 'speak'—in the worst case—exponentially more succinct about metric spaces [5], [14]. Besides, it should be clear that, by letting

$$F := \{\mathsf{A}^{=a}, \mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{>a}_{<b} \mid a, b \in M\},$$

we obtain a language $\mathcal{L}^{\mathcal{O}}_{F}[M]$ which—since it can define all the distance operators from the list Op, e.g., $\mathsf{A}^{>a}_{\leq b}\_\_$ as $\mathsf{A}^{>a}_{<b}\_\_ \wedge \mathsf{A}^{=b}\_\_$, etc.—is as expressive as the full language $\mathcal{L}^{\mathcal{O}}[M]$. We will therefore, for simplicity, concentrate on the languages $\mathcal{L}^{\mathcal{O}}_{F}[M]$ rather than $\mathcal{L}^{\mathcal{O}}[M]$.

The motivation for considering these alternative languages to first-order logic is that their fragments are more easily analysed. For instance, the satisfiability problem for the language $\mathcal{L}_D[M]$, where

$$D = \{\mathsf{A}^{\leq a}, \mathsf{A}^{>a} \mid a \in M\},$$

is decidable in metric spaces, and $\mathcal{L}_D[M]$ even has the finite model property [14].[5] Moreover, note that $\mathcal{L}_D[M]$ is still a very expressive language, for it comprises the difference operator [19] 'everywhere but here' as $\mathsf{A}^{>0}\varphi$, and thus can also define the universal modality, as well as simulate nominals.

As with first-order metric logics, **modal metric logics** are defined semantically, that is, as sets of formulae of some language that are valid in the class of all metric spaces:

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[4]In fact, in the class of all symmetric distance spaces (not necessarily satisfying the triangular inequality).

[5]The language $\mathcal{L}^{\mathcal{O}}_{D}[M]$ is called $\mathcal{MS}^\sharp$ in [14], and $\mathcal{L}_D[M]$ is simply called $\mathcal{L}(M)$ in [13].

DEFINITION 2.1 (Modal metric logics). Given a parameter set $M \subseteq \mathbb{R}^+$ and an operator set O, we define the logics $\mathcal{MS}_O^{\mathcal{O}}[M]$ ($\mathcal{MS}_O[M]$) as the sets of all $\mathcal{L}_O^{\mathcal{O}}[M]$-formulae ($\mathcal{L}_O[M]$-formulae) valid in the class $\mathcal{MS}$.

3. Boolean Distance Logics

An analysis of the expressive completeness result mentioned in the last section relating the languages $\mathcal{L}_F^{\mathcal{O}}[M]$ and $\mathcal{LF}_2[M]$, and, in particular, the expressiveness of the 'ring operators' of the form $\mathsf{A}_{<b}^{>a}$, suggests that the language $\mathcal{L}_F^{\mathcal{O}}$ is closely related to Boolean modal logic. For instance, while the formula $\mathsf{A}_{<b}^{>a}\varphi$ is clearly not equivalent to a conjunction of the form $\mathsf{A}^{>a}\varphi \wedge \mathsf{A}^{<b}\varphi$, the operator $\mathsf{A}_{<b}^{>a}$ does coincide with a Boolean modal operator $[\succ_a \wedge \prec_b]$ build from a conjunction of the symbols $\succ_a$ and $\prec_b$ being interpreting by the distance function d in the obvious way (see below).

Yet, unlike in the case of 'standard' Boolean modal logic, the natural ordering of parameters from $\mathbb{R}^+$ imposes additional structure on Boolean distance operators defined by allowing arbitrary Boolean combinations of symbols from the set

$$\{\approx_a, \prec_a, \succ_a \mid \ a \in M\},$$

and hence restricts the number of 'new' operators obtained in this way.

Furthermore, Boolean modal languages with similar expressive capabilities as the language $\mathcal{L}_F^{\mathcal{O}}$, namely Boolean modal logic enriched with converse modal operators and the difference operator, have been shown to be expressively equivalent to the two-variable fragment of first-order logic, compare [16].

In this section we show that, indeed, the language $\mathcal{L}_F^{\mathcal{O}}$ is expressively equivalent to a natural variant of Boolean modal logic (over the class of all distance spaces) and thus expressively equivalent to two-variable first-order logic interpreted on symmetric distance spaces.

Let us start by defining the languages of Boolean distance logic:

DEFINITION 3.1 (Boolean distance logic). Let M be a parameter set and define a set of modal parameters as:

$$\mathbb{M} := \{\approx_a, \prec_a \mid a \in M\}.$$

Then, let $B(\mathbb{M})$ be the set of all Boolean combinations of symbols from $\mathbb{M}$. Now, the language $\mathcal{LB}[M]$ consists of a denumerably infinite list $\{p_l : l < \omega\}$ of **propositional variables**, a denumerably infinite list $\{i_l : l < \omega\}$ of **nominals**, the **Boolean connectives** $\wedge$ and $\neg$, the **propositional**

constants $\top$ and $\bot$, the **universal modality** $\blacksquare$, as well as the following set of **Boolean distance operators** depending on M:

$$\{[\delta] : \delta \in B(\mathbb{M})\}.$$

The set of well-formed formulae of this language is constructed in the standard way; it will be identified with $\mathcal{LB}[M]$.

Again, other Booleans as well as the dual Boolean distance operators $\langle\delta\rangle$ and the universal diamond $\blacklozenge$ are defined as abbreviations.

DEFINITION 3.2 (Semantics for Boolean distance logic). As before, the models for the language $\mathcal{LB}$ are of the form:

$$\mathfrak{B} = \left\langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \ldots, i_0^{\mathfrak{B}}, i_1^{\mathfrak{B}}, \ldots \right\rangle,$$

where $\langle W, d\rangle$ is a metric space, the $p_i^{\mathfrak{B}}$ are subsets of W and nominals i_l are interpreted by singleton subsets $i_l^{\mathfrak{B}}$.

We just have to define the truth-relation for the new distance operators: let $\mathfrak{B}$ be a model, w a point in W and $[\delta]\varphi$ an $\mathcal{LB}$ formula with $\delta \in B(\mathbb{M})$. First, we define the **extension** of δ with respect to a point $w \in W$, abbreviated as $\|\delta\|^w$, as follows:

$$\|\delta\|^w := \{v \in W \mid \langle w, v\rangle \models \delta\},$$

where $\langle w, v\rangle \models \delta$ is defined inductively as:

- $\langle w, v\rangle \models \approx_a \iff w = v$;
- $\langle w, v\rangle \models \prec_a \iff d(w, v) < a$;
- $\langle w, v\rangle \models \delta \wedge \gamma \iff \langle w, v\rangle \models \delta$ and $\langle w, v\rangle \models \gamma$;
- $\langle w, v\rangle \models \neg\gamma \iff \langle w, v\rangle \not\models \gamma$.

Now set:

- $\langle \mathfrak{B}, w\rangle \models [\delta]\varphi \iff$ for all $v \in \|\delta\|^w$ we have $\langle \mathfrak{B}, v\rangle \models \varphi$.

Let us introduce the notion of **satisfiability** for the Boolean distance operators. A combination δ can be called **satisfiable**, if there is a model $\mathfrak{B}$ and a point w such that $\|\delta\|^w \neq \varnothing$. Deciding satisfiability of a given δ is a very simple problem, we just have to check whether or not a system of equalities and inequalities in one variable, i.e. of the form $x = a_i, x \neq a_j, x < a_k, x \geq a_l$, has a solution.

Clearly, all the distance operators of $\mathcal{L}_F^{\mathcal{O}}[M]$ and the universal modality are definable in $\mathcal{LB}[M]$ with respect to all distance spaces, namely we can translate them as follows:

- $(\mathsf{A}^{=a}\varphi)^\star = [\approx_a]\varphi^\star$;
- $(\mathsf{A}^{<a}\varphi)^\star = [\prec_a]\varphi^\star$;
- $(\mathsf{A}^{>a}\varphi)^\star = [\neg(\prec_a \vee \approx_a)]\varphi^\star = [\neg \prec_a \wedge \neg \approx_a]\varphi^\star$;
- $(\mathsf{A}^{>a}_{<b}\varphi)^\star = [\neg(\prec_a \vee \approx_a) \wedge \prec_b]\varphi^\star = [\neg \prec_a \wedge \neg \approx_a \wedge \prec_b]\varphi^\star$;
- $(\blacksquare\varphi)^\star = (\mathsf{A}^{=a}\varphi)^\star \wedge (\mathsf{A}^{<a}\varphi)^\star \wedge (\mathsf{A}^{>a}\varphi)^\star$.

Note that in the translation of the $\mathcal{L}^{\mathcal{O}}_F[M]$-operators we only needed atomic negation and conjunction. Note also that for each formula of the form $[\bigvee_{l\leq n}\delta_l]\varphi$ we have for all models $\mathfrak{B}$ and points w:

$$\langle\mathfrak{B}, w\rangle \models [\bigvee_{l\leq n}\delta_l]\varphi \iff \langle\mathfrak{B}, w\rangle \models \bigwedge_{l\leq n}[\delta_l]\varphi.$$

Hence, as far as expressivity goes, we can do with atomic negation and conjunction in the definition of Boolean distance operators by bringing δ in disjunctive normal form (DNF) and by replacing the disjunction in δ in favour of conjunctions of formulae. This procedure, however, can result in an exponential blow up of formulae length and is thus not inert with respect to complexity issues, cf. [15].

We can now show that the languages $\mathcal{L}^{\mathcal{O}}_F[M]$ and $\mathcal{LB}[M]$ are equally expressive over the class of all models based on distance spaces.

Theorem 3.3. *The languages $\mathcal{L}^{\mathcal{O}}_F[M]$ and $\mathcal{LB}[M]$ are equally expressive over the class of all models based on distance spaces.*

Proof. We have already shown that all the distance operators of $\mathcal{L}^{\mathcal{O}}_F$ as well as the universal modality are definable in $\mathcal{LB}$, so $\mathcal{LB}$ is at least as expressive as $\mathcal{L}^{\mathcal{O}}_F$. It remains to show that there is a translation $.^\dagger : \mathcal{LB} \longrightarrow \mathcal{L}^{\mathcal{O}}_F$ such that for all formulae $\varphi \in \mathcal{LB}$, models $\mathfrak{B}$ and points w in $\mathfrak{B}$ we have:

$$\langle\mathfrak{B}, w\rangle \models \varphi \quad\iff\quad \langle\mathfrak{B}, w\rangle \models \varphi^\dagger.$$

As sketched above, we can assume without loss of generality that all Boolean distance operators are defined by conjunctions of literals of modal parameters. The translation is defined inductively. As usual we define

- $p_i^\dagger = p_i$;
- $i_k^\dagger = i_k$;
- $(\psi \wedge \chi)^\dagger = \psi^\dagger \wedge \chi^\dagger$;

- $(\neg\psi)^\dagger = \neg\psi^\dagger$.

The remaining case of $\psi = [\delta]\chi$, δ a conjunction of literals, is more difficult. We distinguish several cases.

First we check whether δ is satisfiable. If it is not we have $\|\delta\|^w = \varnothing$ for any model $\mathfrak{B}$ and point w, so we can set:

- $([\delta]\varphi)^\dagger = \top$, if δ is inconsistent;

For the remaining cases we assume δ is satisfiable. If $\approx_a \in \delta$ for some a, then for any model $\mathfrak{B}$ and point $w \in W$ we have $\|\delta\|^w = \| \approx_a \|^w$. Hence we can define

- $([\delta]\varphi)^\dagger = \mathsf{A}^{=a}\varphi^\dagger$, if $\approx_a \in \delta$ for some a and δ is consistent.

It remains to consider the case where

$$\delta = \neg \approx_{a_1} \wedge \ldots \wedge \neg \approx_{a_n} \wedge \neg \prec_{b_1} \wedge \ldots \wedge \neg \prec_{b_m} \wedge \prec_{c_1} \wedge \ldots \wedge \prec_{c_k},$$

with $n, m, k \geq 0$ and $a_1 < \ldots < a_n$, $b_1 < \ldots < b_m$ and $c_1 < \ldots < c_k$. Assume first that $m = k = 0$. Then we can translate

- $([\delta]\varphi)^\dagger = \mathsf{A}^{<a_1}\varphi^\dagger \wedge \mathsf{A}^{>a_1}_{<a_2}\varphi^\dagger \wedge \ldots \wedge \mathsf{A}^{>a_{n-1}}_{<a_n}\varphi^\dagger \wedge \mathsf{A}^{>a_n}\varphi^\dagger$.

We can without loss of generality assume that $m, k \leq 1$. Namely, if $m, k > 0$ set

$$b := max(\{b_i \mid \neg \prec_{b_i} \in \delta\})$$

and

$$c := min(\{c_i \mid \prec_{c_i} \in \delta\}).$$

Then δ is equivalent to δ', where

$$\delta' = \neg \approx_{a_1} \wedge \ldots \wedge \neg \approx_{a_n} \wedge \neg \prec_b \wedge \prec_c .$$

Moreover, since δ is satisfiable, we can assume that the parameters are ordered as follows

$$b \leq a_1 < a_2 < \ldots < a_n \leq c.$$

It should be clear now how we have to translate the remaining cases. For simplicity, we use the operators $\mathsf{A}^{\geq a}_{<b}$ etc. that are definable in $\mathcal{L}^{\mathcal{O}}_F$:

- $([\delta]\varphi)^\dagger = \mathsf{A}^{\geq b}_{<a_1}\varphi^\dagger \wedge \mathsf{A}^{>a_1}_{<a_2}\varphi^\dagger \wedge \ldots \wedge \mathsf{A}^{>a_{n-1}}_{<a_n}\varphi^\dagger \wedge \mathsf{A}^{>a_n}_{<c}\varphi^\dagger \quad (n, m, k > 0)$
- $([\delta]\varphi)^\dagger = \mathsf{A}^{\geq b}_{<a_1}\varphi^\dagger \wedge \mathsf{A}^{>a_1}_{<a_2}\varphi^\dagger \wedge \ldots \wedge \mathsf{A}^{>a_{n-1}}_{<a_n}\varphi^\dagger \wedge \mathsf{A}^{>a_n}\varphi^\dagger \ (n, m > 0, k = 0)$

- $([\delta]\varphi)^\dagger = \mathsf{A}^{<a_1}\varphi^\dagger \wedge \mathsf{A}^{\geq a_1}_{<a_2}\varphi^\dagger \wedge \ldots \wedge \mathsf{A}^{\geq a_{n-1}}_{<a_n}\varphi^\dagger \wedge \mathsf{A}^{\geq a_n}_{<c}\varphi^\dagger$ $(n, k > 0, m = 0)$
- $([\delta]\varphi)^\dagger = \mathsf{A}^{\geq b}_{<c}\varphi^\dagger$ $(m, k > 0, n = 0)$
- $([\delta]\varphi)^\dagger = \mathsf{A}^{<c}\varphi^\dagger$ $(n, m = 0, k > 0)$
- $([\delta]\varphi)^\dagger = \mathsf{A}^{\geq b}\varphi^\dagger$ $(n, k = 0, m > 0)$

■

We could as well show that $\mathcal{LB}[M]$ is expressively complete over the class of all distance spaces (abstracting from interpretations) for a language $\mathcal{L}^{-}\mathcal{B}[M]$ that is like $\mathcal{LB}[M]$, but without nominals or the universal modality. Also, $\mathcal{LB}[M]$ is as expressive as the 2-variable first-order language $\mathcal{LF}_2[M]$ over symmetric models by the expressive completeness result for the language $\mathcal{L}^{\mathcal{O}}_F[M]$. To obtain languages $\mathcal{L}^{+}{}^{\mathcal{O}}_F[M]$ and $\mathcal{L}^{+}\mathcal{B}[M]$ that are as expressive as $\mathcal{LF}_2[M]$ over models based on arbitrary distance spaces, we had to add **inverse distance operators** like $\mathsf{A}^{\leq a}_{-}$ etc., defined by

$$\langle \mathfrak{B}, w\rangle \models \mathsf{A}^{\leq a}_{-}\varphi \iff \langle \mathfrak{B}, u\rangle \models \varphi \text{ for all } u \in W \text{ with } d(u, w) < a,$$

to the languages $\mathcal{L}^{\mathcal{O}}_F[M]$, and **inverse Boolean operators**

$$\langle \mathfrak{B}, w\rangle \models [\delta]_{-}\varphi \iff \text{ for all } v \in \|\delta\|^{w}_{-} \text{ we have } \langle \mathfrak{B}, v\rangle \models \varphi,$$

where $\|\delta\|^{w}_{-} := \{v \in W \mid \langle v, w\rangle \models \delta\}$, to the languages $\mathcal{LB}[M]$, similarly to what has been done in [16], but we leave the details of these enrichments to the reader.

Although the Boolean modal languages $\mathcal{LB}[M]$ thus turn out to be just syntactic variants of the modal languages $\mathcal{L}^{\mathcal{O}}_F[M]$, the reduction carried out in the proof of Theorem 3.3 sheds a clearer light also on the expressiveness of the operators of $\mathcal{L}^{\mathcal{O}}_F[M]$. In fact, the frame conditions needed to define an adequate relational representation of metric spaces for the language $\mathcal{L}^{\mathcal{O}}_F[M]$ are derived from rather elementary validities expressible with Boolean modal distance operators.

4. Frame representation

We can also use standard Kripkean possible worlds semantics to interpret the modal languages at hand. Let M be a parameter set. A polymodal M-frame for the languages $\mathcal{L}^{\mathcal{O}}_O$ is a structure of the form

$$\mathfrak{f} = \left\langle W, \{(R^{o}_{a|b})_{a,b\in M} \mid o \in O\}\right\rangle$$

which consists of a set W (whose members are called 'points') and families $(R^o_{a|b})_{a,b\in M}$ of binary relations on $W \times W$ for each operator symbol $o \in O$. The notation $R^o_{a|b}$ is shorthand for the fact that some operators, e.g. $\mathsf{A}^{>a}$, are indexed by one parameter $a \in M$, while others, e.g. $\mathsf{A}^{>a}_{<b}$, are indexed by two parameters, a and b. A model based on a frame is of the form

$$\mathfrak{M} = \left\langle \mathfrak{f}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots i_0^{\mathfrak{M}}, i_1^{\mathfrak{M}}, \dots \right\rangle,$$

where the $p_n^{\mathfrak{M}}$ are subsets of W and the $i_m^{\mathfrak{M}}$ singleton subsets. If we work in languages without nominals, the interpretations for nominals are omitted. The notions of truth (in a pointed model) and validity in M-models and M-frames are the usual Kripkean ones, with the addition that nominals are interpreted as singleton sets of worlds. For instance,

$$\begin{aligned}
\langle \mathfrak{M}, w\rangle \models \mathsf{A}^{>a}_{<b}\varphi \quad &\Longleftrightarrow \quad \langle \mathfrak{M}, u\rangle \models \varphi \text{ for all } u \in W \text{ such that } wR^{>a}_{<b}u;\\
\langle \mathfrak{M}, w\rangle \models \blacksquare\varphi \quad &\Longleftrightarrow \quad \langle \mathfrak{M}, u\rangle \models \varphi \text{ for all } u \in W;\\
\langle \mathfrak{M}, w\rangle \models i \quad &\Longleftrightarrow \quad i^{\mathfrak{M}} = \{w\}.
\end{aligned}$$

Similarly for the other operators. It should be clear that the truth or falsity of a formula at a point depends only on the propositional variables, nominals, and operators appearing in it. Thus, given a sublanguage $\mathcal{L}^{\mathcal{O}}_{O'}[M'] \subset \mathcal{L}^{\mathcal{O}}_{O}[M]$ with $O' \subset O$ and $M' \subset M$ and a frame

$$\mathfrak{f} = \left\langle W, \{({}^{\mathfrak{f}}R^o_{a|b})_{a,b\in M} \ \mid \ o \in O\}\right\rangle$$

for $\mathcal{L}^{\mathcal{O}}_{O}[M]$, we may define the **frame-reduct** $\mathfrak{f}\restriction_{(O',M')}$ of $\mathfrak{f}$ as

$$\mathfrak{f}\restriction_{(O',M')} := \left\langle W, \{({}^{\mathfrak{f}}R^{o'}_{a'|b'})_{a',b'\in M'} \ \mid \ o' \in O'\}\right\rangle.$$

We then have for every formula φ of $\mathcal{L}^{\mathcal{O}}_{O'}[M']$ and every model $\mathfrak{M}$ based on $\mathfrak{f}$:

$$\left\langle \mathfrak{f}, p_0^{\mathfrak{M}}, \dots, i_0^{\mathfrak{M}}, w\right\rangle \models \varphi \iff \left\langle \mathfrak{f}\restriction_{(O',M')}, p_0^{\mathfrak{M}}, \dots, i_0^{\mathfrak{M}}, w\right\rangle \models \varphi.$$

This observation does not depend on the presence of nominals.

Clearly, frames in general do not need to reflect any properties of distance or metric spaces. However, as Proposition 4.2 below shows, all the logics defined above semantically via validity in metric spaces can also be characterised by specific classes of frames. But let us first introduce some helpful concepts. Given a normal modal (hybrid) logic L in language $\mathcal{L}$ and a class $\boldsymbol{F}$ of frames, we define the expression $Fr(L)$, the **frames of** L, as

$Fr(L) := \{\mathfrak{f} \mid \mathfrak{f} \models L\}$. Similarly, we define $Th(\boldsymbol{F})$, the **theory of $\boldsymbol{F}$**, as $Th(\boldsymbol{F}) := \{\varphi \in \mathcal{L} \mid \boldsymbol{F} \models \varphi\}$.

It is well known from standard modal logic that the theory $Th(\boldsymbol{F})$ of a class $\boldsymbol{F}$ of frames determines a normal modal logic. Also, if the logics L and L' coincide, where $L' = Th(Fr(L))$, then L is frame-complete with respect to standard Kripke semantics. In a similar fashion we can show that all distance logics are complete with respect to frame semantics.

Definition 4.1 (Frame-companion). Let $S = \langle W, d\rangle$ be a distance space and M a parameter set. We define the M-**frame-companion** of S for language $\mathcal{L}_O^{\mathcal{O}}$ or $\mathcal{L}_O$ as

$$\mathfrak{f}_{O,M}(S) = \left\langle W', \{(R^o_{a|b})_{a,b\in M} \mid o \in O\}\right\rangle,$$

by setting $W' := W$ and, for all $u, v \in W$:

$$\begin{array}{ll} uR_{<a}v :\iff d(u,v) < a, & uR_{>a}v :\iff d(u,v) > a, \\ uR_{=a}v :\iff d(u,v) = a, & uR^{>a}_{<b}v :\iff a < d(u,v) < b, \end{array}$$

etc., for those operators appearing in O.

Further, if $\mathfrak{B} = \left\langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \ldots, i_0^{\mathfrak{B}}, i_1^{\mathfrak{B}}, \ldots\right\rangle$ is a model based on the distance space $S = \langle W, d\rangle$, then the Kripke model $\mathfrak{M}_{O,M}(\mathfrak{B})$ based on the M-frame $\mathfrak{f}_{O,M}(S)$ is the structure

$$\mathfrak{M}_{O,M}(\mathfrak{B}) = \left\langle \mathfrak{f}_{O,M}(S), p_0^{\mathfrak{M}_{O,M}(\mathfrak{B})}, \ldots, i_0^{\mathfrak{M}_{O,M}(\mathfrak{B})}, \ldots \right\rangle,$$

with $p_n^{\mathfrak{M}_{O,M}(\mathfrak{B})} := p_n^{\mathfrak{B}}$ and $i_m^{\mathfrak{M}_{O,M}(\mathfrak{B})} := i_m^{\mathfrak{B}}$, for all $n, m < \omega$. $\mathfrak{M}_{O,M}(\mathfrak{B})$ is called the M-**frame-companion model** of $\mathfrak{B}$ for language $\mathcal{L}_O^{\mathcal{O}}[M]$. The same definition applies to languages $\mathcal{L}_O$, but nominals are left out.

Theorem 4.2 (Frame Characterization). *All the logics $\mathcal{MS}_O[M]$ and $\mathcal{MS}_O^{\mathcal{O}}[M]$ for $O \subseteq \mathsf{Op}[M]$, are characterised by classes of frames, i.e., for some fixed parameter set M:*

$$\mathcal{MS}_O = Th(Fr(\mathcal{MS}_O)) \quad \textit{and} \quad \mathcal{MS}_O^{\mathcal{O}} = Th(Fr(\mathcal{MS}_O^{\mathcal{O}})).$$

Proof. That $\mathcal{MS}_O \subseteq Th(Fr(\mathcal{MS}_O))$ and $\mathcal{MS}_O^{\mathcal{O}} \subseteq Th(Fr(\mathcal{MS}_O^{\mathcal{O}}))$ hold should be clear. So it suffices to show that if $\varphi \notin \mathcal{MS}_O$ ($\notin \mathcal{MS}_O^{\mathcal{O}}$) then φ can be refuted in a frame for $\mathcal{MS}_O$ ($\mathcal{MS}_O^{\mathcal{O}}$).

First, by a straightforward structural induction we can show that for all models $\mathfrak{B}$ based on some metric space $S = \langle W, d\rangle$, their frame-companion models $\mathfrak{M}_{O,M}(\mathfrak{B})$, all formulae $\psi \in \mathcal{L}_O^{\mathcal{O}}[M]$ and points $w \in W$:

$$\langle \mathfrak{B}, w\rangle \models \psi \iff \langle \mathfrak{M}_{O,M}(\mathfrak{B}), w\rangle \models \psi.$$

Hence, for every metric space $S \in \mathcal{MS}$ we have $\mathfrak{f}_{O,M}(S) \in Fr(\mathcal{MS}_O[M])$ and if $\mathfrak{B}$ is a model such that $\langle \mathfrak{B}, w \rangle \not\models \varphi$ then $\langle \mathfrak{M}_{O,M}(\mathfrak{B}), w \rangle \not\models \varphi$.

■

Thus, every class of metric (or distance) spaces induces in a canonical way a corresponding class of frames which generates the same set of tautologies. Our next aim is to show that we can in fact define an elementary class of frames that characterises theoremhood in the language $\mathcal{L}_F^{\mathcal{O}}[M]$. The M-**frames** for the language $\mathcal{L}_F^{\mathcal{O}}[M]$ are now structures of the form

$$\mathfrak{f} = \left\langle W, \{R_{<a}, R_{>a}, R_{=a}, R_{<b}^{>a}\}_{a,b \in M} \right\rangle .$$

The following definition of an F-**metric frame** singles out those M-frames that reflect 'enough' properties of metric spaces.

Definition 4.3 (F-Metric Frames). An M-frame $\mathfrak{f}$ is called F-**metric** if it meets the following requirements, for all $u, v, w \in W$, $a, b \in M$, and $a + b \in M$ in (F10)–(F12):

(F1)	$R_{>a} = \overline{R_{<a}} \cap \overline{R_{=a}}$	
(F2)	$R_{<b}^{>a} = \overline{R_{<a}} \cap \overline{R_{=a}} \cap R_{<b}$	
(F3)	$R_{=0} = \{\langle w, w \rangle \mid w \in W\}$	
(F4)	$R_{<0} = \varnothing$	
(F5)	$R_{=a} \cap R_{<b} = \varnothing$	$(a \geq b)$
(F6)	$R_{=a} \cap R_{=b} = \varnothing$	$(a \neq b)$
(F7)	$R_{<a} \subseteq R_{<b}$	$(a \leq b)$
(F8)	$R_{=a} \subseteq R_{<b}$	$(a < b)$
(F9)	$R_{=a}$ and $R_{<a}$ are symmetric	
(F10)	$(uR_{=a}v \wedge vR_{=b}w) \implies (uR_{=a+b}w \vee uR_{<a+b}w)$	
(F11)	$(uR_{<a}v \wedge vR_{<b}w) \implies uR_{<a+b}w$	
(F12)	$(uR_{=a}v \wedge vR_{<b}w) \implies uR_{<a+b}w$	

We denote the class of all F-metric M-frames by $\mathfrak{F}_F[M]$.

As remarked earlier, note that the Conditions (F1)–(F12) correspond rather directly to validities in the Boolean modal language $\mathcal{LB}[M]$. For instance, (F5) corresponds, for $a \geq b$, to the validity $[\approx_a \wedge \prec_b]\bot$, and so on.

Further, for $a \neq 0$, (F3) and (F6) imply that the relation $R_{=a}$ is irreflexive, and (F3) and (F8) imply that $R_{<a}$ is reflexive. Thus, in all F-metric M-frames we have additionally.

(F13)	$R_{=a}$ is irreflexive	$(a \neq 0)$
(F14)	$R_{<a}$ is reflexive	$(a \neq 0)$

We are now in a position to prove a representation theorem which shows that the notion of an F-metric M-frame is sufficient to capture validity in metric spaces. This representation is 'finitary' in the sense that, given a formula $\varphi \in \mathcal{L}_F^{\mathcal{O}}[M]$ satisfiable in some F-metric M-frame $\mathfrak{f}$, we construct a finite parameter set $M(\varphi)$ such that φ is satisfiable in a possibly infinite F-metric $M(\varphi)$-frame $\mathfrak{g}$, but which is based on the finitely many relations induced by $M(\varphi)$, and from which we can construct an 'equivalent' metric space, i.e., one whose frame-companion is $\mathfrak{g}$.

Theorem 4.4 (Representation). *(i) For every finite parameter set M and F-metric M-frame $\mathfrak{f}$ there is a metric space S such that $\mathfrak{f}$ is its frame-companion, i.e., $\mathfrak{f} = \mathfrak{f}_{F,M}(S)$. In particular, if $\mathfrak{f}$ is finite, so is S.*

(ii) For an arbitrary parameter set M we have: An $\mathcal{L}_F^{\mathcal{O}}[M]$-formula φ is satisfiable in a metric space model based on a set W if and only if it is satisfiable in a model based on an F-metric M-frame based on W if and only if it is satisfiable in a model based on an F-metric $M(\varphi)$-frame based on W, with $M(\varphi)$ finite.

Proof. We first prove (i). Let M be a finite parameter set and

$$\mathfrak{f} = \left\langle W, \{R_{<a}, R_{>a}, R_{=a}, R_{<b}^{>a}\}_{a,b\in M} \right\rangle$$

an F-metric M-frame. Enumerate the N elements of M as

$$\langle 0 = a_0, a_1, \ldots, a_{N-2}, a_{N-1} = \gamma \rangle \text{ with } a_i < a_j, \text{ if } i < j.$$

Thus, $\gamma = \max(M)$. If $a = a_i \in M$, we refer to the position i of a in the enumeration also by i_a. Now for the definition of the metric. Let

$$D := \{a_i + a_j - \gamma \mid a_i + a_j > \gamma, a_i, a_j \in M\} \cup \{a_i - a_j \mid a_i > a_j, a_i, a_j \in M\},$$

and let $\mu := \min(D \cup \{1\})$. Next we choose some $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{\mu}{2^N + 1}.$$

Before we proceed to define a metric, let us summarise some properties of ε that we will need later on:

Lemma 4.5. *The following hold:*

1. *$a_j < a_i$ if and only if $a_j < a_i - 2^i \cdot \varepsilon$*
2. *$a_i - 2^i \cdot \varepsilon > 0$ for all $a_i \in M - \{0\}$.*

3. *$a_i + a_j < \gamma + (2^i + 2^j + 1) \cdot \varepsilon$ implies $a_i + a_j \leq \gamma$.*

4. *$a_i + a_j < \gamma + (2^i + 1) \cdot \varepsilon$ implies $a_i + a_j \leq \gamma$.*

5. *$a_i + a_j < \gamma + \varepsilon$ implies $a_i + a_j \leq \gamma$.*

PROOF. (1): $a_i > a_j$ if and only if $\mu \leq a_i - a_j$ and thus

$$\varepsilon < \frac{\mu}{2^N + 1} \leq \frac{a_i - a_j}{2^N + 1} < \frac{a_i - a_j}{2^i},$$

i.e., $a_j < a_i - 2^i \cdot \varepsilon$. (2) follows from (1) with $a_j = 0$. (3): First, note that for all $i, j < \omega$ we have $2^i + 2^j \leq 2 \cdot 2^{\max(i,j)} = 2^{\max(i,j)+1}$. Thus we obtain:

$$a_i + a_j < \gamma + (2^i + 2^j + 1) \cdot \varepsilon \leq \gamma + (2^{\max(i,j)+1} + 1) \cdot \varepsilon \leq \gamma + (2^N + 1) \cdot \varepsilon.$$

So, since by definition of ε we have $(2^N + 1) \cdot \varepsilon < \mu$, we have $a_i + a_j < \gamma + \mu$, which implies $a_i + a_j \leq \gamma$. For if $a + b > \gamma$, then $0 < a + b - \gamma \geq \mu$ by definition of μ, and so $a + b \geq \gamma + \mu$. (4) and (5) are a consequence of (3). ■

Now, define a function d by setting:

$$d(v, w) := \begin{cases} \gamma + \varepsilon & vR_{>a}w \text{ for all } a \in M; \\ a & vR_{=a}w \text{ for some } a \in M; \\ a_i - 2^i \cdot \varepsilon & \text{where } i = \min\{j < N \mid vR_{<a_j}w\}, \text{ otherwise.} \end{cases}$$

We first show that the function d is well-defined and total. Note that in the case $d(v, w) = a_i - 2^i \cdot \varepsilon$, $a_i = 0$ can not occur because of Condition (F4), $R_{<0} = \varnothing$. This together with (2) of Lemma 4.5 shows that $d(v, w) \geq 0$ whenever d is defined.

Moreover, the three cases in the definition of $d(v, w)$ are mutually exclusive, but exhaustive. If for all $a \in M$ we have $vR_{>a}w$ then, for all $a \in M$, $\neg vR_{<a}w$ and $\neg vR_{=a}w$ by Property (F1). If $vR_{=a}w$ for some $a \in M$, then, again by (F1), $\neg vR_{>a}w$. And if for all $a \in M$ we have $\neg vR_{=a}w$ and there is a $b \in M$ such that $\neg vR_{>b}w$, then, by (F1), $vR_{<b}w$. So d is always defined. Lastly, by Property (F6), we cannot have $vR_{=a}w$ and $vR_{=b}w$ for $a \neq b$, which shows that d is well-defined.

Next, we show that d is indeed a metric:

(a): $d(v, w) = 0$ iff $v = w$.

By (2), $a_i - 2^i \cdot \varepsilon > 0$ for all $i > 0$. Hence $d(v, w) = 0$ iff $vR_{=0}w$ iff $v = w$, according to Property (F3) of F-metric frames.

(b): $d(v, w) = d(w, v)$.

If $d(v, w) = \gamma + \varepsilon$, then $vR_{>a}w$ for all $a \in M$, i.e., by Property (F1), $\neg vR_{=a}w$ and $\neg vR_{<a}w$ for all $a \in M$. By Property (F9), $R_{=a}$ and $R_{<a}$ are symmetric, thus $wR_{>a}v$ for all $a \in M$ by (F1), and so $d(w, v) = \gamma + \varepsilon$.

Suppose $d(v, w) = a$ for some $a \in M$. By (1) this is the case if and only if $vR_{=a}w$, and so $wR_{=a}v$ by the symmetry of $R_{=a}$, thus $d(w, v) = a$. The case of $d(v, w) = a_i - 2^i \cdot \varepsilon$ follows similarly from the symmetry of $R_{<a}$, Property (F9).

(c): $d(u, v) + d(v, w) \geq d(u, w)$.

First, we can assume without loss of generality that $d(u, v), d(v, w) \neq 0$. Otherwise, if e.g. $d(u, v) = 0$, we have $u = v$ by (a) and the inequality obtains.

Case (i): If $d(u, v) + d(v, w) \geq \gamma + \varepsilon$, the inequality obtains.

Case (ii): Suppose $d(u, v) = a$ and $d(v, w) = b$ because of $uR_{=a}v$ and $vR_{=b}w$, with $a, b \in M$, and $a + b < \gamma + \varepsilon$. By (5) we then have $a + b \leq \gamma$ and thus $a + b \in M$, since M is a parameter set. By Property (F10), we have either (ii.i) $uR_{=a+b}w$, or (ii.ii) $uR_{<a+b}w$. In Case (ii.i) we have $d(u, w) = a + b$, and the inequality obtains. In Case (ii.ii) we have $d(u, w) \leq a + b - 2^{i_{a+b}} \cdot \varepsilon < a + b$.

Case (iii): $d(u, v) = a_i$, $d(v, w) = a_j - 2^j \cdot \varepsilon$ with $a_i, a_j \in M$, and, by assumption, $a_i + a_j - 2^j \cdot \varepsilon < \gamma + \varepsilon$. By definition of d, $uR_{=a_i}v$, $vR_{<a_j}w$ and $\neg vR_{<a_k}w$ for all $k < j$. Further, by (4), $a_i + a_j \leq \gamma$ and so $a_i + a_j \in M$.

By Property (F12) we obtain $uR_{<a_i+a_j}w$ and so

$$d(u, w) \leq a_i + a_j - 2^{i_{a_i+a_j}} \cdot \varepsilon \leq a_i + a_j - 2^j \cdot \varepsilon = d(u, v) + d(v, w),$$

since $i_{a_i+a_j} \geq \max(i, j) \geq j$.

Case (iv): $d(u, v) = a_i - 2^i \cdot \varepsilon$ and $d(u, v) = a_j$ with $a_i, a_j \in M$. This is similar to case (iii). We use again (F12) and additionally symmetry.

Case (v): $d(u, v) = a_i - 2^i \cdot \varepsilon$ and $d(v, w) = a_j - 2^j \cdot \varepsilon$, with $a_i, a_j \in M$. By definition of d, $uR_{<a_i}v$ and $vR_{<a_j}w$. By assumption, $a_i + a_j - 2^i \cdot \varepsilon - 2^j \cdot \varepsilon < \gamma + \varepsilon$. By (3) of Lemma 4.5 we obtain $a_i + a_j \leq \gamma$ and thus $a_i + a_j \in M$. By Property (F11) we have $uR_{<a_i+a_j}w$. Thus

$$d(u, w) \leq a_i + a_j - 2^{i_{a_i+a_j}} \cdot \varepsilon \leq a_i + a_j - 2^{\max(i,j)+1} \cdot \varepsilon \leq a_i + a_j - 2^i \cdot \varepsilon - 2^j \cdot \varepsilon,$$

since we can assume $i, j \neq 0$ and so $2^{i_{a_i+a_j}} > 2^{\max(i,j)+1} = 2 \cdot 2^{\max(i,j)} \geq 2^i + 2^j$. Hence the inequality follows.

Now that we have shown that d is a metric, we can define the metric space $S = \langle W, d \rangle$ and show that $\mathfrak{f}$ is its M-frame companion, i.e., that $\mathfrak{f}_{F,M}(S) = \mathfrak{f}$. To this end, we have to show that:

(A) $d(u,v) = a \iff uR_{=a}v$, for all $a \in M$;
(B) $d(u,v) < a \iff uR_{<a}v$, for all $a \in M$;
(C) $d(u,v) > a \iff uR_{>a}v$, for all $a \in M$;
(D) $a < d(u,v) < b \iff uR^{>a}_{<b}v$, for all $a, b \in M$.

(A): To prove (A), note that since $\varepsilon > 0$ and by (1) and (2) we have $\gamma + \varepsilon, a_i - 2^i \cdot \varepsilon \neq a$ for all $a \in M$. Thus we immediately obtain from the definition of d that $uR_{=a}v$ for some $a \in M$ if and only if $d(u,v) = a$.

(B): Suppose first that $d(u,v) < a$, then either (i) $d(u,v) = b < a$ and $wR_{=b}v$ for some $b \in M$, or, (ii) $d(u,v) = a_i - 2^i \cdot \varepsilon < a$. In Case (i) we obtain $uR_{<a}v$ by Condition (F8). In Case (ii) we have $uR_{<a_i}v$ with a_i minimal with this property. By (1) of Lemma 4.5 we have $a_i - 2^i \cdot \varepsilon < a$ implies $a_i < a$. Hence, by Condition (F7), $uR_{<a}v$.

Conversely, suppose that $uR_{<a}v$. Then, by (F5), $\neg uR_{=b}v$ for all $b \geq a$. We again have to distinguish two cases. Case (i): There exists a $b < a$ with $uR_{=b}v$. Then $d(u,v) = b < a$. In Case (ii), we have $\neg uR_{=b}v$ for all $b \in M$ and hence $d(u,v) = a_i - 2^i \cdot \varepsilon$ with $a_i \leq a$. Hence $d(u,v) < a$.

(C): Suppose first that $d(u,v) > a$. There are three cases to consider. Case (i): $d(u,v) = \gamma + \varepsilon$. Then $uR_{>b}v$ for all $b \in M$. Hence, in particular, $uR_{>a}v$. Case (ii): $d(u,v) = b > a$ for some $b \in M$ and $uR_{=b}v$. By (F5) we have $\neg uR_{<a}v$ and by (F6) $\neg uR_{=a}v$. Hence, (F1) implies $uR_{>a}v$. Case (iii): $d(u,v) = a_i - 2^i \cdot \varepsilon > a$. Then $\neg uR_{=a}v$ by definition of d and $\neg uR_{<a}v$ since otherwise $d(u,v) < a$. Hence, by (F1), $uR_{>a}v$.

Conversely, suppose $uR_{>a}v$. There are again three cases. But note first that we cannot have $uR_{=b}v$ for $b \leq a$ due to (F1) and (F8). Case (i): For all $b \in M$ we have $uR_{>b}v$. Then $d(u,v) = \gamma + \varepsilon > a$ by definition of d. Case (ii): There is some $b > a$ with $uR_{=b}v$. Then $d(u,v) = b > a$. Case (iii): There is some $b > a$ with $uR_{<b}v$. Then $d(u,v) = a_i - 2^i \cdot \varepsilon$ with $a < a_i \leq b$. But by (1), $a_i - 2^i \cdot \varepsilon > a$, as required.

(D): By (B) and (C), $a < d(u,v) < b$ if and only if $uR_{>a}v$ and $uR_{<b}v$. But by (F1) and (F2), this is the case if and only if $uR^{>a}_{<b}v$, as required.

We can now prove (ii).
Suppose that φ is satisfied in the metric space model

$$\mathfrak{B} = \left\langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \dots, i_0^{\mathfrak{B}}, i_1^{\mathfrak{B}}, \dots \right\rangle$$

based on the metric space $S = \langle W, d \rangle$, i.e., that $\langle \mathfrak{B}, w \rangle \models \varphi$ for some point $w \in W$. By Proposition 4.2, φ is satisfied in the frame-companion model $\mathfrak{M}_F(\mathfrak{B})$. It is easily checked that the relations of the frame companion $\mathfrak{f}_F(S)$ satisfy properties (F1)–(F14). Thus, $\mathfrak{f}_F(S)$ is an F-metric M-frame in which φ is satisfiable.

Conversely, suppose that φ is satisfied in an F-metric frame model

$$\mathfrak{M} = \left\langle \mathfrak{f}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \ldots, i_0^{\mathfrak{M}}, i_1^{\mathfrak{M}}, \ldots \right\rangle$$

based on the F-metric M-frame $\mathfrak{f} = \left\langle W, \{R_{<a}, R_{>a}, R_{=a}, R_{<b}^{>a}\}_{a,b \in M} \right\rangle$. We first define a finite parameter set $M(\varphi)$ and an F-metric $M(\varphi)$-frame $\mathfrak{f}^{\dagger}$ such that φ is satisfiable in $\mathfrak{f}$ if and only if it is satisfiable in $\mathfrak{f}^{\dagger}$.

Let

$$Par(\varphi) := \{a \in M \mid a \text{ occurs in } \varphi\}$$

and let $\gamma_\varphi := \max(Par(\varphi)) + 1$ and define $M(\varphi)$ as follows:

$$M(\varphi) := \{b \in M : \gamma_\varphi > b = b_1 + \ldots + b_n, b_i \in Par(\varphi), n < \omega\}.$$

It is easily seen that $M(\varphi)$ is a finite parameter set, for the number k of summands in a sum

$$b_1 + \ldots + b_k < \gamma_\varphi$$

is bounded by γ_φ / c, where $c \neq 0$ is the minimal positive number in $Par(\varphi)$. So $|M(\varphi)| < |Par(\varphi)|^{\frac{\gamma_\varphi}{c}}$. Further, $0 \in M(\varphi)$ and if $a, b \in M(\varphi)$ and $a + b < \max(M(\varphi)) < \gamma_\varphi$, then $a = a_1 + \ldots + a_k$ and $b = b_1 + \cdots + b_l$ with $a_i, b_j \in Par(\varphi)$. So $a + b \in M(\varphi)$.

Now we define the frame $\mathfrak{f}^{\dagger}$ as the frame-reduct of $\mathfrak{f}$ with respect to $M(\varphi)$, that is

$$\mathfrak{f}^{\dagger} := \mathfrak{f} \restriction_{(F, M(\varphi))} .$$

As remarked on Page 12, since $\varphi \in \mathcal{L}_F^{\mathcal{O}}[M(\varphi)]$, φ is satisfiable in $\mathfrak{f}$ if and only if it is satisfiable in $\mathfrak{f} \restriction_{(F, M(\varphi))}$. By (i) there is a metric space S such that $\mathfrak{f}^{\dagger}$ is its frame-companion, i.e., $\mathfrak{f}_{F, M(\varphi)}(S) = \mathfrak{f} \restriction_{(F, M(\varphi))}$. By Proposition 4.2, φ is satisfiable in S, which had to be shown.

■

5. Axiomatisation

In this section, we will present an axiomatisation of the logic $\mathcal{MS}_F^{\mathcal{O}}[M]$ (for some fixed parameter set $M \subseteq \mathbb{R}^+$)—thus axiomatising the two-variable logic $\mathcal{FM}_2[M]$ (via translation)—and show it to be weakly complete with respect to metric spaces.

Theorem 4.4 implies that, to axiomatise the logic $\mathcal{MS}_F^{\mathcal{O}}[M]$ (or $\mathcal{MS}_F[M]$), it suffices to axiomatise the class $\mathfrak{F}_F[M]$ of F-metric M-frames. More specifically, it allows us to transfer *weak completeness* from F-metric M-frames to metric spaces. That this is all we can hope for in general follows from the fact that *strong completeness* implies compactness, and the non-compactness of $\mathcal{MS}_F[M]$ for infinite (unbounded) M, which we will discuss in Section 6.

In [13], we gave an 'orthodox' axiomatisation of the logic $\mathcal{MS}_D[M]$ in the sense that the axiomatic system given was a standard (modal) Hilbert calculi comprising as rules of proof just *modus ponens* and *necessitation*. We proceeded by applying finite filtrations to the canonical models and by 'repairing' the resulting models to obtain standard metric models still refuting a given formula.

Unfortunately, as concerns the distance logic $\mathcal{MS}_F[M]$, this proof technique is rather difficult to apply. First, while in the case of D-metric frames we had to deal only with one condition that was not definable in the language, the relational representation of metric spaces for the language $\mathcal{L}_F[M]$ given in the last section comprises several frame conditions not definable in $\mathcal{L}_F[M]$. Second, by a result of [14], the language $\mathcal{L}_F[M]$ does not have the finite model property, and so we cannot expect to be able to apply a filtration technique similar to the one employed for the language $\mathcal{L}_D[M]$ in [13].

However, we can axiomatise the logic $\mathcal{MS}_F^{\mathcal{O}}[M]$ by making use of its *hybrid* character, i.e., the presence of both, nominals and the universal modality, and by using general results from hybrid completeness theory involving the use of non-standard rules of inference, namely (a simplified version of) the *covering rule* (COV) used, e.g., in [10].

Similarly to formulae from the *Sahlqvist fragment*, pure formulae, that is formulae containing *only* nominals (rather than propositional variables), define frame classes that are always first-order definable (see, e.g., [7, Proposition 3.1]). As early as in [3], it was realised that axiomatisations with pure formulae give rise to 'easy' completeness proofs. Very roughly, completeness proofs for languages involving nominals and universal modality or the @-operator proceed by combining the techniques of canonical models from modal logic and a Henkin construction as in first-order logic. In [2], we find such a completeness proof for languages containing nominals and @-operator, and in [7], we find a completeness proof for languages with nominals and the universal modality.

Although many first-order conditions that are modally or Sahlqvist definable are definable by pure formulae, e.g. reflexivity to name one of the simplest examples, this is not true in general. The Church-Rosser property

$$\forall u \forall v \forall w (uRv \wedge uRw \rightarrow \exists x (vRx \wedge wRx))$$

Axiom schemata for $\mathsf{MS}_F^{\mathcal{O}}[M]$

(CL)	Axioms of propositional calculus	
$(\mathsf{K}_{\bigcirc})$	$\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$, where $\bigcirc \in \{\blacksquare, \mathsf{A}^{=a}, \mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{>a}_{<b}\}$	$(a, b \in M)$
$(\mathsf{Def}^{>})$	$\mathsf{E}^{>a} i \leftrightarrow (\mathsf{A}^{<a}\neg i \wedge \mathsf{A}^{=a}\neg i)$	$(a \in M)$
$(\mathsf{Def}^{>}_{<})$	$\mathsf{E}^{>a}_{<b} i \leftrightarrow (\mathsf{A}^{<a}\neg i \wedge \mathsf{A}^{=a}\neg i \wedge \mathsf{A}^{>b}\neg i \wedge \mathsf{A}^{=b}\neg i)$	$(a, b \in M)$
$(\mathsf{Dis}_{\leqq})$	$\mathsf{E}^{=a} i \to \mathsf{A}^{<b}\neg i$	$(a \geq b)$
$(\mathsf{Dis}_{=})$	$\mathsf{E}^{=a} i \to \mathsf{A}^{=b}\neg i$	$(a \neq b)$
$(\mathsf{T}_{=0})$	$\mathsf{A}^{=0}\varphi \leftrightarrow \varphi$	$(0 \in M)$
$(\mathsf{Bot}_{<0})$	$\mathsf{A}^{<0}\bot$	$(0 \in M)$
$(\mathsf{Mon}_{<})$	$\mathsf{A}^{<a}\varphi \to \mathsf{A}^{<b}\varphi$	$(a \geq b)$
$(\mathsf{Mon}_{\leqq})$	$\mathsf{A}^{<a}\varphi \to \mathsf{A}^{=b}\varphi$	$(a > b)$
(Tra_1)	$(\mathsf{A}^{<a+b}\varphi \wedge \mathsf{A}^{=a+b}\varphi) \to \mathsf{A}^{=a}\mathsf{A}^{=b}\varphi$	$(a + b \in M)$
(Tra_2)	$\mathsf{A}^{<a+b}\varphi \to \mathsf{A}^{<a}\mathsf{A}^{<b}\varphi$	$(a + b \in M)$
(Tra_3)	$\mathsf{A}^{<a+b}\varphi \to \mathsf{A}^{<a}\mathsf{A}^{=b}\varphi$	$(a + b \in M)$
$(\mathsf{B}_{\bigcirc})$	$\varphi \to \bigcirc\neg\bigcirc\neg\varphi$, $\quad \bigcirc \in \{\mathsf{A}^{=a}, \mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{>a}_{<b}\}$	$(a, b \in M)$
$(\mathsf{Inc}_{\bigcirc})$	$\blacksquare\varphi \to \bigcirc\varphi$, $\quad \bigcirc \in \{\mathsf{A}^{=a}, \mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{>a}_{<b}\}$	$(a, b \in M)$
$(\mathsf{4}_{\blacksquare})$	$\blacksquare\varphi \to \blacksquare\blacksquare\varphi$	
$(\mathsf{B}_{\blacksquare})$	$\varphi \to \blacksquare\blacklozenge\varphi$	
$(\mathsf{T}_{\blacksquare})$	$\blacksquare\varphi \to \varphi$	
(Nom_1)	$\blacklozenge i$	
(Nom_2)	$\blacklozenge(i \wedge \varphi) \to \blacksquare(i \to \varphi)$	

Inference rules

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathrm{MP}) \qquad \frac{\varphi}{\blacksquare\varphi}\ (\mathrm{RN}) \qquad \frac{i \to \varphi}{\varphi}, i \notin \varphi\ (\mathrm{COV}_0) \qquad (\mathrm{SSUB})$$

Table 1. The axiomatic system $\mathsf{MS}_F^{\mathcal{O}}[M]$.

is modally definable by the Sahlqvist formula

$$\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi,$$

which is not equivalent to any pure formula [11].

Now, to axiomatise the class $\mathfrak{F}_F$ using the language $\mathcal{L}_F^{\mathcal{O}}[M]$, we have a number of options. First, we can check that, indeed, all the frame conditions from Definition 4.3 are definable by pure formulae. This gives us an immediate completeness result since it is known that any extension with pure axioms of the basic logic of the universal modality enriched with nominals is complete with respect to the class of frames defined by the pure formulae when using the additional *covering rule* of inference (COV) [11]. (COV) was first introduced in [18] to axiomatise PDL with nominals, and further examined most notably in [8] and [7].

Usually, (COV) takes on a rather complicated form, being formulated with the help of *universal forms* $u(\sharp)$ (compare [9] and [8]), which are, up to propositional equivalence, formulae of the shape

$$\varphi_0 \rightarrow \triangledown_1^{k_1}(\varphi_1 \rightarrow \ldots \rightarrow \triangledown_{n-1}^{k_{n-1}}(\varphi_{n-1} \rightarrow \triangledown_n^{k_n}(\varphi_n \rightarrow \sharp))\ldots)$$

where the $\triangledown_i^{k_i}$ are sequences of k_i 'universal' distance operators from the list $\{\mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{=a}, \mathsf{A}^{>a}_{<b} \mid a, b \in M\}$ or the universal modality $\blacksquare$, and some of the φ_i may be $\top$, when necessary. The standard (COV) rule that is needed in general for pure extensions states that if $u(\neg i)$ is derivable for some universal form $u(\sharp)$ and nominal i not appearing in $u(\sharp)$, then infer $u(\bot)$.

Yet, we can do a bit better. Notice that all the modalities in the set F are symmetric by Conditions (F1), (F2) and (F9). Fortunately, symmetric modalities are a special case of the *versatile similarity types* of [20] or the *reversive languages* of [10]. Moreover, it is clear that the languages $\mathcal{L}_F^{\mathcal{O}}[M]$ (having nominals and the universal modality) and $\mathcal{L}_F[M]$ (having the 'difference operator' $\mathsf{E}^{>0}$) have the same expressive power when it comes to frame definability [19]. Thus, we can choose between giving an axiomatisation over the language $\mathcal{L}_F[M]$ (using the general completeness theorems for languages employing the difference operator and Sahlqvist axioms from [20]), and a (mixed) axiomatisation using pure formulae and Sahlqvist schemes (using the (generalised) Sahlqvist completeness theorem from [11]). Note, in particular, that in the case of symmetric modalities, universal forms are no longer necessary: given the rule (COV_0), that is, infer φ from $i \rightarrow \varphi$, ($i \notin \varphi$), the rule (COV) becomes derivable [10].

We give here the axiomatisation for the language with nominals which we feel to be the most elegant. The axiomatic system, which is listed in Table 1, will be denoted by $\mathsf{MS}_F^{\mathcal{O}}[M]$.

Some comments on the choice of axioms might be in order. First, we have the standard S5 and 'inclusion' axioms for the universal modality, as well as the Axioms (Nom_1) and (Nom_2) which suffice to axiomatise nominals in the presence of the universal modality, compare [7]. The remaining axioms are derived from the Representation Theorem, Theorem 4.4. They precisely define the first-order conditions given for F-metric M-frames, i.e., those that are needed to construct an appropriate metric space from a frame.

The inference rules of the system $\mathsf{MS}_F^{\mathcal{O}}[M]$ are **sorted substitution** (SSUB), i.e., arbitrary formulae may be substituted for propositional variables, and nominals for nominals, and **modus ponens**, **necessitation** for $\blacksquare$, as well as (COV_0).

Note that in the presence of the Inclusion Axioms ($\mathsf{Inc}_{\bigcirc}$) all of the rules of necessitation

$$\frac{\varphi}{\bigcirc\varphi}\ (\mathrm{RN}\bigcirc), \qquad \bigcirc \in \{\mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{=a}, \mathsf{A}^{>a}_{<b} \mid a, b \in M\}$$

are derivable in $\mathsf{MS}_F^{\mathcal{O}}$.

Thus, the details of the proof of the following theorem are easily spelled out:

Theorem 5.1 (Strong frame completeness). *For every $\mathcal{L}_F^{\mathcal{O}}[M]$-formula φ and set of formulae Γ:*

$$\Gamma \vdash_{\mathsf{MS}_F^{\mathcal{O}}} \varphi \quad \Longleftrightarrow \quad \Gamma \models_{\mathfrak{F}_F[M]} \varphi.$$

Next, as a corollary to Theorem 4.4 and Theorem 5.1 we obtain:

Theorem 5.2 (Weak metric completeness). *For every $\mathcal{L}_F^{\mathcal{O}}[M]$-formula φ:*

$$\vdash_{\mathsf{MS}_F^{\mathcal{O}}} \varphi \quad \Longleftrightarrow \quad \varphi \in \mathcal{MS}_F^{\mathcal{O}}.$$

6. Compactness

To clarify the relationship between Theorems 5.1 and 5.2, let us briefly discuss the compactness property in metric spaces. We have shown in Theorem 4.4 (i) that, for finite parameter sets M, there is a precise correspondence between F-metric M-frames and metric spaces, that is, there are maps

$$\mathcal{MS} \xrightarrow{h} \mathfrak{F}_F[M], \text{ with } h(S) := \mathfrak{f}_{F,M}(S)$$

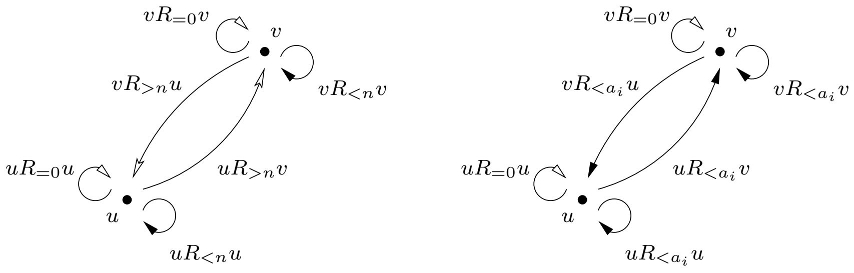
and

$$\mathfrak{F}_F[M] \xrightarrow{g} \mathcal{MS}, \text{ with } h \circ g = \mathsf{id}_{\mathfrak{F}_F[M]},$$

where $\mathsf{id}_{\mathfrak{F}_F[M]}$ is the identity function on the class $\mathfrak{F}_F[M]$ (but note that, in general, $g \circ h(S) \neq S$). This means that, for finite M, the local consequence relations $\models_{\mathcal{MS}}$ and $\models_{\mathfrak{F}_F[M]}$ coincide and are, by strong completeness with respect to F-metric M-frames, both compact. This picture changes radically, however, when we move to infinite parameter sets. For instance, let $M = \mathbb{N}$ and consider the set

$$\Gamma = \{\neg \mathsf{E}^{<n} p \mid n < \omega\} \cup \{\blacklozenge p\}.$$

Then every finite subset Γ_0 of Γ is satisfiable in some metric space, but Γ is not. This shows that the local consequence relation $\models_{\mathcal{MS}}$ is not compact for the language $\mathcal{L}_F^{\mathcal{O}}[\mathbb{N}]$. Since the local consequence relation $\models_{\mathfrak{F}_F[M]}$ is compact



¬(†) ¬(‡)

Figure 1. Non-standard frames with 'points at infinity' and 'infinitesimal points'.

independently of M, Γ is satisfiable in some F-metric M-frame. Indeed, define a 'non-standard' F-metric M-frame $\mathfrak{f}$ that is based on the set $W = \{u, v\}$ by setting $uR_{>n}v$, for all $n \in \mathbb{N}$, $uR_{=0}u$, $vR_{=0}v$, $uR_{<n}u$ and $vR_{<n}v$, for all $n > 0$, and $R_{<m}^{>n} = \varnothing$, for all n, m, see Figure 1.

Then $\mathfrak{f}$ is an F-metric $[\mathbb{N}]$-frame and Γ is satisfiable in u. But $\mathfrak{f}$ is not the frame-companion of any metric space S, since any such frame-companion satisfies the additional infinitary condition

$$(\dagger) \qquad \bigcup_{n<\omega} R_{<n} = W \times W.$$

This condition holds generally in any frame-companion of some metric space whenever the parameter set M is **unbounded**: since the distance between any two points is functionally determined by the metric, the **Archimedean axiom** for the real numbers

$$\forall a, b \in \mathbb{R}^+ \exists n \in \mathbb{N} : n \cdot a > b$$

guarantees that, eventually, we will find some $n \in \mathbb{N}$ such that $d(u, v) < n$, which rules out frames as the one defined above.

Similarly, suppose M is a **dense** subset of $\mathbb{R}^+$, e.g. $M = \mathbb{Q}^+$. Suppose $\mathfrak{a} = (a_n)_{n<\omega}$ is some strictly decreasing sequence of numbers from M, i.e., $a_{n+1} < a_n$ for all $n < \omega$, with $\inf(\mathfrak{a}) = 0$. Then every frame-companion of some metric space $\langle W, d\rangle$ satisfies the following condition:

$$(\ddagger) \qquad \forall v, w \in W : v \neq w \implies \inf(\{a_i \mid a_i \in \mathfrak{a} \text{ and } vR_{<a_i}w\}) > 0.$$

On the other hand, there are F-metric $\mathbb{Q}^+$-frames that violate it, and which are hence not the frame-companion of a metric space, compare, again, Figure 1.

It should be now rather clear that the concept of 'metric space' is not first-order definable on frames: an adequate stronger relational representation of metric spaces also has to 'represent' the theory of real numbers. In particular, given an arbitrary F-metric M-frame $\mathfrak{f}$ with an infinite parameter set M, it is in general not possible to find an equivalent metric space S, i.e, such that $\mathsf{Th}(\mathfrak{f}) = \mathsf{Th}(S)$. At this point, we can proceed in different ways. One possibility is to enrich the language $\mathcal{L}_F^{\mathcal{O}}[M]$ by numerical variables $x, y, \ldots$ that range over M and can take the place of parameters, and to allow explicit quantification over these variables, with the obvious semantic interpretation.[6] Then, for instance, the formula $\mathsf{E}^{>0}\varphi \to \mathsf{E}^{\exists x.<x}\varphi$ taken as an extra axiom corresponds to the frame-condition $\forall u, v.\exists a \in M.uR_{<a}v$, thus expressing (†).

7. Craig Interpolation

By algebraically manipulating the frame conditions given for F-metric M-frames into equivalent forms taking other operator sets as primitive (in the sense that $\mathsf{A}^{=a}$ and $\mathsf{A}^{<a}$ were primitive in Definition 4.3), the frame representation for the language $\mathcal{L}_F^{\mathcal{O}}[M]$ can be used to derive corresponding representation theorems for various sublanguages $\mathcal{L}_O^{\mathcal{O}}[M]$, $O \subseteq \mathsf{Op}$, and to

[6] A similar extension with variables but not allowing quantification was considered for weaker decidable logics of distance in [21], and it was shown that adding variables ranging over parameters and linear inequalities as constraints preserves decidability.

give respective sound and complete axiom systems (for details compare [12]). In particular, we can derive various positive and negative results about Craig interpolation for languages without nominals. If some subset of $\{\mathsf{A}^{<a}, \mathsf{A}^{\leq a}\}$ is taken as primitive, we obtain frame representations comprising universal Horn conditions that are Sahlqvist axiomatisable, which yields, by a result of [17], Craig interpolation for all these logics.

THEOREM 7.1. *The logics* $\mathcal{MS}_L[M]$, $L \subseteq \{\mathsf{A}^{<a}, \mathsf{A}^{\leq a} \mid a \in M\}$, *have Craig interpolation.*

Curiously, whereas the operators $\mathsf{A}^{<a}$ and $\mathsf{A}^{\leq a}$ are incomparable with respect to expressivity over metric spaces—'talking' about closed and open balls respectively (compare [22])—they generate exactly the same sets of tautologies (when ignoring the 'trivial' operators $\mathsf{A}^{<0}$ and $\mathsf{A}^{\leq 0}$). Sometimes, a more restricted version of Craig interpolation is investigated, where interpolants not only have to be build from shared propositional variables, but also from shared modal operators. This, however, makes no sense in this general form in the context of distance logics. Consider, for instance, the formula

$$\mathsf{A}^{<2}\varphi \rightarrow \mathsf{A}^{<1}\mathsf{A}^{<1}\varphi.$$

While the language containing the operators of type $\mathsf{A}^{<a}$ have Craig interpolation in all classes of distance spaces, the above formula—being tautological in triangular spaces—has clearly no interpolant using no distance operators at all. The question, on the other hand, exactly which parameters from a parameter set M are needed in an interpolant, given the parameters appearing in the antecedent and consequent of some valid implication, is a non-trivial and interesting question.

Failure of Craig interpolation, though, is the norm for distance logics: as shown in [4], any language that has Craig interpolation over a class of frames and extends the basic logic of the difference operator is at least as expressive as the first-order correspondence language. Thus, all distance logics containing the operator $\mathsf{E}^{>0}$ fail to have Craig interpolation.

Even worse, if we consider the languages $\mathcal{L}_D[M \setminus \{0\}]$ comprising the operators $\mathsf{A}^{\leq a}$ and $\mathsf{A}^{>a}$ but leaving out the difference operator $\mathsf{E}^{>0}$, we can still construct a counterexample for Craig interpolation following the lines of the proof for failure of interpolation in Humberstone's inaccessibility logic (comprising modal operators for a binary relation and its complement) [1].

Let us exemplary prove this result. As shown in [13], the following definition allows us to prove the analogue of Theorem 4.4 for the language $\mathcal{L}_D$:

call an M-frame $\mathfrak{f}$ of the form

$$\mathfrak{f} = \langle W, (R_{\leq a})_{a\in M}, (R_{>a})_{a\in M}\rangle$$

D-**metric**, if the following conditions hold for all $a, b \in M$ and $w, u, v \in W$:

(D1) $R_{\leq a} \cup R_{>a} = W \times W$;
(D2) $R_{\leq a} \cap R_{>a} = \varnothing$;
(D3) If $uR_{\leq a}v$ and $a \leq b$, then $uR_{\leq b}v$;
(D4) $uR_{\leq 0}v \iff u = v$;
(D5) $uR_{\leq a}v \iff vR_{\leq a}u$;
(D6) If $uR_{\leq a}v$ and $vR_{\leq b}w$, then $uR_{\leq a+b}w$, whenever $a + b \in M$.

Next, we need the following variant of Lemma 2.5 in [1]:[7]

LEMMA 7.2 (Areces and Marx). *Suppose there are finite D-metric frames $\mathfrak{g}, \mathfrak{h}$ and a frame $\mathfrak{f}$ for $\mathcal{L}_D$ containing just one world such that*

1. *There are surjective p-morphisms m, n such that $\mathfrak{g} \overset{m}{\twoheadrightarrow} \mathfrak{f} \overset{n}{\twoheadleftarrow} \mathfrak{h}$;*
2. *There is no D-metric frame $\mathfrak{j}$ with commuting surjective p-morphisms g and h from $\mathfrak{j}$ onto $\mathfrak{g}$ and $\mathfrak{h}$, i.e., such that $\mathfrak{g} \overset{g}{\twoheadleftarrow} \mathfrak{j} \overset{h}{\twoheadrightarrow} \mathfrak{h}$ and $m \circ g = n \circ h$.*

Then Craig interpolation fails.

The proof of this criterion depends on the fact that finite frames can be characterised syntactically up to bisimulation [6] and that a counterexample for interpolation can be explicitly constructed from the formulae describing the frames and functions $\mathfrak{g}$, $\mathfrak{h}$, m and n.

THEOREM 7.3. *The logics $\mathcal{MS}_D[M]$ and $\mathcal{MS}_D[M \setminus \{0\}]$, $|M| < \omega$, fail to have Craig interpolation.*

PROOF. We only have to prove the result for the case of 'non-standard' parameter sets excluding 0. First, let us define two finite D-metric frames $\mathfrak{g}$ and $\mathfrak{h}$. Define

$$\mathfrak{g} = \left\langle U, (R_{\leq a})_{a\in M\setminus\{0\}}, (R_{>a})_{a\in M\setminus\{0\}}\right\rangle$$

[7] When compared to this Lemma, note that since in D-metric M-frames the relations $R_{>a}$ are the complement of $R_{\leq a}$, for all $a \in M$, D-metric frames are point-generated by every point. Moreover, the class of D-metric frames is elementary and obviously closed under point-generated subframes.

and

$$\mathfrak{h} = \left\langle V, (S_{\leq a})_{a\in M\setminus\{0\}}, (S_{>a})_{a\in M\setminus\{0\}}\right\rangle$$

by letting

$$U := \{u, v, w\}, R_{\leq a} := \{\langle u, u\rangle, \langle v, v\rangle, \langle w, w\rangle\} \text{ and } R_{>a} := (U \times U) \setminus R_{\leq a},$$

and

$$V := \{u', v'\}, S_{\leq a} := \{\langle u', u'\rangle, \langle v', v'\rangle\} \text{ and } S_{>a} := (V \times V) \setminus S_{\leq a},$$

for every $a \in M \setminus \{0\}$. Note that by D-metric frames for parameter sets $M \setminus \{0\}$ we still mean frames satisfying all of the conditions of D-metric frames even if the relation $R_{>0}$ is not explicitly present in the frame. Thus, for instance, the relations $R_{>a}$ are assumed to be irreflexive. Further, define the one-point frame $\mathfrak{f}$ based on the set $W = \{x\}$ by setting $T_{\leq a} = T_{>a} = \{\langle x, x\rangle\}$, for every $a \in M \setminus \{0\}$. Finally, define functions m, n with $\mathfrak{g} \overset{m}{\twoheadrightarrow} \mathfrak{f}$ and $\mathfrak{h} \overset{n}{\twoheadrightarrow} \mathfrak{f}$ by setting $m(u) = m(v) = m(w) = x$ and $n(u') = n(v') = x$. Obviously, m and n are surjective p-morphisms. In Figure 2 below, white arrows represent the relation $R_{\leq a}$, black arrows represent $R_{>a}$ (holding for all a), and the functions m and n are shown as dotted lines.

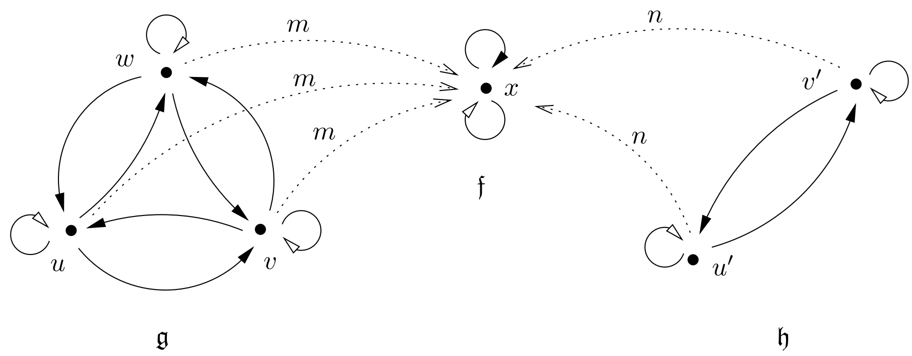


Figure 2. Counterexample for interpolation in $\mathcal{L}_D[M \setminus \{0\}]$.

Now, suppose that there is a D-metric frame

$$\mathfrak{j} = \left\langle J, (P_{\leq a})_{a\in M\setminus\{0\}}, (P_{>a})_{a\in M\setminus\{0\}}\right\rangle$$

with commuting surjective p-morphisms g and h from $\mathfrak{j}$ onto $\mathfrak{g}$ and $\mathfrak{h}$, i.e., such that $\mathfrak{g} \overset{g}{\twoheadleftarrow} \mathfrak{j} \overset{h}{\twoheadrightarrow} \mathfrak{h}$ and $m \circ g = n \circ h$. Pick some y_1 in $\mathfrak{j}$ such that $g(y_1) = u$. This exists by surjectivity of g. Then we have, for some a, $g(y_1) R_{>a} v R_{>a} w$, and so there is a y_2 in $\mathfrak{j}$ with $g(y_2) = v$ and $y_1 P_{>a} y_2$. Since $\mathfrak{j}$ is assumed to be D-metric, $P_{>a}$ is irreflexive and so $y_1 \neq y_2$. Repeating this argument shows that there is a y_3 in $\mathfrak{j}$ with $y_2 P_{>a} y_3$, $g(y_3) = w$, and $y_2 \neq y_3$. Since $g(y_1) = u \neq w = g(y_3)$, we also have $y_1 \neq y_3$. Note that, at this point, we directly obtain a contradiction if $\mathsf{E}^{>0}$ is in the signature: y_1, y_2, y_3 are all distinct, which implies that $y_1 P_{>0} y_2 P_{>0} y_3 P_{>0} y_1$, and since h is a p-morphism, we obtain

$$h(y_1) S_{>0} h(y_2) S_{>0} h(y_3) S_{>0} h(y_1),$$

which implies that $\mathfrak{h}$ has at least 3 points, which is a contradiction.

Let us now continue with the case of $M \setminus \{0\}$. Without loss of generality, we may assume $h(y_1) = u'$. Then $h(y_2) = v'$, for $y_1 P_{>a} y_2$ implies $h(y_1) S_{>a} h(y_2)$ and so, by irreflexivity, $h(y_1) \neq h(y_2)$. Similarly, it follows that $h(y_3) \neq h(y_2)$, and so $h(y_3) = u'$. It follows that $h(y_1) S_{\leq a} h(y_3)$ and so $y_1 P_{\leq a} y_3$, for otherwise we would have $y_1 P_{>a} y_3$ by condition (D1) of D-standard frames, and thus $h(y_1) S_{>a} h(y_3)$ contrary to the definition of $\mathfrak{h}$. But $y_1 P_{\leq a} y_3$ implies $g(y_1) R_{\leq a} g(y_3)$, i.e., $u R_{\leq a} w$, contradicting condition (D2) of D-metric frames.

■

Note that the proof of the last theorem can be readily adapted to prove failure of interpolation for the language $\mathcal{L}_F[M]$. Simply notice that the modality $\mathsf{E}^{\leq a}$ is definable in $\mathcal{L}_F[M]$ and that the modalities $\mathsf{E}^{=a}$ and $\mathsf{E}^{>a}_{<b}$ can be 'trivialised' in the frames used in the proof: just set $R^{>a}_{<b} = S^{>a}_{<b} = \varnothing$, $R_{=a} = S_{=a} = \varnothing$ for $a \neq 0$, and $R_{=0} = S_{=0} = \mathsf{id}$.

It is open, however, whether Craig interpolation still fails over metric spaces, if we only have 'weak' difference operators in the form of $\mathsf{E}^{>a}$ saying 'somewhere at least a far away', but not their complements (Craig interpolation is obtained for this language if we leave out the triangular inequality, since then all frame conditions are again universal Horn).

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