Logics of metric spaces

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We investigate the expressive power and computational properties of two different types of languages intended for speaking about distances. First, we consider a first-order language $\mathcal{FM}$ the two-variable fragment of which turns out to be undecidable in the class of distance spaces validating the triangular inequality as well as in the class of all metric spaces. Yet, this two-variable fragment is decidable in various weaker classes of distance spaces. Second, we introduce a variable-free 'modal' language $\mathcal{MS}$ which, when interpreted in metric spaces, has the same expressive power as the two-variable fragment of $\mathcal{FM}$. We determine natural and expressive fragments of $\mathcal{MS}$ which are decidable in various classes of distance spaces validating the triangular inequality, in particular, the class of all metric spaces.

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1. INTRODUCTION

This paper investigates the expressive power and computational properties of languages designed for speaking about distances. 'Distances' can be induced by dif-

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ferent measures. We may be interested in the physical distance between two cities a and b, i.e., in the length of the straight (or geodesic) line between a and b. More pragmatic would be to bother about the length of the railroad connecting a and b, or even better, the time it takes to go from a to b by train (plane, ship, etc.). But we can also define the distance as the number of cities (stations, friends to visit, etc.) on the way from a to b, as the difference in altitude between a and b, and so forth.

The standard mathematical models, capturing common features of various notions of distance, are known as metric spaces. A *metric space* is a pair $\langle W, d \rangle$, where W is a set (of points) and d a function from $W \times W$ into the set $\mathbb{R}^+$ (of non-negative real numbers) satisfying the following axioms

$$d(x, y) = 0 \text{ iff } x = y, \tag{1}$$
$$d(x, z) \leq d(x, y) + d(y, z), \tag{2}$$
$$d(x, y) = d(y, x) \tag{3}$$

for all $x, y, z \in W$. The value $d(x, y)$ is called the *distance* from the point x to the point y. The perhaps most 'popular' metric spaces are the n-dimensional Euclidean spaces $\langle \mathbb{R}^n, d_n \rangle$ with the metric

$$d_n(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Although acceptable in many cases, the concept of metric space is not universally applicable to all interesting measures of distance between points, especially those used in everyday life. Consider, for instance, the following two examples:

(i) If $d(x, y)$ is the flight-time from x to y then, as we know it too well, d is not necessarily symmetric, even approximately (just go from London to Tokyo and back).

(ii) Often we do not measure distances by means of real numbers but rather use more fuzzy notions such as 'short,' 'medium' and 'long.' To represent these measures we can, of course, take functions d from $W \times W$ into the subset $\{1, 2, 3\}$ of $\mathbb{R}^+$ and define *short* := 1, *medium* := 2, and *long* := 3. So we can still regard these distances as real numbers. However, for measures of this type the triangular inequality (2) usually doesn't hold (short plus short can still be short, but it can also be medium or long).

Metric spaces as well as more general *distance spaces* $\langle W, d \rangle$ satisfying only axiom (1) are the intended models of the languages we construct in this paper.

We begin by considering the first-order languages $\mathcal{FM}[M]$, for $M \subseteq \mathbb{R}^+$, with monadic predicates (for subsets of W), individual constants (for points in W), and the binary predicates $\boldsymbol{\delta}(x, y) < a$ and $\boldsymbol{\delta}(x, y) = a$, $a \in M$, saying that the distance between x and y is smaller than a or equal to a, respectively. Typical sets M of possible distances will be $\mathbb{Q}^+$ (the non-negative rational numbers) and $\mathbb{N}$ (the natural numbers including 0).

The following example will be used to illustrate the expressive power of our languages.

EXAMPLE 1.1. Imagine that you are going to buy a house in London. You then inform your estate agent about your intention and provide her with a number of constraints:

(A) The house should not be too far from your college, say, not more than 10 miles.

(B) The house should be close to shops and restaurants; they should be reachable, say, within 1 mile.

(C) There should be a 'green zone' around the house, at least within 2 miles in each direction.

(D) Factories and motorways must be far from the house, not closer than 5 miles.

(E) There must be a sports center around, and moreover, all sports centers of the district should be reachable on foot, i.e., they should be within, say, 3 miles.

(F) Public transport should be easily accessible: whenever you are not more than 8 miles away from home, there should be a bus stop or a tube station within a distance of 2 miles.

(G) And, of course, there must be a tube station around, not too close, but not too far either—somewhere between 0.5 and 1 mile.

The constraints in Example 1.1 can be formalized in $\mathcal{FM}[\mathbb{Q}^+]$ by the following formulas:

(A′) $\boldsymbol{\delta}(college, house) \leq 10$, where *college* and *house* are constants.

(B′) $\exists x\big(\boldsymbol{\delta}(house, x) \leq 1 \wedge \mathsf{shop}(x)\big)$ and $\exists x\big(\boldsymbol{\delta}(house, x) \leq 1 \wedge \mathsf{restaurant}(x)\big)$, where shop and restaurant are unary predicates.

(C′) $\forall x\big(\boldsymbol{\delta}(house, x) \leq 2 \rightarrow \mathsf{green\_zone}(x)\big)$, where green_zone is a unary predicate.

(D′) $\forall x\big(\mathsf{factory}(x) \vee \mathsf{motorway}(x) \rightarrow \boldsymbol{\delta}(house, x) > 5\big)$, where factory and motorway are unary predicates.

(E′) $\exists x\big(\boldsymbol{\delta}(house, x) \leq 3 \wedge \mathsf{district\_sports\_center}(x)\big) \wedge \forall x\big(\boldsymbol{\delta}(house, x) > 3 \rightarrow \neg\mathsf{district\_sports\_center}(x)\big)$, where district_sports_center is a unary predicate.

(F′) $\forall x\big(\boldsymbol{\delta}(house, x) \leq 8 \rightarrow \exists y\big(\boldsymbol{\delta}(x, y) \leq 2 \wedge \mathsf{public\_transport}(y)\big)\big)$, where public_transport is a unary predicate, and

(G′) $\exists x\big(\boldsymbol{\delta}(house, x) > 0.5 \wedge \boldsymbol{\delta}(house, x) \leq 1 \wedge \mathsf{tube\_station}(x)\big)$, where tube_station is a unary predicate.

As one might expect, the satisfiability problem for $\mathcal{FM}[\mathbb{Q}^+]$- and $\mathcal{FM}[\mathbb{N}]$-formulas in any class of distance spaces containing the class $\mathcal{M}$ of all metric spaces is undecidable (see Theorem 2.1 below). Trying to find decidable but still reasonably expressive sublanguages of $\mathcal{FM}[\mathbb{Q}^+]$, we then consider its two-variable fragment $\mathcal{FM}^2[\mathbb{Q}^+]$ consisting of all $\mathcal{FM}[\mathbb{Q}^+]$-formulas with the variables x and y only. (All formulas in the example above belong to this fragment.) The two-variable fragment of classical first-order logic is known to be decidable (which was proved for the language without equality in [Scott 1962] and for the language with equality in [Mortimer 1975]) and **NExpTime**-complete [Fürer 1984; Grädel et al. 1997] (we refer the reader to [Grädel and Otto 1999; Börger et al. 1997] for more information). We use this result to show that the satisfiability problem for $\mathcal{FM}^2[\mathbb{Q}^+]$-formulas is decidable

—in the class $\mathcal{D}$ of arbitrary distance spaces, and

—in the class $\mathcal{D}_{sym}$ of all distance spaces satisfying (3).

Unfortunately, this does not hold any more as soon as we add the triangular inequality (2): we show that the satisfiability problem for $\mathcal{FM}^2[\mathbb{Q}^+]$-formulas is undecidable both in

—the class $\mathcal{M}$ of all metric spaces and in
—the class $\mathcal{D}_{tr}$ of distance spaces satisfying the triangular inequality.

We then introduce variable-free languages $\mathcal{MS}[M]$, $M \subseteq \mathbb{R}^+$, which instead of first-order quantifiers use *distance operators* 'somewhere in the circle of radius a,' 'somewhere outside the circle of radius b,' etc., where $a, b \in M$. This brings us close to the field of temporal, modal, and description logics, which also avoid the use of first-order quantification by replacing it with various kinds of 'modal' operators like 'sometime in the future,' 'it is possible,' etc. The constraints in Example 1.1 can be formulated in $\mathcal{MS}$ as follows. As before, we treat '*house*' and '*college*' as constants representing certain points in the space; however, 'shop,' 'restaurant' and other unary predicates are now understood as *set variables* interpreted as subsets of the domain of the distance space.

(A″) $\boldsymbol{\delta}(house, college) \leq 10$.

(B″) $house \sqsubseteq (\mathsf{E}^{\leq 1}\mathsf{shop} \sqcap \mathsf{E}^{\leq 1}\mathsf{restaurant})$.

(C″) $house \sqsubseteq \mathsf{A}^{\leq 2}\mathsf{green\_zone}$.

(D″) $house \sqsubseteq \neg \mathsf{E}^{\leq 5}(\mathsf{factories} \sqcup \mathsf{motorways})$.

(E″) $house \sqsubseteq (\mathsf{E}^{\leq 3}\mathsf{district\_sports\_center} \sqcap \mathsf{A}^{>3}\neg\mathsf{district\_sports\_center})$.

(F″) $house \sqsubseteq \mathsf{A}^{\leq 8}\mathsf{E}^{\leq 2}\mathsf{public\_transport}$.

(G″) $house \sqsubseteq \mathsf{E}^{>0.5}_{\leq 1}\mathsf{tube\_station}$.

The intended meaning of the set term constructors above is as follows. The set $\mathsf{E}^{\leq 1}\mathsf{shop}$ contains all points in the domain from which at least one shop is reachable within 1 mile. Likewise, for every point x in $\mathsf{A}^{\leq 2}\mathsf{green\_zone}$, the whole circle of radius 2 around x belongs to the green zone, whereas $\mathsf{E}^{>0.5}_{\leq 1}\mathsf{tube\_station}$ denotes the set of all points located in a distance between 0.5 and 1 mile (excluding 0.5) from at least one tube station.[1]

By replacing quantifiers with distance operators, we do not loose expressive power as compared with $\mathcal{FM}^2[M]$. In fact, we show that $\mathcal{MS}[M]$ is *expressively complete* for $\mathcal{FM}^2[M]$ in the class $\mathcal{M}$ of all metric spaces, for any $M \subseteq \mathbb{R}^+$. This theorem (the proof of which is similar to proofs in [Etessami et al. 1997] and [Lutz et al. 2001]) has two interesting consequences. First, any (decidable) fragment of $\mathcal{FM}^2[M]$ can be obtained as a (decidable) fragment of $\mathcal{MS}[M]$. And second, since the translation from $\mathcal{FM}^2[M]$ into $\mathcal{MS}[M]$ is effective, decidable fragments of $\mathcal{MS}[M]$ have to be proper, in particular, $\mathcal{MS}[\mathbb{Q}^+]$ itself is undecidable when interpreted in distance spaces satisfying the triangular inequality.

[1] By the way, the end of the imaginary story about buying a house in London was not satisfactory. Having checked her knowledge base, the estate agent said: "Unfortunately, your constraints (A)–(G) are not satisfiable in London, where we have

$$\mathsf{tube\_station} \sqsubseteq \mathsf{E}^{\leq 3.5}(\mathsf{factory} \sqcup \mathsf{motorway}).$$

In view of the triangular inequality, this contradicts constraints (D) and (G)."

We prove two results concerning fragments of $\mathcal{MS}[M]$. The first one identifies a rather expressive and natural fragment $\mathcal{MS}^{\#}[M]$, which has the finite model property (even for parameters from $\mathbb{R}^+$) and is decidable (if parameters are taken from $\mathbb{Q}^+$). All the constraints in Example 1.1 save (G) can be formulated in $\mathcal{MS}^{\#}[\mathbb{Q}^+]$. The second result shows that seemingly weak fragments of $\mathcal{MS}[\mathbb{N}]$ are already undecidable. Roughly speaking, we loose decidability as soon as we are able to speak about 'rings,' as in constraint (G).

Table 1 summarizes the main decidability results of this paper: $+$ ($-$) means that the satisfiability problem for the corresponding language in the corresponding class of structures is decidable (undecidable). The results do not depend on whether the parameters are from $\mathbb{N}$ or $\mathbb{Q}^+$. For various fragments we will also obtain **NExpTime** upper bounds for the computational complexity. The fragments $\mathcal{MS}_i[\mathbb{Q}^+/\mathbb{N}]$ are defined in Section 3.

	$\mathcal{D}$	$\mathcal{D}_{sym}$	$\mathcal{D}_{tr}$	$\mathcal{M}$
$\mathcal{FM}[\mathbb{Q}^+/\mathbb{N}]$	$-$	$-$	$-$	$-$
$\mathcal{FM}^2[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$-$	$-$
$\mathcal{MS}[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$-$	$-$
$\mathcal{MS}_i[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$-$	$-$
$\mathcal{MS}^{\#}[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$+$	$+$

Table I. The satisfiability problem for metric logics.

The structure of the paper is as follows: Section 2 introduces the syntax and semantics of both the first-order and the 'modal' languages of metric spaces. Here, we also establish the expressive completeness result for $\mathcal{FM}^2[M]$. In Section 3, we prove the undecidability of $\mathcal{MS}_i[\mathbb{Q}^+/\mathbb{N}]$ by means of a reduction to the undecidable $\mathbb{N}\times\mathbb{N}$-tiling problem. Finally, in Sections 4 and 5, we prove our decidability results for metric and weaker distance spaces.

The idea of constructing logical formalisms capable of speaking about distances is not new. For example, somewhat weaker spatial 'modal logics of distance' were introduced in [Rescher and Garson 1968; von Wright 1979; Segerberg 1980; Jansana 1994; Lemon and Pratt 1998]. However, their computational behavior has remained unexplored (see Section 6 for some interesting open problems). More attention has recently been devoted to metric (or quantitative) temporal logics (see e.g. [Alur and Henzinger 1992; Montanari 1996; Henzinger 1998; Hirshfeld and Rabinovich 1999]), which clearly reflects the fact that temporal logic in general is more developed than spatial logic. For example, starting with Kamp's [Kamp 1968] classical result on the expressive completeness of temporal logic with respect to monadic first-order logic, a beautiful theory comparing the expressive power of first-order, second-order and temporal languages for trees and linear orderings has been developed [Gabbay et al. 1994; Rabinovich 2000]. Nothing like this has been done for spatial logics. We hope this paper, which has grown up from [Suzuki 1997; Sturm et al. 2000], will help to fill the gap.

2. THE LOGICS

2.1 First-order metric logic $\mathcal{FM}[M]$

Suppose that $M \subseteq \mathbb{R}^+$ contains 0; we will call such sets of reals *parameter sets*. The language $\mathcal{FM}[M]$ (of first-order metric logic) contains a countably infinite set $c_1, c_2, \ldots$ of constant symbols, a countably infinite set $x_1, x_2, \ldots$ of individual variables, a countably infinite set $P_1, P_2, \ldots$ of unary predicate symbols, the equality symbol $\doteq$, two (possibly infinite) sets of binary predicates

$$\boldsymbol{\delta}(\_,\_) < a \quad \text{and} \quad \boldsymbol{\delta}(\_,\_) = a \quad (a \in M),$$

the Booleans (including the propositional constants $\top$ for *verum* and $\bot$ for *falsum*), and the quantifier $\exists x_i$ for every variable x_i. Thus, the atomic formulas of $\mathcal{FM}[M]$ are of the form

$$\top, \quad \bot, \quad \boldsymbol{\delta}(t,t') < a, \quad \boldsymbol{\delta}(t,t') = a, \quad t \doteq t,' \quad \text{and} \quad P_i(t),$$

where t and t' are *terms*, i.e., variables or constants, and $a \in M$. Compound $\mathcal{FM}[M]$-formulas are obtained from atomic ones by applying the Booleans and quantifiers in the usual way:

$$\varphi ::= \text{atom} \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \exists x_i \varphi.$$

We use $\boldsymbol{\delta}(t_1,t_2) > a$ as an abbreviation for $\neg(\boldsymbol{\delta}(t_1,t_2) < a) \wedge \neg(\boldsymbol{\delta}(t_1,t_2) = a)$. If $M = \mathbb{Q}^+$, we usually write $\mathcal{FM}$ instead of $\mathcal{FM}[\mathbb{Q}^+]$. The same applies to the languages $\mathcal{MS}[M]$ introduced below. $\mathcal{FM}^2[M]$ denotes the two-variable fragment of $\mathcal{FM}[M]$, that is, the set of all $\mathcal{FM}[M]$-formulas containing occurrences of at most two variables, say, x and y.

$\mathcal{FM}[M]$-formulas are interpreted in structures of the form

$$\mathfrak{A} = \left\langle W, d, P_1^{\mathfrak{A}}, \ldots, c_1^{\mathfrak{A}}, \ldots \right\rangle,$$

where $\langle W, d\rangle$ is a distance space, the $P_i^{\mathfrak{A}}$ are subsets of W interpreting the unary predicates P_i, and the $c_i^{\mathfrak{A}}$ are elements of W interpreting the constants c_i. An *assignment* $\mathfrak{a}$ in $\mathfrak{A}$ is a function assigning elements of W to variables. The pair $\mathfrak{M} = \langle \mathfrak{A}, \mathfrak{a}\rangle$ will be called an $\mathcal{FM}[M]$-*model*. For a term t, let $t^{\mathfrak{M}}$ denote $c_i^{\mathfrak{A}}$ if t is the constant c_i, and $\mathfrak{a}(x)$ if t is the variable x. Now, the *truth-relation* $\mathfrak{M} \models \varphi$, for an $\mathcal{FM}[M]$-formula φ, is defined inductively as follows:

—$\mathfrak{M} \models \top$ and $\mathfrak{M} \not\models \bot$;
—$\mathfrak{M} \models \boldsymbol{\delta}(t_1,t_2) < a$ iff $d(t_1^{\mathfrak{M}}, t_2^{\mathfrak{M}}) < a$;
—$\mathfrak{M} \models \boldsymbol{\delta}(t_1,t_2) = a$ iff $d(t_1^{\mathfrak{M}}, t_2^{\mathfrak{M}}) = a$;
—$\mathfrak{M} \models t_1 \doteq t_2$ iff $t_1^{\mathfrak{M}} = t_2^{\mathfrak{M}}$;
—$\mathfrak{M} \models P_i(t)$ iff $t^{\mathfrak{M}} \in P_i^{\mathfrak{A}}$;
—$\mathfrak{M} \models \exists x_i\, \varphi$ iff $\langle \mathfrak{A}, \mathfrak{b}\rangle \models \varphi$ for some assignment $\mathfrak{b}$ in $\mathfrak{A}$ that may differ from $\mathfrak{a}$ only on x_i;
—$\mathfrak{M} \models \neg\varphi$ iff $\mathfrak{M} \not\models \varphi$;
—$\mathfrak{M} \models \varphi \wedge \psi$ iff $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \models \psi$.

Unfortunately, from the computational point of view, the constructed logic turns out to be too expressive. We have the following undecidability result, where $\mathcal{M}$ is the class of all metric spaces and $\mathcal{D}_{tr}$ the class of all distance spaces satisfying the triangular inequality, (2). Recall that the notation $\mathcal{FM}[\mathbb{Q}^+/\mathbb{N}]$ means that M can be either of $\mathbb{Q}^+$ and $\mathbb{N}$.

THEOREM 2.1. (i) *Let $\mathcal{K}$ be any class of distance spaces containing $\mathcal{M}$. Then the satisfiability problem for $\mathcal{FM}[\mathbb{Q}^+/\mathbb{N}]$-formulas in (models based on spaces from) $\mathcal{K}$ is undecidable.*

(ii) *The satisfiability problem for $\mathcal{FM}^2[\mathbb{Q}^+/\mathbb{N}]$-formulas in any class $\mathcal{C}$ of distance spaces such that $\mathcal{C} \subseteq \mathcal{D}_{tr}$ and $\langle \mathbb{R}^2, d_2 \rangle \in \mathcal{C}$ is undecidable as well.*

PROOF. To prove the former claim, it suffices to observe that $\mathcal{FM}[\mathbb{N}]$ is powerful enough to interpret the theory of graphs (i.e., the theory of structures $\langle W, R\rangle$, where R is a symmetric and reflexive binary relation on W), which is known to be hereditarily undecidable[2] [Rabin 1965]. Indeed, let $\varphi(x, y)$ be the formula

$$\boldsymbol{\delta}(x, y) = 1 \vee \boldsymbol{\delta}(x, y) = 0.$$

Given a graph $\langle W, R\rangle$, we can define a metric space $\langle W, d\rangle$ by taking, for all $a, b \in W$,

$$d(a, b) = \begin{cases} 0, & \text{if } a = b, \\ 1, & \text{if } a \neq b \text{ and } aRb, \\ 2, & \text{if not } aRb. \end{cases}$$

We then clearly have $\langle W, d\rangle \models \varphi[a, b]$ iff aRb. For a formula γ in the signature of graph theory, denote by $\gamma^\bullet$ the result of replacing every occurrence of an atom $R(x, y)$ in γ by $\varphi(x, y)$. Obviously, $\gamma^\bullet$ is an $\mathcal{FM}[\mathbb{N}]$-formula and, for every graph $\langle W, R\rangle$, the formula γ is satisfiable in $\langle W, R\rangle$ iff $\gamma^\bullet$ is satisfiable in $\langle W, d\rangle$. Now consider the set Γ of formulas γ in the signature of graph theory such that $\gamma^\bullet$ is true in all $\mathcal{FM}[\mathbb{N}]$-models based on distance spaces in $\mathcal{K}$. By the result of [Rabin 1965] mentioned above, the theory Γ is undecidable, which yields (i).

(ii) follows from Theorems 2.2 (i) and 3.1 to be proved below. □

2.2 'Modal' metric logic $\mathcal{MS}[M]$

As an alternative to the first-order language $\mathcal{FM}[M]$, where M is a parameter set, we now introduce a purely propositional language $\mathcal{MS}[M]$, whose 'distance operators' are similar to various operators considered in modal logic. The alphabet of $\mathcal{MS}[M]$ contains the following symbols:

—an infinite list of *set* (or *region*) *variables* $X_1, X_2, \dots$;

—an infinite list of *location constants* $c_1, c_2, \dots$;

—*atoms* (propositional constants) $\boldsymbol{\delta}(c, d) = a$ and $\boldsymbol{\delta}(c, d) < a$ for every $a \in M$ and location constants c, d;

—a *set constant* $\{c_i\}$ for every location constant c_i;

—the *set constants* $\top$ and $\bot$;

—the Boolean operators for set terms ($\sqcap$ and $\neg$) and formulas ($\wedge$ and $\neg$);

—the equality symbol $\doteq$ for set terms as well as the symbol $\sqsubseteq$ for elementship;

[2]This means that every subtheory of graph theory is undecidable.

—the set term constructors $\mathsf{E}^{<a}$, $\mathsf{E}^{>a}$, $\mathsf{E}^{=a}$ and $\mathsf{E}^{>a}_{<b}$ (and their duals $\mathsf{A}^{<a}$, $\mathsf{A}^{>a}$, $\mathsf{A}^{=a}$, $\mathsf{A}^{>a}_{<b}$), where $a, b \in M$ and $a < b$.

Set terms s of $\mathcal{MS}[M]$ are defined as follows

$$s ::= X_i \mid \{c_i\} \mid \top \mid \bot \mid \neg s \mid s_1 \sqcap s_2 \mid \mathsf{E}^{<a} s \mid \mathsf{E}^{>a} s \mid \mathsf{E}^{=a} s \mid \mathsf{E}^{>a}_{<b} s.$$

Set variables and set constants are called *atomic set terms*. *Atomic formulas* of $\mathcal{MS}[M]$ are of the form:

—$c \mathrel{\underline{\sqsubset}} s$, where c is a location constant and s a set term,
—$s \doteq t$, where s and t are set terms,
—$\boldsymbol{\delta}(c_1, c_2) = a$ and $\boldsymbol{\delta}(c_1, c_2) < a$, where c_1, c_2 are location constants and $a \in M$.

Finally, an $\mathcal{MS}[M]$-*formula* φ is simply a Boolean combination of atomic ones, i.e.,

$$\varphi ::= c \mathrel{\underline{\sqsubset}} s \mid s \doteq t \mid \boldsymbol{\delta}(c_1, c_2) = a \mid \boldsymbol{\delta}(c_1, c_2) < a \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2.$$

As we have already mentioned, the language $\mathcal{MS}[M]$ contains a number of constructors known from modal and description logic. First, we have an analogue of the difference operator [de Rijke 1990]: $\mathsf{A}^{>0} t$ (i.e., $\neg \mathsf{E}^{>0} \neg t$) says that t holds 'everywhere but here'. The universal modalities of [Goranko and Passy 1992], denoted here by $\Box$ ('everywhere') and $\Diamond$ ('somewhere'), can be defined by putting

$$\Box t := t \sqcap \mathsf{A}^{>0} t \qquad \text{and} \qquad \Diamond t := t \sqcup \mathsf{E}^{>0} t,$$

where $\sqcup$ is the dual of $\sqcap$ (i.e., $s \sqcup t = \neg(\neg s \sqcap \neg t)$). Furthermore, the set constants $\{c\}$ play the role of nominals [Blackburn 1993]. Using these we can state, for example, that

$$(\mathsf{E}^{\leq 1100}\{\mathit{Leipzig}\} \sqcap \mathsf{E}^{\leq 1100}\{\mathit{Malaga}\}) \sqsubseteq \mathit{France},$$

i.e., 'if you are not more than 1100 km away from Leipzig and not more than 1100 km away from Malaga, then you are in France.' Here, $s \sqsubseteq t$ stands for $s \sqcap t \doteq s$.

An $\mathcal{MS}[M]$-*model* is a structure of the form

$$\mathfrak{B} = \left\langle W, d, X_1^{\mathfrak{B}}, X_2^{\mathfrak{B}}, \ldots, c_1^{\mathfrak{B}}, c_2^{\mathfrak{B}} \ldots \right\rangle,$$

where $\langle W, d\rangle$ is a distance space, the $X_i^{\mathfrak{B}}$ are subsets of W, and the $c_i^{\mathfrak{B}}$ are elements of W. Thus, $\mathfrak{B}$ defines explicitly the values of set variables and location constants. The *value* $s^{\mathfrak{B}}$ of an arbitrary $\mathcal{MS}[M]$-term in $\mathfrak{B}$ is computed inductively as follows:

—$\{c_i\}^{\mathfrak{B}} = \{c_i^{\mathfrak{B}}\}$, where $\{c_i\}$ is a set constant;
—$(\top)^{\mathfrak{B}} = W$ and $(\bot)^{\mathfrak{B}} = \emptyset$;
—$(s_1 \sqcap s_2)^{\mathfrak{B}} = s_1^{\mathfrak{B}} \cap s_2^{\mathfrak{B}}$, where s_1 and s_2 are set terms;
—$(\neg s)^{\mathfrak{B}} = W - s^{\mathfrak{B}}$;
—$(\mathsf{E}^{=a} s)^{\mathfrak{B}} = \{x \in W : \exists y \in W \left(d(x,y) = a \;\&\; y \in s^{\mathfrak{B}}\right)\}$;
—$(\mathsf{E}^{<a} s)^{\mathfrak{B}} = \{x \in W : \exists y \in W \left(d(x,y) < a \;\&\; y \in s^{\mathfrak{B}}\right)\}$;
—$(\mathsf{E}^{>a} s)^{\mathfrak{B}} = \{x \in W : \exists y \in W \left(d(x,y) > a \;\&\; y \in s^{\mathfrak{B}}\right)\}$;
—$(\mathsf{E}^{>a}_{<b} s)^{\mathfrak{B}} = \{x \in W : \exists y \in W \left(a < d(x,y) < b \;\&\; y \in s^{\mathfrak{B}}\right)\}$.

The truth-relation $\mathfrak{B} \models \varphi$, φ an $\mathcal{MS}[M]$-formula, is defined in the expected way:

—$\mathfrak{B} \models c \sqsubseteq s$ iff $c^{\mathfrak{B}} \in s^{\mathfrak{B}}$,

—$\mathfrak{B} \models s_1 \doteq s_2$ iff $s_1^{\mathfrak{B}} = s_2^{\mathfrak{B}}$,

—$\mathfrak{B} \models \boldsymbol{\delta}(c_1, c_2) = a$ iff $d(c_1^{\mathfrak{B}}, c_2^{\mathfrak{B}}) = a$,

—$\mathfrak{B} \models \boldsymbol{\delta}(c_1, c_2) < a$ iff $d(c_1^{\mathfrak{B}}, c_2^{\mathfrak{B}}) < a$,

plus the standard definitions for the Boolean connectives.

In what follows we will be using abbreviations like $\mathsf{E}^{\leq a}s$, $\mathsf{E}^{\geq a}$, $\mathsf{E}^{\geq a}_{\leq b}s$, $\mathsf{E}^{\geq a}_{<b}$ and $\mathsf{E}^{>a}_{\leq b}$, the meaning of which should be clear. For instance, $\mathsf{E}^{\geq a}_{\leq b}s$ stands for

$$\mathsf{E}^{>a}_{<b}s \sqcup \mathsf{E}^{=a}s \sqcup \mathsf{E}^{=b}s.$$

Every $\mathcal{FM}[M]$-structure $\mathfrak{A} = \left\langle W, d, P_1^{\mathfrak{A}}, \dots, c_1^{\mathfrak{A}}, \dots \right\rangle$ gives rise to its $\mathcal{MS}[M]$ counterpart

$$\mathfrak{A}_* = \left\langle W, d, X_1^{\mathfrak{A}_*}, \dots, c_i^{\mathfrak{A}_*}, \dots \right\rangle,$$

where $X_i^{\mathfrak{A}_*} = P_i^{\mathfrak{A}}$ and $c_i^{\mathfrak{A}_*} = c_i^{\mathfrak{A}}$ for all i. This correspondence is clearly bijective. If an $\mathcal{FM}[M]$-structure (or an $\mathcal{MS}[M]$-model) is based on a metric space, we call it a *metric* $\mathcal{FM}[M]$-*structure* ($\mathcal{MS}[M]$-*model*).

The theorem we are about to prove shows that, when speaking about metric spaces, $\mathcal{MS}[M]$ is expressively complete for (i.e., has the same expressive power as) the two-variable fragment of $\mathcal{FM}[M]$.

THEOREM 2.2. (i) *For every $\mathcal{MS}[M]$-formula φ there exists an $\mathcal{FM}^2[M]$-sentence $\varphi^\dagger$ such that its length is linear in the length of φ and, for any $\mathcal{FM}[M]$-structure $\mathfrak{A}$, we have*

$$\mathfrak{A} \models \varphi^\dagger \quad \textit{iff} \quad \mathfrak{A}_* \models \varphi.$$

(ii) *For every $\mathcal{FM}^2[M]$-sentence φ there is an $\mathcal{MS}[M]$-formula $\varphi^\ddagger$ such that its length is exponential in the length of φ and, for any metric $\mathcal{FM}$-structure $\mathfrak{A}$, we have*

$$\mathfrak{A} \models \varphi \quad \textit{iff} \quad \mathfrak{A}_* \models \varphi^\ddagger.$$

PROOF. Assume, for simplicity, that $M = \mathbb{Q}^+$.

(i) The proof of the first claim is pretty standard; cf. [Gabbay 1971]. We first translate set terms occurring in φ into $\mathcal{FM}^2$-formulas with at most one free variable and then extend the translation to subformulas of φ using only two variables, x and y.

Let z and z' be metavariables ranging over $\{x, y\}$. The translation $\cdot^\dagger$ of set terms is defined inductively as follows:

$(X_i)^\dagger = P_i(x)$;
$(\top)^\dagger = \top$ and $(\bot)^\dagger = \bot$;
$(\{c_i\})^\dagger = (c_i \doteq x)$;
$(s_1 \sqcap s_2)^\dagger = s_1^\dagger[x/z] \wedge s_2^\dagger[x/z'])$, where z, z' are free in $s_1^\dagger$ and $s_2^\dagger$, respectively;
$(\neg s)^\dagger = \neg s^\dagger$;
$(\mathsf{E}^{<a} s)^\dagger = \exists z\, (\boldsymbol{\delta}(z', z) < a \wedge s^\dagger(z))$, where $z \neq z'$ is free in $s^\dagger$;
$(\mathsf{E}^{>a} s)^\dagger = \exists z\, (\boldsymbol{\delta}(z', z) > a \wedge s^\dagger(z))$, where $z \neq z'$ is free in $s^\dagger$;
$(\mathsf{E}^{=a} s)^\dagger = \exists z\, (\boldsymbol{\delta}(z', z) = a \wedge s^\dagger(z))$, where $z \neq z'$ is free in $s^\dagger$;
$(\mathsf{E}^{>a}_{<b} s)^\dagger = \exists z\, (a < \boldsymbol{\delta}(z', z) < b \wedge s^\dagger(z))$, where $z \neq z'$ is free in $s^\dagger$.

Now, turning to $\mathcal{MS}$-formulas, we define $\cdot^\dagger$ as commuting with the Booleans and by taking

$(c \sqsubseteq s)^\dagger = s^\dagger[c/z]$, where z is free in $s^\dagger$;
$(s_1 \doteq s_2)^\dagger = \forall x\, (s_1^\dagger[x/z_1] \leftrightarrow s_2^\dagger[x/z_2])$, z_1, z_2 free in $s_1^\dagger, s_2^\dagger$, respectively;
$(\boldsymbol{\delta}(c_1, c_2) = a)^\dagger = (\boldsymbol{\delta}(c_1, c_2) = a)$;
$(\boldsymbol{\delta}(c_1, c_2) < a)^\dagger = (\boldsymbol{\delta}(c_1, c_2) < a)$.

By a straightforward induction one can easily check that $\mathfrak{A} \models \varphi^\dagger$ iff $\mathfrak{A}_* \models \varphi$.

(ii) To define the converse translation, we first observe that the following transformations of an $\mathcal{FM}^2$-formula $\varphi(x, y)$ result in an equivalent formula with respect to metric spaces: every occurrence of equality $t_1 \doteq t_2$ can be replaced by $\boldsymbol{\delta}(t_1, t_2) = 0$; $\boldsymbol{\delta}(t, t) = 0$ by $\top$; $\boldsymbol{\delta}(t, t) < 0$ by $\bot$; $\boldsymbol{\delta}(t, t) = a$ by $\bot$ if $a > 0$; $\boldsymbol{\delta}(t, t) < a$ by $\top$ if $a > 0$; $\boldsymbol{\delta}(y, t) = a$ by $\boldsymbol{\delta}(t, y) = a$; $\boldsymbol{\delta}(y, t) < a$ by $\boldsymbol{\delta}(t, y) < a$; $\boldsymbol{\delta}(t, x) = a$ by $\boldsymbol{\delta}(x, t) = a$, and $\boldsymbol{\delta}(t, x) < a$ by $\boldsymbol{\delta}(x, t) < a$. In what follows, we assume that these transformations have been applied to all our formulas, in particular, to φ.

We distinguish between three types of atomic formulas in $\mathcal{FM}^2$: *binary atoms* are of the form $\boldsymbol{\delta}(x, y) < a$ or $\boldsymbol{\delta}(x, y) = a$ (they have two free variables); *unary atoms* are of the form $\boldsymbol{\delta}(x, c) < a$, $\boldsymbol{\delta}(x, c) = a$, $P_i(x)$, $P_i(y)$, $\boldsymbol{\delta}(c, y) < a$ or $\boldsymbol{\delta}(c, y) = a$ (having only one free variable); atoms without free variables can be called *nullary*.

Given an $\mathcal{FM}^2$-sentence φ, we first translate it into a set term $\varphi^\star$ by inductively defining a map $\cdot^\star$ from subformulas of φ with ≤ 1 free variable into $\mathcal{MS}$-set terms (using the 'universal modalities' $\Box$ and $\Diamond$ defined on page 8):

(1) If $\psi = \top$ then $\psi^\star = \top$, and if $\psi = \bot$ then $\psi^\star = \bot$.
(2) If $\psi \in \{P_i(x), P_i(y)\}$ then $\psi^\star = X_i$.
(3) If $\psi = P_i(c)$ then $\psi^\star = \Box(\{c\} \to X_i)$.
(4) If $\psi \in \{\boldsymbol{\delta}(x, c) = a,\ \boldsymbol{\delta}(c, y) = a\}$ then $\psi^\star = \mathsf{E}^{=a}\{c\}$.
(5) If ψ is $\boldsymbol{\delta}(c_1, c_2) = a$ then $\psi^\star = \Box(\{c_1\} \to \mathsf{E}^{=a}\{c_2\})$.
(6) If ψ is $\boldsymbol{\delta}(c_1, c_2) < a$ then $\psi^\star = \Box(\{c_1\} \to \mathsf{E}^{<a}\{c_2\})$.
(7) If $\psi \in \{\boldsymbol{\delta}(x, c) < a,\ \boldsymbol{\delta}(c, y) < a\}$ then $\psi^\star = \mathsf{E}^{<a}\{c\}$.

(8) If $\psi = \chi_1 \wedge \chi_2$ then $\psi^\star = \chi_1^\star \sqcap \chi_2^\star$.

(9) If $\psi = \neg\chi$ then $\psi^\star = \neg(\chi^\star)$.

The remaining cases of $\psi = \exists y \chi(x, y)$ and $\psi = \exists x \chi(x, y)$ are more sophisticated. We consider only the former. The formula $\chi(x, y)$ can be regarded as a Boolean combination of binary atoms β_i and formulas $\upsilon_i(x)$ and $\xi_i(y)$ with at most one free variable. Denote this Boolean combination by κ, i.e.,

$$\chi(x, y) = \kappa(\beta_1, \dots, \beta_r, \upsilon_1(x), \dots, \upsilon_l(x), \xi_1(y), \dots, \xi_s(y)).$$

Let us first move all components in κ without free y out of the scope of the outmost $\exists y$ in ψ. Then ψ can be equivalently rewritten as

$$\bigvee_{\langle \nu_1, \dots, \nu_l \rangle \in \{\top, \bot\}^l} \Big(\exists y\, \kappa(\beta_1, \dots, \beta_r, \nu_1, \dots, \nu_l, \xi_1, \dots, \xi_s) \wedge \bigwedge_{1 \le i \le l} (\upsilon_i \leftrightarrow \nu_i) \Big).$$

Now let $0 = a_0 < a_1 < \dots < a_n$ be the list of all rational numbers occurring in ψ together with 0. So this list is non-empty. Consider the set $\mathcal{R}_\psi$ containing the following formulas:

—$\boldsymbol{\delta}(x, y) = a_i$, for $i \le n$;

—$a_i < \boldsymbol{\delta}(x, y) < a_{i+1}$, for $i < n$;

—$\boldsymbol{\delta}(x, y) > a_n$.

For every $\beta \in \mathcal{R}_\psi$ and every binary atom β_i in ψ, we have either $\beta \models \beta_i$ or $\beta \models \neg\beta_i$. In other words, by assigning a truth-value to some β in $\mathcal{R}_\psi$, we fix the truth values of all binary atoms in ψ. Let $\beta_i^\beta = \top$ if $\beta \models \beta_i$, and $\beta_i^\beta = \bot$ otherwise. Then ψ is equivalent to the formula

$$\bigvee_{\langle \nu_1, \dots, \nu_l \rangle \in \{\top, \bot\}^l} \Big(\bigvee_{\beta \in \mathcal{R}_\psi} \exists y \left(\beta \wedge \kappa(\beta_1^\beta, \dots, \beta_r^\beta, \nu_1, \dots, \nu_l, \xi_1, \dots, \xi_s)\right) \wedge \bigwedge_{1 \le i \le l} (\upsilon_i \leftrightarrow \nu_i) \Big).$$

Next, we replace each $\beta \in \mathcal{R}_\psi$ with the distance operator β^* defined by taking

—$(\boldsymbol{\delta}(x, y) = a_i)^* = \mathsf{E}^{=a_i}$, for $i \le n$;

—$(a_i < \boldsymbol{\delta}(x, y) < a_{i+1})^* = \mathsf{E}^{>a_i}_{<a_{i+1}}$, for $i < n$;

—$(\boldsymbol{\delta}(x, y) > a_n)^* = \mathsf{E}^{>a_n}$,

delete the quantifier $\exists y$ and recursively compute the values of $\upsilon_i^\star$ and $\xi_i^\star$. This yields

$$\psi^\star = \bigsqcup_{\langle \nu_1, \dots, \nu_l \rangle \in \{\top, \bot\}^l} \Big(\bigsqcup_{\beta \in \mathcal{R}_\psi} \beta^* \left(\kappa(\beta_1^\beta, \dots, \beta_r^\beta, \nu_1, \dots, \nu_l, \xi_1^\star, \dots, \xi_s^\star)\right) \sqcap$$

$$\sqcap \bigsqcap_{1 \le i \le l} (\upsilon_i^\star \leftrightarrow \nu_i) \Big).$$

Finally, we put $\varphi^\ddagger = (\varphi^\star \doteq \top)$. It should be clear from the construction that

$$\mathfrak{A} \models \varphi \quad \text{iff} \quad \mathfrak{A}_* \models \varphi^\ddagger.$$

The reader can restore details of the proof using the example below. □

EXAMPLE 2.3. Consider the $\mathcal{FM}^2$-sentence

$$\varphi = \exists y\Big(\exists x\big(\boldsymbol{\delta}(x,y) > 0 \land P_i(x)\big) \land \neg P_i(y)\Big).$$

Let $\xi_1(y) = \exists x\big(\boldsymbol{\delta}(x,y) > 0 \land P_i(x)\big)$ and $\xi_2(y) = \neg P_i(y)$. Then we represent φ as

$$\exists y\big(\xi_1(y) \land \xi_2(y)\big)$$

which is equivalent to

$$\exists y\big(\boldsymbol{\delta}(x,y) = 0 \land \xi_1(y) \land \xi_2(y)\big) \lor \exists y\big(\boldsymbol{\delta}(x,y) > 0 \land \xi_1(y) \land \xi_2(y)\big).$$

Thus, we obtain the $\mathcal{MS}$-set term

$$\varphi^\star = \mathsf{E}^{=0}(\xi_1^\star \sqcap \xi_2^\star) \sqcup \mathsf{E}^{>0}(\xi_1^\star \sqcap \xi_2^\star),$$

where $\xi_1^\star = \mathsf{E}^{=0}(\bot \sqcap X_i) \sqcup \mathsf{E}^{>0}(\top \sqcap X_i)$ or, equivalently, $\xi_1^\star = \mathsf{E}^{>0} X_i$, and $\xi_2^\star = \neg X_i$. So the resulting translation is

$$\varphi^\star = \mathsf{E}^{=0}(\mathsf{E}^{>0} X_i \sqcap \neg X_i) \sqcup \mathsf{E}^{>0}(\mathsf{E}^{>0} X_i \sqcap \neg X_i).$$

Using the universal $\Diamond$, we finally obtain

$$\varphi^\ddagger = \big(\Diamond(\mathsf{E}^{>0} X_i \sqcap \neg X_i) \doteq \top\big).$$

The reader can easily check that φ and $\varphi^\ddagger$ indeed say the same.

3. UNDECIDABILITY

In this section we show that the satisfiability problem for fragments of $\mathcal{MS}[\mathbb{Q}^+/\mathbb{N}]$ containing distance operators like $\mathsf{E}^{\geq 0}_{\leq a}$ is undecidable in natural classes of spaces satisfying the triangular inequality. Consider the following languages:

—$\mathcal{MS}_1[\mathbb{Q}^+/\mathbb{N}]$ is the fragment of $\mathcal{MS}[\mathbb{Q}^+/\mathbb{N}]$ whose set terms are constructed from set variables, the operators $\sqcap$, $\neg$, and $\mathsf{E}^{\geq 0}_{\leq b}$ for $b \in \mathbb{Q}^+/\mathbb{N}$, and whose formulas are Boolean combinations of atoms of the form $s \sqsubseteq t$.

—$\mathcal{MS}_2[\mathbb{Q}^+/\mathbb{N}]$ results from $\mathcal{MS}_1[\mathbb{Q}^+/\mathbb{N}]$ by replacing $\mathsf{E}^{\geq 0}_{\leq b}$ with $\mathsf{E}^{\geq 0}_{< b}$.

—$\mathcal{MS}_3[\mathbb{Q}^+/\mathbb{N}]$ results from $\mathcal{MS}_1[\mathbb{Q}^+/\mathbb{N}]$ by replacing $\mathsf{E}^{\geq 0}_{\leq b}$ with $\mathsf{E}^{\geq 1}_{\leq b}$.

—$\mathcal{MS}_4[\mathbb{N}]$ results from $\mathcal{MS}_1[\mathbb{Q}^+/\mathbb{N}]$ by replacing $\mathsf{E}^{\geq 0}_{\leq b}$ with $\mathsf{E}^{\geq 1}_{< b}$.

THEOREM 3.1. *Let $\mathcal{K} \subseteq \mathcal{D}_{tr}$ contain $\langle \mathbb{R}^2, d_2\rangle$. Then the satisfiability problem for $\mathcal{MS}_i[\mathbb{Q}^+/\mathbb{N}]$-formulas in (models based on spaces from) $\mathcal{K}$ is undecidable for any $1 \leq i \leq 4$.*

PROOF. We consider only $\mathcal{MS}_1[\mathbb{N}]$; the other languages are treated analogously. The proof is by reduction to the undecidable $\mathbb{N} \times \mathbb{N}$ tiling problem (see [van Emde Boas 1997; Börger et al. 1997] and references therein). We remind the reader that the tiling problem for $\mathbb{N} \times \mathbb{N}$ is formulated as follows: given a finite set $\mathcal{T} = \{T_1, \ldots, T_l\}$ of tile types (i.e., squares T_i with colors $left(T_i)$, $right(T_i)$, $up(T_i)$, and $down(T_i)$ on their edges), decide whether the grid $\mathbb{N} \times \mathbb{N}$ can be covered with tiles, each of a type from $\mathcal{T}$, in such a way that the colors of adjacent edges on adjacent tiles match, or, more precisely, whether there exists a function $\tau : \mathbb{N} \times \mathbb{N} \to \mathfrak{T}$ such that, for all $n, m \in \mathbb{N}$, we have

$$right(\tau(n,m)) = left(\tau(n+1,m)),$$
$$up(\tau(n,m)) = down(\tau(n,m+1)).$$

So suppose a set of tile types $\mathfrak{T} = \{T_1, \dots, T_l\}$ is given. Our aim is to construct an $\mathcal{MS}_1[\mathbb{N}]$-formula which is satisfiable in $\mathcal{K}$ iff $\mathfrak{T}$ can tile $\mathbb{N} \times \mathbb{N}$.

Note that $\mathsf{E}^{\leq b}t$ is definable in $\mathcal{MS}_1[\mathbb{N}]$ as $t \sqcup \mathsf{E}^{>0}_{\leq b}t$. Hence, $\mathsf{A}^{\leq b}$ is definable as well. Take set variables $Z_1, \dots, Z_l$, $X_0, \dots, X_4, Y_0, \dots, Y_4$. Let $\chi_{i,j} = \mathsf{A}^{\leq 9}(X_i \sqcap Y_j)$, for $i, j \leq 4$, and let Γ be the set of the following formulas, where $i, j \leq 4$ and $k \leq l$:

$$X_i \sqcap Y_j \sqsubseteq \mathsf{E}^{\leq 9}\chi_{i,j}, \quad \chi_{i,j} \sqsubseteq \mathsf{A}^{>0}_{\leq 80}\neg\chi_{i,j}, \quad \chi_{i,j} \sqsubseteq \neg\chi_{m,n} \ ((i,j) \neq (m,n)), \tag{4}$$

$$\chi_{i,j} \sqsubseteq \bigsqcup_{k \leq l} \mathsf{A}^{\leq 9}Z_k, \quad Z_m \sqsubseteq \neg Z_n \ (n \neq m), \tag{5}$$

$$\chi_{i,j} \sqcap Z_k \sqsubseteq \mathsf{E}^{\leq 20}(\mathsf{E}^{\leq 20}\chi_{i,j} \sqcap \chi_{i+_5 1,j} \sqcap \bigsqcup_{right(T_k)=left(T_m)} Z_m), \tag{6}$$

$$\chi_{i,j} \sqcap Z_k \sqsubseteq \mathsf{E}^{\leq 20}(\mathsf{E}^{\leq 20}\chi_{i,j} \sqcap \chi_{i,j+_5 1} \sqcap \bigsqcup_{up(T_k)=down(T_m)} Z_m), \tag{7}$$

where $+_5$ denotes addition modulo 5.[3] The first formula in (4) is satisfied in a model $\mathfrak{B}$ iff $X_i^{\mathfrak{B}} \cap Y_j^{\mathfrak{B}}$ is the union of a set of spheres of radius 9. The second one is satisfied in $\mathfrak{B}$ iff the distance between any two distinct centers of spheres of radius 9, all points in which belong to $X_i^{\mathfrak{B}} \cap Y_j^{\mathfrak{B}}$, is more than 80, while the third formula guarantees that the sets $\chi_{i,j}^{\mathfrak{B}}$ are pairwise disjoint. We think of $\chi_{i,j}^{\mathfrak{B}}$, for $i, j \leq 4$, as a finite family of infinite sets making up the grid for the tiling (see Fig. 1). The formulas in (5) ensure that every point of the grid is covered by some tile and that different tiles never cover the same point. Finally, formulas (6) and (7) ensure the tiling conditions in the horizontal and vertical directions, respectively.

Note that if $x \in \chi_{i,j}^{\mathfrak{B}}$ then, in view of (6), there exist $y \in \chi_{i+_5 1,j}^{\mathfrak{B}}$ and $z \in \chi_{i,j}^{\mathfrak{B}}$ for which $d(x,y) \leq 20$ and $d(y,z) \leq 20$. But then, by the triangular inequality, $d(x,z) \leq 40$, and so, by the second formula in (4), $x = z$. The situation in the vertical direction is similar.

We are going to show that the conjunction of formulas in $\{\neg(\chi_{0,0} \sqsubseteq \bot)\} \cup \Gamma$ is satisfiable in $\mathcal{K}$ iff $\mathfrak{T}$ can tile $\mathbb{N} \times \mathbb{N}$. This will be done in two steps.

Lemma 3.2. *If $\mathfrak{T}$ tiles $\mathbb{N} \times \mathbb{N}$, then $\{\neg(\chi_{0,0} \sqsubseteq \bot)\} \cup \Gamma$ is satisfiable in the 2-dimensional Euclidean space $\langle \mathbb{R}^2, d_2 \rangle$.*

Proof. Suppose $\tau : \mathbb{N} \times \mathbb{N} \to \mathfrak{T}$ is a tiling. For $r \in \mathbb{R}^2$, put

$$S(r) = \{y \in \mathbb{R}^2 : d_2(r,y) \leq 9\}.$$

[3]The first conjunct in the right hand sides of (6) and (7) is redundant if $\mathcal{K}$ consists of symmetric spaces only.

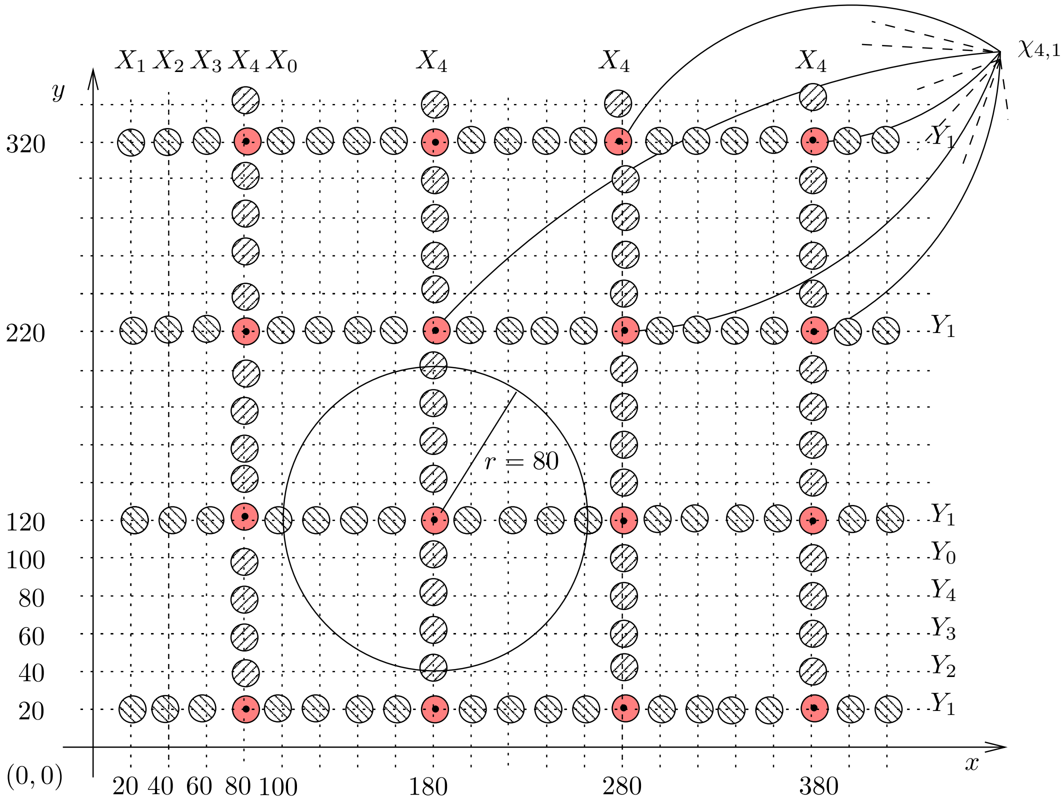

Fig. 1. Building the grid.

Define a model $\mathfrak{B}$ on $\langle \mathbb{R}^2, d_2 \rangle$ by taking, for $i, j \leq 4$ and $k \leq l$,

$$X_i^{\mathfrak{B}} = \bigcup_{m,n \in \mathbb{N}} S(20(5n + i), 20m),$$

$$Y_j^{\mathfrak{B}} = \bigcup_{m,n \in \mathbb{N}} S(20n, 20(5m + j)),$$

$$Z_k^{\mathfrak{B}} = \bigcup_{\tau(n,m) = T_k} S(20n, 20m).$$

It is not difficult to see that this model satisfies $\{\neg(\chi_{0,0} \sqsubseteq \bot)\} \cup \Gamma$; see Fig. 1. $\square$

LEMMA 3.3. *Suppose that a model $\mathfrak{B}$ based on a space $\langle W, d \rangle \in \mathcal{D}_{tr}$ satisfies the conjunction of $\{\neg(\chi_{0,0} \sqsubseteq \bot)\} \cup \Gamma$. Then there exists a function $f : \mathbb{N} \times \mathbb{N} \to W$ such that, for all $i, j \leq 4$ and $k_1, k_2 \in \mathbb{N}$,*

(a) $f(5k_1 + i, 5k_2 + j) \in \chi_{i,j}^{\mathfrak{B}}$;

(b) $d(f(k_1, k_2), f(k_1 + 1, k_2)) \leq 20$ *and* $d(f(k_1 + 1, k_2), f(k_1, k_2)) \leq 20$;

(c) $d(f(k_1, k_2), f(k_1, k_2 + 1)) \leq 20$ *and* $d(f(k_1, k_2 + 1), f(k_1, k_2)) \leq 20$.

The map $\tau : \mathbb{N} \times \mathbb{N} \to \mathfrak{T}$ defined by taking $\tau(n,m) = T_k$ iff $f(n,m) \in Z_k^{\mathfrak{B}}$, for all $k \leq l$ and all $n, m \in \mathbb{N}$, is a tiling.

PROOF. We define f inductively. Pick some $f(0,0) \in \chi_{0,0}^{\mathfrak{B}}$. By (6), we can find then a sequence $w_n \in W$, for $n \in \mathbb{N}$, such that

—$w_0 = f(0,0)$,
—$w_{5k+i} \in (\chi_{i,0})^{\mathfrak{B}}$ for all $i \leq 4$ and $k \in \mathbb{N}$,
—$d(w_n, w_{n+1}) \leq 20$ and $d(w_{n+1}, w_n) \leq 20$.

We put $f(n,0) = w_n$ for all $n \in \mathbb{N}$. Similarly, by (7), we find a sequence v_n, for $n \in \mathbb{N}$, such that

—$v_0 = f(0,0)$,
—$v_{5k+j} \in (\chi_{0,j})^{\mathfrak{B}}$ for all $j \leq 4$ and $k \in \mathbb{N}$,
—$d(v_n, v_{n+1}) \leq 20$ and $d(v_{n+1}, v_n) \leq 20$.

Put $f(0,m) = v_m$ for all $m \in \mathbb{N}$. Suppose now that f has been defined for all (m', n') with $m' + n' < m + n$ so that it satisfies conditions (a)–(c). Without loss of generality we can assume that $n = 5k_1$, $m = 5k_2 + 1$, for some $k_1, k_2 \in \mathbb{N}$. Then $f(n, m-1) \in (\chi_{0,0})^{\mathfrak{B}}$, and hence $f(n, m-1) \in (\mathsf{E}^{\leq 20}\chi_{0,1})^{\mathfrak{B}}$. So we can find a $w' \in (\chi_{0,1})^{\mathfrak{B}}$ with $d(f(n, m-1), w') \leq 20$ and $d(w', f(n, m-1)) \leq 20$. We then put $f(n,m) = w'$. It remains to prove that f still satisfies (a)–(c). To this end it suffices to show that $d(f(n-1, m), w') \leq 20$ and $d(w', f(n-1, m)) \leq 20$. We have $f(n-1, m) \in (\chi_{4,1})^{\mathfrak{B}}$, and so there exists a $w'' \in (\chi_{0,1})^{\mathfrak{B}}$ such that $d(f(n-1, m), w'') \leq 20$ and $d(w'', f(n-1, m)) \leq 20$. Thus it is enough to show that $w' = w''$. Suppose otherwise. Then we have

—$d(w'', f(n-1, m)) \leq 20$;
—$d(f(n-1), m), f(n-1, m-1)) \leq 20$;
—$d(f(n-1, m-1), f(n, m-1)) \leq 20$;
—$d(f(n, m-1), w') \leq 20$.

By the triangular inequality, it follows that $d(w'', w') \leq 80$, contrary to the second formula in (4). It is readily seen now that τ is a tiling. □

This completes the proof of Theorem 3.1.

4. DECIDABLE LOGICS OF METRIC SPACES

Consider the language $\mathcal{MS}^{\#}[M]$ whose set term constructors are the Booleans, $\mathsf{E}^{>a}$ and $\mathsf{E}^{\leq a}$, for all $a \in M$, their duals $\mathsf{A}^{>a}$ and $\mathsf{A}^{\leq a}$, as well as the nominal constructor which gives the set term $\{c\}$ for any location constant c. Thus, the $\mathcal{MS}^{\#}[M]$ set terms s are:

$$s ::= X_i \mid \{c_i\} \mid \top \mid \bot \mid \neg s \mid s_1 \sqcap s_2 \mid \mathsf{E}^{>a}s \mid \mathsf{E}^{\leq a}s \mid \mathsf{A}^{>a}s \mid \mathsf{A}^{\leq a}s.$$

The atomic formulas of $\mathcal{MS}^{\#}[M]$ are $\boldsymbol{\delta}(c,d) < a$ and $\boldsymbol{\delta}(c,d) = a$, for $a \in M$, $c \sqsubseteq s$ and $s_1 \doteq s_2$, where c, d are constants and s_1, s_2 set terms. Complex $\mathcal{MS}^{\#}[M]$-formulas are Boolean combinations of atoms:

$$\varphi ::= c \sqsubseteq s \mid s_1 \doteq s_2 \mid \boldsymbol{\delta}(c_1, c_2) = a \mid \boldsymbol{\delta}(c_1, c_2) < a \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2.$$

Note that in $\mathcal{MS}^{\#}[\mathbb{Q}^+]$ we can express all constraints from Example 1.1, save (G) (the formula $house \sqsubseteq \mathsf{E}^{>0.5}\mathsf{tube\_station} \sqcap \mathsf{E}^{\leq 1}\mathsf{tube\_station}$ is clearly not equivalent to $house \sqsubseteq \mathsf{E}^{>0.5}_{\leq 1}\mathsf{tube\_station}$). Note also that the difference operator and the universal modality are still definable in $\mathcal{MS}^{\#}[M]$.

The satisfiability problem for arbitrary $\mathcal{MS}^{\#}[M]$-formulas can be reduced to the satisfiability problem for $\mathcal{MS}^{\#}[M]$-formulas without the nominal constructor. Indeed, suppose that $c_1, \dots, c_n$ are all location constants occurring in an $\mathcal{MS}^{\#}[M]$-formula φ as set terms $\{c_i\}$. Take fresh set variables $X_1, \dots, X_n$ and let φ' be the result of replacing all the $\{c_i\}$ in φ with X_i. It is readily checked that φ is satisfiable in a model based on a distance space $\mathfrak{D}$ iff the formula

$$\varphi' \wedge \bigwedge_{i \leq n} \Diamond (X_i \wedge \mathsf{A}^{>0} \neg X_i) \doteq \top$$

is satisfiable in $\mathfrak{D}$.[4] Since our main concern is the finite model property and decidability, we will, for purely technical reasons, from now on assume that no nominals $\{c\}$ occur in $\mathcal{MS}^{\#}[M]$-formulas.

Our aim in the remaining part of this section is to prove that $\mathcal{MS}^{\#}[\mathbb{R}^+]$ interpreted in metric spaces has the finite model property and that $\mathcal{MS}^{\#}[\mathbb{Q}^+]$ interpreted in metric spaces is decidable. But before turning to the details of the proof, we introduce a relational semantics that enables us to use tools and techniques from standard modal logic.

4.1 Relational semantics

As we have two kinds of 'modal operators' in $\mathcal{MS}^{\#}[\mathbb{R}^+]$, namely, $\mathsf{E}^{\leq a}$ and $\mathsf{E}^{>a}$, *relational metric M-models* of $\mathcal{MS}^{\#}[\mathbb{R}^+]$ should be quadruples of the form

$$\mathfrak{S} = \langle W, (R_a)_{a \in M}, (R_{\overline{a}})_{a \in M}, \mathfrak{a} \rangle , \tag{8}$$

where W is a non-empty set, $(R_a)_{a \in M}$ and $(R_{\overline{a}})_{a \in M}$ are two families of binary relations on W, M is a parameter set, and $\mathfrak{a}$ is an assignment in W associating with every set variable X_i a subset $\mathfrak{a}(X_i)$ of W and with every location constant c_i an element $\mathfrak{a}(c_i)$ of W. We understand relations $uR_a v$ and $uR_{\overline{a}}v$ as 'v is at most a (units) far from u' and 'v is more than a (units) far from u,' respectively.

The *value* $t^{\mathfrak{S}}$ of a set term t in $\mathfrak{S}$ is now inductively defined in the standard Kripkean way:

$$(\mathsf{E}^{\leq a} t)^{\mathfrak{S}} = \{w \in W : \exists v \in t^{\mathfrak{S}} \ wR_a v\},$$
$$(\mathsf{E}^{>a} t)^{\mathfrak{S}} = \{w \in W : \exists v \in t^{\mathfrak{S}} \ wR_{\overline{a}} v\}.$$

The values of $\mathsf{A}^{\leq a}t$ and $\mathsf{A}^{>a}t$ are defined dually. (As we have no explicit metric in the relational model, there is no straightforward way to interpret atoms of the form $\boldsymbol{\delta}(c,d) < a$ or $\boldsymbol{\delta}(c,d) = a$. Satisfiability of these formulas will be simulated by other constructors introduced in the next section.)

Aiming to represent metric models by means of relational models, we have to impose a number of restrictions on the accessibility relations. Namely, we say that

[4]Recall that we always have $0 \in M$.

a model $\mathfrak{S}$ of the form (8) is *M-standard* if the following conditions are satisfied for all $a, b \in M$ and $w, u, v \in W$:

(i) $R_a \cup R_{\overline{a}} = W \times W$,

(ii) $R_a \cap R_{\overline{a}} = \emptyset$,

(iii) if $uR_a v$ and $a \leq b$, then $uR_b v$,

(iv) if $uR_{\overline{a}} v$ and $a \geq b$, then $uR_{\overline{b}} v$,

(v) $uR_0 v$ iff $u = v$,

(vi) if $uR_a v$ and $vR_b w$, then $uR_{a+b} w$ whenever $a + b \in M$,

(vii) $uR_a v$ iff $vR_a u$ and $uR_{\overline{a}} v$ iff $vR_{\overline{a}} u$.

Properties (v), (vi) and (vii) reflect axioms (1)–(3) of metric spaces. Note that as a consequence of (i), (ii) and (vi) we have:

(viii) if $uR_a v$ and $uR_{\overline{a+b}} w$ then $vR_{\overline{b}} w$ whenever $a + b \in M$.

With every metric space model $\mathfrak{B} = \langle W, d, X_1^{\mathfrak{B}}, \ldots, c_1^{\mathfrak{B}}, \ldots \rangle$ we can associate the relational metric M-model

$$\mathfrak{S}(\mathfrak{B}) = \langle W, (R_a)_{a \in M}, (R_{\overline{a}})_{a \in M}, \mathfrak{a} \rangle$$

in which the relations R_a and $R_{\overline{a}}$ are defined by taking, for all $w, v \in W$,

$$wR_a v \quad \text{iff} \quad d(w, v) \leq a,$$
$$wR_{\overline{a}} v \quad \text{iff} \quad d(w, v) > a,$$

$\mathfrak{a}(X_i) = X_i^{\mathfrak{B}}$ and $\mathfrak{a}(c_i) = c_i^{\mathfrak{B}}$. Clearly, $\mathfrak{S}(\mathfrak{B})$ is M-standard. Moreover, the following obvious lemma shows that $\mathfrak{S}(\mathfrak{B})$ can be regarded as a relational representation of $\mathfrak{B}$.

LEMMA 4.1. *For every metric space model $\mathfrak{B}$ and every $\mathcal{MS}^{\#}[M]$ set term t, the value of t in $\mathfrak{B}$ coincides with the value of t in $\mathfrak{S}(\mathfrak{B})$.*

At the end of Section 4.2 (Step 5) we will show how under certain conditions a *finite* M-standard model can be transformed into a finite metric model. (However, the technique we use does not apply to infinite models.)

4.2 The finite model property

In this section we prove the following

THEOREM 4.2. *An $\mathcal{MS}^{\#}[\mathbb{R}^+]$-formula φ is satisfiable in a metric space model iff it is satisfiable in a finite metric space model.*

PROOF. We first outline the idea of the proof which consists of five steps. Suppose $\mathfrak{B} \models \varphi$ for some metric $\mathcal{MS}$-model $\mathfrak{B} = \langle W, d, X_1^{\mathfrak{B}}, \ldots, c_1^{\mathfrak{B}}, \ldots \rangle$.

Step 1. Depending on $\mathfrak{B}$, we transform φ into a set Φ with $\mathfrak{B} \models \Phi$, containing only formulas of the form $c \sqsubseteq t$, $s \doteq t$, $s \not\doteq t$, and $\boldsymbol{\delta}(c, d) = a$, in such a way that φ is satisfiable in a finite model whenever Φ is finitely satisfiable (see Lemma 4.3). Starting from Φ, we compute a finite set $M[\Phi]$ of real numbers containing, in particular, all the numbers occurring in Φ.

Step 2. We replace the metric d by a new metric d' with the (finite) range $M[\Phi]$ and obtain a new model $\mathfrak{B}_1$ which still satisfies Φ.

Step 3. The next step is to filtrate (as in modal logic; see e.g. [Chagrov and Zakharyaschev 1997]) the relational metric model $\mathfrak{S} = \mathfrak{S}(\mathfrak{B}_1)$ through some suitable set of terms $cl(\Phi)$. To define $cl(\Phi)$, for each $\boldsymbol{\delta}(c,d) = a$ in Φ we add to Φ the terms X^d, X^c, and $\mathsf{A}^{\geq b}\neg X^d$, where $b = \max\{a' \in M[\Phi] \ : \ a' < a\}$ and X^c, X^d are fresh set variables (these additional terms are required to prove Lemma 4.7 (2) below). The set $cl(\Phi)$ is the closure of the resulting set of terms under rules that are similar to the rules of the Fischer–Ladner closure for **PDL**-formulas (cf. [Harel 1984]). As a result of the filtration we get a *finite* relational metric model $\mathfrak{S}^f$.

Step 4. However, unlike $\mathfrak{S}$, in general $\mathfrak{S}^f$ is not $M[\Phi]$-standard, which means that we cannot directly transform it into a finite metric space model. In fact, $\mathfrak{S}^f$ satisfies all the properties (i)–(viii) save (ii): there may exist a $v \in W^f$ such that $wR_a v$ and $wR_{\overline{a}} v$, for some $w \in W^f$ and $a \in M[\Phi]$. To 'cure' these defects, we make copies of such 'bad' points v and modify the relations R_a and $R_{\overline{a}}$ in $\mathfrak{S}^f$ obtaining a finite standard relational metric model $\mathfrak{S}^*$. (The 'copying-method' was developed by the Bulgarian school of modal logic; see [Gargov et al. 1988; Vakarelov 1991]. Our technique follows [Goranko 1990].)

Step 5. The final step is to transform $\mathfrak{S}^*$ into a finite metric $\mathcal{MS}$-model $\mathfrak{B}^*$ and to show that $\mathfrak{B}^*$ satisfies Φ.

Let us now turn to technical details. Suppose $\mathfrak{B} \models \varphi$.

Step 1. Denote by $term(\varphi)$ the set of all set terms occurring in φ including all subterms; $sub(\varphi)$ stands for the set of all subformulas of φ. Define a set $\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3$ by taking:

$$\begin{aligned}
\Phi_1 &= \{c \sqsubseteq t : (c \sqsubseteq t) \in sub(\varphi),\ \mathfrak{B} \models c \sqsubseteq t\} \cup \{c \sqsubseteq \neg t : (c \sqsubseteq t) \in sub(\varphi),\ \mathfrak{B} \not\models c \sqsubseteq t\},\\
\Phi_2 &= \{s \doteq t : (s \doteq t) \in sub(\varphi),\ \mathfrak{B} \models s \doteq t\} \cup \{s \not\doteq t : (s \doteq t) \in sub(\varphi),\ \mathfrak{B} \models s \not\doteq t\},\\
\Phi_3 &= \{\boldsymbol{\delta}(c,d) = a : \boldsymbol{\delta}(c,d) \text{ occurs in } \varphi,\ a = d(\mathfrak{a}(c), \mathfrak{a}(d))\}.
\end{aligned}$$

Note that the set of parameters from $\mathbb{R}^+$ that occur in Φ_3 depends on the model $\mathfrak{B}$ and not just on the initial formula φ. It should be clear from the definition that we have the following:

LEMMA 4.3. *Suppose Φ is associated with the model $\mathfrak{B}$ satisfying φ. Then the following hold:*
(1) $\mathfrak{B} \models \Phi$.
(2) *For every metric $\mathcal{MS}$-model $\mathfrak{B}'$, if $\mathfrak{B}' \models \Phi$ then $\mathfrak{B}' \models \varphi$.*

Next we construct $M[\Phi]$. Let

$$M(\Phi) = \{a \in \mathbb{R} : a \text{ occurs in } \Phi\} \cup \{0, 1\}.$$

So $M(\Phi)$ depends on $\mathfrak{B}$, whereas the cardinality of $M(\Phi)$ can be bounded in terms of φ. Denote by γ the smallest natural number that is greater than all numbers in $M(\Phi)$ and define $M[\Phi]$ as follows:

$$M[\Phi] = \{\gamma, 0\} \cup \{a \in \mathbb{R} : a = a_1 + \cdots + a_n < \gamma,\ a_1, \ldots, a_n \in M(\Phi),\ n < \omega\}.$$

Let $\mu = \min\{M(\Phi) - \{0\}\}$ and let χ be the least natural number such that $\chi \geq \gamma/\mu$.

LEMMA 4.4. $|M[\Phi]| \leq |M(\Phi)|^{\chi}$.

PROOF. For any $a_1, \dots, a_n \in M(\Phi) - \{0\}$ with $a_1 + \cdots + a_n \leq \gamma$ we have $n \leq \chi$ (for otherwise, if $n > \chi$, we would have $\gamma \leq \chi\mu < n\mu \leq \gamma$, which is a contradiction). The claim follows immediately. □

Step 2. We show now that Φ is satisfied in a metric $\mathcal{MS}$-model

$$\mathfrak{B}_1 = \left\langle W, d', X_1^{\mathfrak{B}_1}, \dots, c_1^{\mathfrak{B}_1}, \dots \right\rangle$$

such that the range of d' is a subset of $M = M[\Phi]$. Indeed, define d' by taking, for all $w, v \in W$,

$$d'(w,v) = \min\bigl(\{\gamma\} \cup \{a \in M : d(w,v) \leq a\}\bigr),$$

$X_i^{\mathfrak{B}_1} = X_i^{\mathfrak{B}}$ for all X_i, and $c_i^{\mathfrak{B}_1} = c_i^{\mathfrak{B}}$ for all c_i.

Clearly, the range of d' is a subset of M. Let us check that d' is a metric. It satisfies (1) because $0 \in M$. That d' is symmetric follows from the symmetry of d. To show (2), we prove first that

$$\{a \in M : d'(w,v) + d'(v,u) \leq a\} \subseteq \{a \in M : d'(w,u) \leq a\}. \tag{9}$$

Suppose $d'(w,v) + d'(v,u) \leq a$, for $a \in M$. If $d'(w,v) = \gamma$ then $d'(v,u) = 0$, and so $d(v,u) = 0$ and $v = u$. Similarly, $d'(v,u) = \gamma$ implies $w = v$. Hence we may assume that both $d'(w,v) < \gamma$ and $d'(v,u) < \gamma$. Then there are $a_1, a_2 \in M$ such that $d'(w,v) = a_1$, $d'(v,u) = a_2$. Moreover, $d(w,v) \leq a_1$, $d(v,u) \leq a_2$ and $a_1 + a_2 < a$. Thus $d'(w,u) \leq a$, which proves (9). Now, if $d'(w,u) > d'(w,v) + d'(v,u)$ then $\gamma > d'(w,v)$ and $\gamma > d'(v,u)$. Hence there are $a_1, a_2 \in M$ such that $d'(w,v) = a_1$, $d'(v,u) = a_2$ and $\gamma > a_1 + a_2$. Thus $a_1 + a_2 \in M$ and $d'(w,u) \leq a_1 + a_2$, which is a contradiction. It follows that $d'(w,u) \leq d'(w,v) + d'(v,u)$.

LEMMA 4.5. *The set Φ is satisfied in $\mathfrak{B}_1$.*

PROOF. Clearly, for each $(\boldsymbol{\delta}(c,d) = a) \in \Phi_3$, $d(\mathfrak{a}(c), \mathfrak{a}(d)) = d'(\mathfrak{a}(c), \mathfrak{a}(d)) = a$. So $\mathfrak{B}_1 \models \Phi_3$. To show $\mathfrak{B}_1 \models \Phi_1 \cup \Phi_2$, it suffices to prove that

$$\forall w \in W \forall t \in \mathit{term}(\Phi_1 \cup \Phi_2)\ (w \in t^{\mathfrak{B}} \leftrightarrow w \in t^{\mathfrak{B}_1}).$$

This can be done by a straightforward induction on the construction of t. The basis of induction and the case of Booleans are trivial. So suppose that t is $\mathsf{A}^{\leq a} s$ (whence $a \in M$). Then we have:

$$\begin{aligned} w \in t^{\mathfrak{B}} &\Leftrightarrow_1 \forall v \in W\ (d(w,v) \leq a \rightarrow v \in s^{\mathfrak{B}}) \\ &\Leftrightarrow_2 \forall v \in W\ (d'(w,v) \leq a \rightarrow v \in s^{\mathfrak{B}_1}) \\ &\Leftrightarrow_3 w \in t^{\mathfrak{B}_1}. \end{aligned}$$

The equivalences $\Leftrightarrow_1$ and $\Leftrightarrow_3$ are obvious; $\Leftrightarrow_2$ holds by the induction hypothesis and the fact that, for all $w, v \in W$ and $a \in M$, $d(x,y) \leq a$ iff $d'(x,y) \leq a$. The case of $\mathsf{A}^{>a} s$ is considered in a similar way. □

Step 3. For each location constant d occurring in Φ_3 we pick a new set variable X^d and define

$$\begin{aligned} t(\Phi) = {} & \mathit{term}(\Phi) \cup \{X^d : d \text{ occurs in } \Phi_3\} \cup \{\neg X^d : d \text{ occurs in } \Phi_3\} \cup \\ & \{\mathsf{A}^{\leq b} \neg X^d : (\boldsymbol{\delta}(c,d) = a) \in \Phi_3,\ b = \max\{a' \in M[\Phi] : a' < a\}\}. \end{aligned}$$

Clearly, $t(\Phi)$ is closed under subterms.

Since the X^d, for $d \in \Phi_3$, do not occur in Φ, we may assume that $(X^d)^{\mathfrak{B}_1} = \{d^{\mathfrak{B}_1}\}$ for all constants d in Φ_3.

Define the *closure* $cl(\Phi)$ of $t(\Phi)$ as the smallest set T of terms such that $t(\Phi) \subseteq T$ and

(1) T is closed under subterms;

(2) if $t \in T$, then $\mathsf{A}^{\leq 0} t \in T$ whenever t is not of the form $\mathsf{A}^{\leq 0} s$;

(3) if $\mathsf{A}^{\leq a} t \in T$ and $a \geq a_1 + \cdots + a_n$, for $a_i \in M[\Phi] - \{0\}$, then $\mathsf{A}^{\leq a_1} \ldots \mathsf{A}^{\leq a_n} t \in T$;

(4) if $\mathsf{A}^{>a} t \in T$ and $b \in M[\Phi]$, then $\neg \mathsf{A}^{\leq b} \neg \mathsf{A}^{>a} t \in T$;

(5) if $\mathsf{A}^{>a} t \in T$ and $b > a$ ($b \in M[\Phi]$), then $\mathsf{A}^{>b} t \in T$ and if $c + a \in M[\Phi]$ ($c \in M[\Phi] - \{0\}$), then $\neg \mathsf{A}^{>a+c} \neg \mathsf{A}^{>a} t \in T$.

LEMMA 4.6. $|cl(\Phi)| \leq S(\Phi) = 2^{\chi+3} \cdot |t(\Phi)| \cdot |M[\Phi]|^{2\chi+1}$.

PROOF. Observe that $cl(\Phi)$ can be obtained from $t(\Phi)$ step-by-step as follows:

First, take the closure of $t(\Phi)$ under subterms and (5) and denote the result by $cl_1(\Phi)$. Second, take the closure of $cl_1(\Phi)$ under subterms and (4) and denote the result by $cl_2(\Phi)$, which is still closed under (5). Third, take the closure of $cl_2(\Phi)$ under subterms and (3), denote the result by $cl_3(\Phi)$ and notice that $cl_3(\Phi)$ is closed under (4) and (5). Finally, take the closure of $cl_3(\Phi)$ under (2). This is closed under (1)–(5).

The following is now readily checked:

—$|cl_1(\Phi)|$ is bounded by $|t(\Phi)| \cdot 2^{\chi} \cdot |M[\Phi]|^{\chi}$, because the introduced terms are of the form $(\neg)\mathsf{A}^{>a_1}(\neg)\mathsf{A}^{>a_2}(\neg)\ldots(\neg)\mathsf{A}^{>a_k} t$, with $a_i - a_{i+1} \geq \mu$ and $(\neg)$ marking a possible occurrence of $\neg$. The length k of such sequences of parameters a_i is bounded by χ, because $a_1 \leq \gamma$.

—$|cl_2(\Phi)|$ is bounded by $4 \cdot |cl_1(\Phi)| \cdot |M[\Phi]|$.

—$|cl_3(\Phi)|$ is bounded by $|cl_2(\Phi)| \cdot |M[\Phi]|^{\chi}$ because, as follows from the proof of Lemma 4.4, no chain $\mathsf{A}^{\leq a_1} \cdots \mathsf{A}^{\leq a_n}$ of length $> \chi$ is introduced when taking the closure under (3).

—$|cl(\Phi)|$ is bounded by $2 \cdot |cl_3(\Phi)|$.

So we obtain that $|cl(\Phi)|$ is bounded by $S(\Phi) = 2^{\chi+3} \cdot |t(\Phi)| \cdot |M[\Phi]|^{2\chi+1}$. □

Recall that $\mathfrak{B}_1 \models \Phi$. Consider now the relational counterpart of $\mathfrak{B}_1$, i.e., the model

$$\mathfrak{S}(\mathfrak{B}_1) = \langle W, (R_a)_{a \in M}, (R_{\overline{a}})_{a \in M}, \mathfrak{b} \rangle$$

which, for brevity, will be denoted by $\mathfrak{S}$. We are going to filtrate $\mathfrak{S}$ through $\Theta = cl(\Phi)$. Define an equivalence relation $\equiv$ on W by taking $u \equiv v$ if $u \in t^{\mathfrak{S}}$ iff $v \in t^{\mathfrak{S}}$, for all $t \in \Theta$. Let $[u] = \{v \in W : u \equiv v\}$. Note that if d is a location constant in Φ_3, then $[\mathfrak{b}(d)] = \{\mathfrak{b}(d)\}$, since $X^d \in \Theta$.

Construct a filtration $\mathfrak{S}^f = \left\langle W^f, (R^f_a)_{a \in M}, (R^f_{\overline{a}})_{a \in M}, \mathfrak{b}^f \right\rangle$ of $\mathfrak{S}$ through Θ by taking

—$W^f = \{[u] : u \in W\}$;

—$\mathfrak{b}^f(c) = [\mathfrak{b}(c)]$;

—$\mathfrak{b}^f(X) = \{[u] : u \in \mathfrak{b}(X)\}$;
—$[u]R_a^f[v]$ iff for all terms $\mathsf{A}^{\leq a}t \in \Theta$,
—$u \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$ implies $v \in t^{\mathfrak{S}}$ and
—$v \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$ implies $u \in t^{\mathfrak{S}}$;
—$[u]R_{\overline{a}}^f[v]$ iff for all terms $\mathsf{A}^{>a}t \in \Theta$,
—$u \in (\mathsf{A}^{>a}t)^{\mathfrak{S}}$ implies $v \in t^{\mathfrak{S}}$ and
—$v \in (\mathsf{A}^{>a}t)^{\mathfrak{S}}$ implies $u \in t^{\mathfrak{S}}$.

Since Θ is finite, W^f is finite as well. Note also that we have $\mathfrak{b}^f(X^d) = \{\mathfrak{b}^f(d)\}$ whenever d is a location constant in Φ_3.

LEMMA 4.7. (1) *For every* $t \in \Theta$ *and every* $u \in W$, $u \in t^{\mathfrak{S}}$ *iff* $[u] \in t^{\mathfrak{S}^f}$.
(2) *For every* $(\boldsymbol{\delta}(c,d) = a) \in \Phi_3$, $a = \min\{b \in M : \mathfrak{b}^f(c)R_b^f\mathfrak{b}^f(d)\}$.
(3) $\mathfrak{S}^f$ *satisfies* (i) *and* (iii)–(viii) *from Section 4.1.*

PROOF. Claim (1) is proved by an easy induction on the construction of t. To prove (2), take $(\boldsymbol{\delta}(c,d) = a) \in \Phi_3$. We must show that $\mathfrak{b}^f(c)R_a^f\mathfrak{b}^f(d)$ and $\neg\mathfrak{b}^f(c)R_b^f\mathfrak{b}^f(d)$, for all $b \in M$ such that $a > b$. Notice first that uR_av implies $[u]R_a^f[v]$ and $uR_{\overline{a}}v$ implies $[u]R_{\overline{a}}^f[v]$. Since $\mathfrak{B}_1 \models \Phi$, we have $\mathfrak{B}_1 \models \boldsymbol{\delta}(c,d) = a$, and so $d'(\mathfrak{b}(c), \mathfrak{b}(d)) = a$. Hence $\mathfrak{b}(c)R_a\mathfrak{b}(d)$ and $\mathfrak{b}^f(c)R_a^f\mathfrak{b}^f(d)$. Suppose now that $b' \in M$ is maximal with $b' < a$ and consider $\mathsf{A}^{\leq b'}\neg X^d$. By definition, $\mathfrak{b}(X^d) = \{\mathfrak{b}(d)\}$. Hence $\mathfrak{b}(d) \notin (\neg X^d)^{\mathfrak{S}}$. On the other hand, we have $b' < d'(\mathfrak{b}(c), \mathfrak{b}(d))$, from which $\mathfrak{b}(c) \in (\mathsf{A}^{\leq b'}\neg X^d)^{\mathfrak{S}}$. Since $(\mathsf{A}^{\leq b'}\neg X^d) \in \Theta$, we then obtain $\neg\mathfrak{b}^f(c)R_{b'}^f\mathfrak{b}^f(d)$. Then, for arbitrary $b \in M$ such that $b < a$, it follows by (3)(iii) that $\neg\mathfrak{b}^f(c)R_b^f\mathfrak{b}^f(d)$.

To prove (3), let us check conditions (i) and (iii)–(viii).

(i): We have to show that $R_a^f \cup R_{\overline{a}}^f = W^f \times W^f$. Let $\neg[u]R_a^f[v]$. Then $\neg uR_av$, and so $uR_{\overline{a}}v$, since $\mathfrak{S}$ satisfies (i). Thus $[u]R_{\overline{a}}^f[v]$.

(iii): If $[u]R_a^f[v]$ and $a \leq b$ then $[u]R_b^f[v]$. Let $[u]R_a^f[v]$ and $a < b$, for $b \in M$. Suppose $u \in (\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$. By the definition of $\Theta = cl(\Phi)$, $\mathsf{A}^{\leq a}t \in \Theta$, and so $u \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$. Hence $v \in t^{\mathfrak{S}}$. That $v \in (\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$ implies $u \in t^{\mathfrak{S}}$ is shown in the same way.

(iv): If $[u]R_{\overline{a}}^f[v]$ and $a \geq b$ then $[u]R_{\overline{b}}^f[v]$. Let $[u]R_{\overline{a}}^f[v]$ and $a > b$. Suppose $u \in (\mathsf{A}^{>b}t)^{\mathfrak{S}}$. Then $\mathsf{A}^{>a}t \in \Theta$, $u \in (\mathsf{A}^{>a}t)^{\mathfrak{S}}$, and so $v \in t^{\mathfrak{S}}$. Again, the other direction is treated analogously.

(v): $[u]R_0^f[v]$ iff $[u] = [v]$. The implication ($\Leftarrow$) is obvious. So suppose $[u]R_0^f[v]$. Take some $t \in \Theta$ with $u \in t^{\mathfrak{S}}$. Without loss of generality we may assume that t is not of the form $\mathsf{A}^{\leq 0}s$. Then, by the definition of Θ, $u \in (\mathsf{A}^{\leq 0}t)^{\mathfrak{S}}$ and $\mathsf{A}^{\leq 0}t \in \Theta$. Hence $v \in t^{\mathfrak{S}}$. In precisely the same way one can show that for all $t \in \Theta$, $v \in t^{\mathfrak{S}}$ implies $u \in t^{\mathfrak{S}}$. Therefore, $[u] = [v]$.

(vi): If $[u]R_a^f[v]$ and $[v]R_b^f[w]$, then $[u]R_{a+b}^f[w]$, for $(a+b) \in M$. Suppose $u \in (\mathsf{A}^{\leq a+b}t)^{\mathfrak{S}}$. Then $\mathsf{A}^{\leq a}\mathsf{A}^{\leq b}t \in \Theta$ and $u \in (\mathsf{A}^{\leq a}\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$. So $v \in (\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$, whence $w \in t^{\mathfrak{S}}$. Now suppose that $w \in (\mathsf{A}^{\leq a+b}t)^{\mathfrak{S}}$. Again, we have $\mathsf{A}^{\leq b}\mathsf{A}^{\leq a}t \in \Theta$ and $w \in (\mathsf{A}^{\leq b}\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$. Then $v \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$, whence $u \in t^{\mathfrak{S}}$.

(vii): $[w]R_a^f[u]$ iff $[u]R_a^f[w]$ and $[w]R_{\overline{a}}^f[u]$ iff $[u]R_{\overline{a}}^f[w]$ hold by definition.

(viii): If $[u]R^f_a[v]$ and $[u]R^f_{\overline{a+b}}[w]$, then $[v]R^f_{\overline{b}}[w]$, for $(a+b) \in M$. Suppose that $v \in (\mathsf{A}^{>b}t)^{\mathfrak{S}}$. Then $\neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}t \in \Theta$ and $u \in (\neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$. Hence $u \in (\mathsf{A}^{>(a+b)}t)^{\mathfrak{S}}$ and so $w \in t^{\mathfrak{S}}$. For the other direction suppose $w \in (\mathsf{A}^{>b}t)^{\mathfrak{S}}$. Then $u \in (\neg\mathsf{A}^{>(a+b)}\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$ and $\neg\mathsf{A}^{>(a+b)}\neg\mathsf{A}^{>b}t \in \Theta$. Hence $u \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$ and so $v \in t^{\mathfrak{S}}$. □

Step 4. Unfortunately, $\mathfrak{S}^f$ does not necessarily satisfy (ii) which is required to construct the model $\mathfrak{B}^*$ we need: it may happen that for some points $[u]$, $[v]$ in W^f and $a \in M$, we have both $[u]R^f_a[v]$ and $[u]R^f_{\overline{a}}[v]$. To 'cure' these defects, we have to perform some surgery. The defects form the set

$$D(W^f) = \{v \in W^f : \exists a \in M \exists u \in W^f \ (uR^f_a v \ \& \ uR^f_{\overline{a}} v)\}.$$

Let

$$W^* = \{\langle v, i\rangle : v \in D(W^f), i \in \{0,1\}\} \cup \{\langle u, 0\rangle : u \in W^f - D(W^f)\}.$$

So for each $v \in D(W^f)$ we now have two copies $\langle v, 0\rangle$ and $\langle v, 1\rangle$. Define an assignment $\mathfrak{b}^*$ in W^* by taking

—$\mathfrak{b}^*(c) = \left\langle \mathfrak{b}^f(c), 0\right\rangle$ and
—$\mathfrak{b}^*(X) = \{\langle u, i\rangle \in W^* : u \in \mathfrak{b}^f(X)\}$.

Finally, we define accessibility relations R^*_a and $R^*_{\overline{a}}$ as follows:

—if $a > 0$ then $\langle u, i\rangle \, R^*_a \, \langle v, j\rangle$ iff either
  —$uR^f_a v$ and $\neg uR^f_{\overline{a}} v$, or
  —$uR^f_a v$ and $i = j$;
—if $a = 0$ then $\langle u, i\rangle \, R^*_a \, \langle v, j\rangle$ iff $\langle u, i\rangle = \langle v, j\rangle$;
—$R^*_{\overline{a}}$ is defined as the complement of R^*_a, i.e., $\langle u, i\rangle \, R^*_{\overline{a}} \, \langle v, j\rangle$ iff $\neg \langle u, i\rangle \, R^*_a \, \langle v, j\rangle$.

Lemma 4.8. *$\mathfrak{S}^* = \langle W^*, (R^*_a)_{a\in M}, (R^*_{\overline{a}})_{a\in M}, \mathfrak{b}^*\rangle$ is an M-standard relational metric model.*

Proof. That $\mathfrak{S}^*$ satisfies (i), (ii), and (v) follows immediately from the definition. Let us check the remaining conditions.

(iii) Suppose that $\langle u, i\rangle \, R^*_a \, \langle v, j\rangle$ and $b \in M$ is such that $a < b$. If $i = j$ then clearly $\langle u, i\rangle \, R^*_b \, \langle v, j\rangle$. So assume $i \neq j$. Then, by definition, $uR^f_a v$ and $\neg uR^f_{\overline{a}} v$. Since $\mathfrak{S}^f$ satisfies (iii) and (iv), we obtain $uR^f_b v$ and $\neg uR^f_{\overline{b}} v$. Thus $\langle u, i\rangle \, R^*_b \, \langle v, j\rangle$.

(iv) Suppose that $\langle u, i\rangle \, R^*_{\overline{a}} \, \langle v, j\rangle$ and $b \in M$ is such that $a \geq b$, but $\neg \langle u, i\rangle \, R^*_{\overline{b}} \, \langle v, j\rangle$. By (i), $\langle u, i\rangle \, R^*_b \, \langle v, j\rangle$. And by (iii), $\langle u, i\rangle \, R^*_a \, \langle v, j\rangle$. Finally, (ii) yields $\neg \langle u, i\rangle \, R^*_{\overline{a}} \, \langle v, j\rangle$, which is a contradiction.

(vi) Suppose $\langle u, i\rangle \, R^*_a \, \langle v, j\rangle$, $\langle v, j\rangle \, R^*_b \, \langle w, k\rangle$ and $a+b \in M$. Then $uR^f_a v$ and $vR^f_b w$. As $\mathfrak{S}^f$ satisfies (vi), we have $uR^f_{a+b} w$. If $i = k$ then clearly $\langle u, i\rangle \, R^*_{a+b} \, \langle w, k\rangle$. So assume $i \neq k$. If $i = j \neq k$ then, using (viii) for $\mathfrak{S}^f$, $\neg uR^f_{\overline{a+b}} w$, since $uR^f_a v$ and $\neg vR^f_{\overline{b}} w$. The case $i \neq j = k$ is considered analogously using the fact that the relations in $\mathfrak{S}^f$ are symmetric.

(vii) follows from the symmetry of R^f_a and $R^f_{\overline{a}}$.

Now, the symmetry of $R^*_{\overline{a}}$ follows from the symmetry of R^*_a and (i), (ii). □

Lemma 4.9. *For all $\langle v,i\rangle \in W^*$ and $t \in \Theta$, we have $\langle v,i\rangle \in t^{\mathfrak{S}^*}$ iff $v \in t^{\mathfrak{S}^f}$.*

Proof. The proof is by induction on t. The basis of induction and the case of Booleans are trivial (we remind the reader that Θ contains no set constants $\{c\}$). The cases $t = (\mathsf{A}^{\leq a}s)$ and $t = (\mathsf{A}^{>a}s)$ are consequences of the following claims:

Claim 1: If $uR^f_a v$ and $\langle u,i\rangle \in W^*$ ($i \in \{0,1\}$), then there exists a j such that $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$. Indeed, this is clear for $i=0$. Suppose $i=1$. If v was duplicated, then $\langle v,1\rangle$ is as required. If v was not duplicated, then $\neg uR^f_{\overline{a}}v$, and so $\langle v,0\rangle$ is as required.

Claim 2: If $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$, then $uR^f_a v$. This should be obvious.

Claim 3: If $uR^f_{\overline{a}}v$ and $\langle u,i\rangle \in W^*$ ($i \in \{0,1\}$), then there exists a j such that $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$. Suppose $i=0$. If v was not duplicated, then $\neg uR^f_a v$. Hence $\neg\,\langle u,0\rangle\, R^*_a\, \langle v,0\rangle$. If v was duplicated, then $\neg\,\langle u,0\rangle\, R^*_a\, \langle v,1\rangle$. In the case of $i=1$ we have $\neg\,\langle u,1\rangle\, R^*_a\, \langle v,0\rangle$, i.e., $\langle u,1\rangle\, R^*_{\overline{a}}\, \langle v,0\rangle$.

Claim 4: If $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$, then $uR^f_{\overline{a}}v$. Indeed, if $i=j$ then $\neg uR^f_a v$ and so $uR^f_{\overline{a}}v$. And if $i \neq j$, then $uR^f_{\overline{a}}v$. □

Step 5. To complete the proof, we transform $\mathfrak{S}^*$ into a finite metric $\mathcal{MS}$-model and show that this model satisfies Φ. Let

$$\mathfrak{B}^* = \langle W^*, d^*, X^*_1, \ldots, c^*_1, \ldots\rangle,$$

where for all $w,v \in W^*$, set variables X_i, and constants c_i,

$$d^*(w,v) = \min(\{\gamma\} \cup \{a \in M : wR^*_a v\}), \quad X^*_i = \mathfrak{b}^*(X_i), \quad c^*_i = \mathfrak{b}^*(c_i).$$

As M is finite, d^* is well-defined. Using (v)–(vii), it is easy to see that d^* is a metric with range $M[\Phi]$. So $\mathfrak{B}^*$ is a finite metric space model. It remains to show that $\mathfrak{B}^*$ satisfies Φ. Observe first that

(✠) for all $w \in W^*$ and $t \in t(\Phi)$, we have $w \in t^{\mathfrak{S}^*}$ iff $w \in t^{\mathfrak{B}^*}$.

This is proved by induction on t. The basis of induction and the case of Booleans are clear. So let $t = (\mathsf{A}^{\leq a}s)$ for some $a \in M$. Then

$$\begin{aligned} w \in (\mathsf{A}^{\leq a}s)^{\mathfrak{S}^*} &\Leftrightarrow_1 \forall v\ (wR^*_a v \to v \in s^{\mathfrak{S}^*})\\ &\Leftrightarrow_2 \forall v\ (wR^*_a v \to v \in s^{\mathfrak{B}^*})\\ &\Leftrightarrow_3 \forall v\ (d^*(w,v) \leq a \to v \in s^{\mathfrak{B}^*})\\ &\Leftrightarrow_4 w \in (\mathsf{A}^{\leq a}s)^{\mathfrak{B}^*}. \end{aligned}$$

Equivalences $\Leftrightarrow_1$ and $\Leftrightarrow_4$ are obvious; $\Leftrightarrow_2$ holds by the induction hypothesis; $\Leftarrow_3$ is an immediate consequence of the definition of d^*, and $\Rightarrow_3$ follows from (iii). The case $t = (\mathsf{A}^{>a}s)$ is considered analogously.

We can now show that $\mathfrak{B}^* \models \Phi$. Let $(c \sqsubseteq t) \in \Phi_1$. Then we have:

$$\mathfrak{B}^* \models c \sqsubseteq t \Leftrightarrow_1 c^* \in t^{\mathfrak{B}^*} \Leftrightarrow_2 \mathfrak{b}^*(c) \in t^{\mathfrak{S}^*} \Leftrightarrow_3 \left\langle \mathfrak{b}^f(c), 0\right\rangle \in t^{\mathfrak{S}^*} \Leftrightarrow_4$$
$$\mathfrak{b}^f(c) \in t^{\mathfrak{S}^f} \Leftrightarrow_5 [\mathfrak{b}(c)] \in t^{\mathfrak{S}^f} \Leftrightarrow_6 \mathfrak{b}(c) \in t^{\mathfrak{S}} \Leftrightarrow_7 c^{\mathfrak{B}_1} \in t^{\mathfrak{B}_1} \Leftrightarrow_8 \mathfrak{B}_1 \models c \sqsubseteq t.$$

Equivalences $\Leftrightarrow_1$ and $\Leftrightarrow_8$ are obvious; $\Leftrightarrow_2$ follows from (✠); $\Leftrightarrow_3$ and $\Leftrightarrow_5$ hold by definition; $\Leftrightarrow_4$ follows from Lemma 4.9, $\Leftrightarrow_6$ from Lemma 4.7, and $\Leftrightarrow_7$ from Lemma 4.1.

Since $\mathfrak{B}_1 \models \Phi$, we have $\mathfrak{B}^* \models \Phi_1$. That $\mathfrak{B}^* \models \Phi_2$ is proved analogously using (✠).

It remains to show that $\mathfrak{B}^* \models \Phi_3$. Take any $\boldsymbol{\delta}(c,d) = a$ from Φ_3. We must prove that $d^*(c^*, d^*) = a$. By Lemma 4.7 (2),

$$a = \min\{b \in M : \mathfrak{b}^f(c) R_b^f \mathfrak{b}^f(d)\}.$$

So $a = \min\{b \in M : \langle \mathfrak{b}^f(c), 0\rangle \, R_b^* \, \langle \mathfrak{b}^f(d), 0\rangle\}$. By the definition of $\mathfrak{b}^*$, we have $a = \min\{b \in M : \mathfrak{b}^*(c) R_b^* \mathfrak{b}^*(d)\}$, which means that $d^*(\mathfrak{b}^*(c), \mathfrak{b}^*(d)) = a$.

We have proved the following:

THEOREM 4.10. *Φ is satisfied in a metric $\mathcal{MS}$-model*

$$\mathfrak{B}^* = \langle W^*, d^*, X_1^*, \dots, c_1^*, \dots\rangle$$

such that $|W^*| \leq 2 \cdot 2^{S(\Phi)}$ and the range of d^* is a subset of $M[\Phi]$.

From Theorem 4.10 and Lemma 4.3 (2), it follows that φ is satisfied in the finite model $\mathfrak{B}^*$, which completes the proof of Theorem 4.2.

4.3 Decidability

The main result of this section is the following:

THEOREM 4.11. (i) *The satisfiability problem for $\mathcal{MS}^{\#}[\mathbb{Q}^+]$-formulas in the class $\mathcal{M}$ of metric spaces is decidable.*

(ii) *Let $q \in \mathbb{N}$. The satisfiability problem for $\mathcal{MS}^{\#}[\{0,\dots,q\}]$-formulas in $\mathcal{M}$ is decidable in* **NExpTime**.

We will first concentrate on (i). Note that the finite model property of $\mathcal{MS}^{\#}[\mathbb{R}^+]$ proved above is not enough to establish the decidability of $\mathcal{MS}^{\#}[\mathbb{Q}^+]$: we still do not know an effectively computable upper bound for the size of a finite model satisfying a given formula φ. Indeed, the set $M(\Phi)$ depends not only on φ, but also on the initial model $\mathfrak{B}$ satisfying φ because of the possible introduction of new parameters $a \in \mathbb{R}$ by expressions of the form $\boldsymbol{\delta}(c,d)$ occurring in φ. Note however that by Lemmas 4.4 and 4.6, an upper bound for the size of $\mathfrak{B}^*$ can be computed effectively from the maximum of $M(\Phi)$, the minimum of $M(\Phi) - \{0\}$, and φ. Thus, to obtain an effective upper bound, it suffices to start the construction with a model satisfying φ for which both the maximum of $M(\Phi)$ and the minimum of $M(\Phi) - \{0\}$ are bounded. The next lemma shows how to obtain such a model. Let n_φ and m_φ be the minimal and the maximal positive numbers occurring in φ, respectively; if no such numbers exist, then put $m_\varphi = n_\varphi = 1$.

LEMMA 4.12. *Suppose that an $\mathcal{MS}^{\#}[\mathbb{Q}^+]$-formula φ is satisfied in a metric $\mathcal{MS}$-model $\mathfrak{B} = \langle W, d, X_1^{\mathfrak{B}}, \dots, c_1^{\mathfrak{B}}, \dots\rangle$. Denote by D the set of all expressions of the form $\boldsymbol{\delta}(c,d)$ occurring in φ and assume that $D \neq \emptyset$. Then there is a metric d' on W such that φ is satisfied in $\mathfrak{B}' = \langle W, d', X_1^{\mathfrak{B}}, \dots, c_1^{\mathfrak{B}}, \dots\rangle$ and*

$$\min\{d'(c^{\mathfrak{B}}, d^{\mathfrak{B}}) > 0 : \boldsymbol{\delta}(c,d) \in D\} \geq n_\varphi/2,$$
$$\max\{d'(c^{\mathfrak{B}}, d^{\mathfrak{B}}) : \boldsymbol{\delta}(c,d) \in D\} \leq 2m_\varphi.$$

PROOF. Let $a = n_\varphi$ and $b = m_\varphi$. Set

$$a' = \min\{d(c^{\mathfrak{B}}, d^{\mathfrak{B}}) > 0 : \boldsymbol{\delta}(c,d) \in D\},$$
$$b' = \max\{d(c^{\mathfrak{B}}, d^{\mathfrak{B}}) : \boldsymbol{\delta}(c,d) \in D\}.$$

We consider the case where $a' < a/2$ and $2b < b'$. The other cases are even easier and left to the reader. Define d' by taking for all $v, w \in W$

$$d'(v,w) := \begin{cases} d(v,w) & \text{if } a \leq d(v,w) \leq b \text{ or } d(v,w) = 0, \\ b + (b/(b'-b)) \cdot (d(v,w) - b) & \text{if } d(v,w) > b, \\ a + (a/2(a-a')) \cdot (d(v,w) - a) & \text{if } 0 < d(v,w) < a. \end{cases}$$

One can easily compute that if $d(v,w) > b$ then $d'(v,w) < d(v,w)$, and if $0 < d(v,w) < a$ then $d'(v,w) > d(v,w)$. It is a routine exercise now to show that d' is a metric. Clearly, it satisfies conditions (1) and (3). Let us see that for all $u, v, w \in W$, we have

$$d'(u,w) \leq d'(u,v) + d'(v,w). \tag{10}$$

We consider here only two cases and leave the remaining ones to the reader.

Case 1: $d(u,w) > b$ and $0 < d(u,v), d(v,w) < a$. Then, as was observed above, we have $d'(u,w) < d(u,w)$, $d(u,v) < d'(u,v)$ and $d(u,v) < d'(u,v)$, which together with $\langle W, d\rangle$ satisfying the triangular inequality yields (10).

Case 2: $d(u,w) > b$, $0 < d(u,v) < a$ and $d(v,w) > b$. Note first that we again have $d'(u,v) > d(u,v)$, and in view of (2), $d(u,v) \geq d(u,w) - d(v,w)$. It remains to observe that $0 < b/(b'-b) < 1$ and

$$d'(u,w) - d'(v,w) = \frac{b}{b'-b} \cdot (d(u,w) - d(v,w)),$$

which yields $d'(u,v) \geq d'(u,w) - d'(v,w)$, i.e., (10).

To complete the proof, it remains to observe that for every parameter a occurring in φ, every relation $\approx$ in $\{=, <, \leq, >, \geq\}$, and all $x, y \in W$, we have

$$d(x,y) \approx a \quad \text{iff} \quad d'(x,y) \approx a.$$

It follows that $t^{\mathfrak{B}} = t^{\mathfrak{B}'}$ for every term t occurring in φ, and so φ is satisfied in $\mathfrak{B}'$. □

It follows that we can start the filtration with a model $\mathfrak{B}$ for which we obtain (by Lemmas 4.4 and 4.6) the following upper bound for $cl(\Phi)$:

—$|cl(\Phi)|$ is bounded by $l(\varphi)^{p(m_\varphi/n_\varphi)}$, where p is a polynomial function of degree 2 not depending on φ and $l(\varphi)$ is the length of φ.[5]

Summarizing the results obtained so far, we have

THEOREM 4.13. *There exists a quadratic polynomial p such that every $\mathcal{MS}^{\#}[\mathbb{Q}^+]$-formula φ which is satisfiable in a metric space model is satisfiable in a metric space whose domain is bounded by*

$$f(\varphi) = 2 \cdot 2^{l(\varphi)^{p(\frac{m_\varphi}{n_\varphi})}}.$$

[5] This is done as follows. First, by Lemma 4.12 and the definition of χ, we obtain that $\chi \leq \frac{4m_\varphi+2}{n_\varphi} + 1$. Further, we clearly have $|M(\Phi)| \leq l(\varphi)$ and $|t(\Phi)| \leq 4 \cdot l(\varphi)$, whence, by Lemma 4.4, $|M[\Phi]| \leq l(\varphi)^\chi$. Hence, by Lemma 4.6, we obtain $|cl(\Phi)| \leq 2^{\chi+3} \cdot 2^2 \cdot l(\varphi) \cdot l(\varphi)^{\chi\cdot(2\chi+1)} \leq l(\varphi)^{\chi+6} \cdot l(\varphi)^{2\chi^2+\chi} = l(\varphi)^{2\chi^2+2\chi+6}$.

In contrast to many standard satisfiability problems even this result does not directly imply the decidability of the satisfiability problem for $\mathcal{MS}^{\#}[\mathbb{Q}^+]$–formulas, because there are infinitely many (even uncountably many) different metric spaces based on a finite set. We now address this problem.

Fix a formula φ and $n \leq f(\varphi)$. Put $W = \{1, \dots, n\}$. Suppose that φ contains constants $C(\varphi) = \{c_1, \dots, c_k\}$, set variables $V(\varphi) = \{X_1, \dots, X_l\}$ and parameters $P(\varphi) = \{a_0, a_1, \dots, a_p\}$, where $a_0 = 0$ belongs to $P(\varphi)$ even if it does not occur in φ. Assume that $0 < a_1 < a_2 < \dots < a_p$. Further, take variables x_{ij}, for every $i, j \in W$. These variables are intended to 'simulate' the distance $d(i, j)$ between i and j.

Let I_1, I_2 and I_3 be a partition of $W \times W$, and k a function from $W \times W$ to $\{0, 1, \dots, p\}$. There are only finitely many pairs $\mathfrak{E} = (\mathfrak{T}, \mathfrak{C})$ whose first component is a structure

$$\mathfrak{T} = \left\langle W, (X^{\mathfrak{T}} : X \in V(\varphi)), (c^{\mathfrak{T}} : c \in C(\varphi)) \right\rangle$$

and the second one is a set of 'constraints' of the form

$$\begin{aligned} \mathfrak{C} = \{ & x_{ij} = a_{k(ij)} : (i,j) \in I_1 \} \\ & \cup \{ x_{ij} > a_p : (i,j) \in I_2 \} \\ & \cup \{ a_{k(ij)} < x_{ij} < a_{k(ij)+1} : (i,j) \in I_3 \}, \end{aligned}$$

where $X^{\mathfrak{T}} \subseteq \{1, \dots, n\}$ for every set variable X of φ, $c^{\mathfrak{T}} \in \{1, \dots, n\}$ for every constant c of φ. The constraints in $\mathfrak{C}$ specify for every ordered pair of elements i, j from W whether the distance between i and j is equal to some $a_{k(ij)}$, greater than a_p or strictly between some $a_{k(ij)}$ and $a_{k(ij)+1}$. Pairs $\mathfrak{E}$ of this type will be called *(n-)constraint systems for* φ. Constraint systems specify a class of models based on the domain W in such a way that it is possible to determine from the system the value of all those terms which contain parameters from $P(\varphi)$ only. Define the extension $s^{\mathfrak{E}}$ of a term s containing parameters from $P(\varphi)$ only by induction:

—$X^{\mathfrak{E}} = X^{\mathfrak{T}}$ for every set variable X of φ;
—$(s_1 \sqcap s_2)^{\mathfrak{E}} = s_1^{\mathfrak{E}} \cap s_2^{\mathfrak{E}}$;
—$(\neg s)^{\mathfrak{E}} = W - s^{\mathfrak{E}}$;
—$(\mathsf{E}^{\leq a} s)^{\mathfrak{E}} = \{ i \in W : \exists j \in W \left((x_{ij} = a \in \mathfrak{C} \ \&\ j \in s^{\mathfrak{E}}) \vee (x_{ij} < a \in \mathfrak{C} \ \&\ j \in s^{\mathfrak{E}}) \right) \}$;
—$(\mathsf{E}^{> a} s)^{\mathfrak{E}} = \{ i \in W : \exists j \in W \left(a < x_{ij} \in \mathfrak{C} \ \&\ j \in s^{\mathfrak{E}} \right) \}$.

The truth-relation $\mathfrak{E} \models \varphi$, φ an $\mathcal{MS}[M]$-formula with parameters from $P(\varphi)$, is defined as expected (we list only the interesting clauses):

—$\mathfrak{E} \models \boldsymbol{\delta}(c_1, c_2) = a$ iff $x_{ij} = a \in \mathfrak{C}$ for $i = c_1^{\mathfrak{T}}$ and $j = c_2^{\mathfrak{T}}$;
—$\mathfrak{E} \models \boldsymbol{\delta}(c_1, c_2) < a$ iff $x_{ij} < a \in \mathfrak{C}$ for $i = c_1^{\mathfrak{T}}$ and $j = c_2^{\mathfrak{T}}$.

Say that $\mathfrak{E} = (\mathfrak{T}, \mathfrak{C})$ *satisfies* φ if $\mathfrak{E} \models \varphi$. Of course, if φ is satisfiable in a model of size n, then φ is satisfied in an n-constraint system for φ. The converse does not hold, because it could be that there does not exist a metric d on W which conforms to $\mathfrak{C}$, where a metric d *conforms to* $\mathfrak{C}$ if by setting $x_{ij} = d(i, j)$, for all $i, j \in W$, all constraints in $\mathfrak{C}$ are satisfied.

So, say that $\mathfrak{E} = (\mathfrak{T}, \mathfrak{C})$ is *satisfiable* if the constraints in $\mathfrak{C}$ together with the following set of equalities and inequalities has a solution in $\mathbb{R}^+$:

—$x_{ii} = 0$, for all $i \in W$;
—$x_{ij} = x_{ji}$, for all $i, j \in W$ (symmetry);
—$x_{ik} + x_{kj} \geq x_{ij}$, for all $i, j, k \in W$ (triangular inequality).

The following is now easily checked:

LEMMA 4.14. *A formula $\varphi \in \mathcal{MS}^{\#}[\mathbb{Q}^{+}]$ is satisfiable in a metric space model of size n iff there exists a satisfiable n-constraint system for φ which satisfies φ.*

LEMMA 4.15. *It is decidable in polynomial time $\rho(n)$ whether an n-constraint system $\mathfrak{C}$ for φ is satisfiable and satisfies φ.*

PROOF. Clearly, given a satisfiable n-constraint system $\mathfrak{C}$ for φ, it is decidable in polynomial time whether φ is indeed satisfied in $\mathfrak{C}$.

Hence, it remains to show that checking satisfiability of $\mathfrak{C}$ can be done in polynomial time. First, notice that the decidability of this problem follows from Tarski's result on the decidability of the theory of real closed fields [Tarski 1951]. On the other hand, the problem can be understood as a standard problem of *linear programming*, where we can choose some arbitrary *objective function* to be maximized. In fact, we are only interested in the question whether this system of equalities and inequalities has a common solution, i.e., in the *linear programming feasibility problem.* Furthermore, since all parameters in the constraints are from $\mathbb{Q}$, a solution exists in $\mathbb{R}$ iff a solution exists in $\mathbb{Q}$, because the set of solutions can be represented as a (possibly unbounded) convex polyhydron. Hence we can restrict ourselves to searching for rational solutions. This problem has been shown, e.g. in [Blum et al. 1998], to be solvable in polynomial time measured in the number of variables, i.e., in n. □

Theorem 4.11 (i) follows from Theorem 4.13 and Lemmas 4.14 and 4.15. Theorem 4.11 (ii) follows from Theorem 4.13 and Lemmas 4.14 and 4.15, because for $\varphi \in \mathcal{MS}^{\#}[\{1, \ldots, q\}]$ the number q is an upper bound for m_{φ}/n_{φ}. Now the decision procedure is as follows: given $\varphi \in \mathcal{MS}^{\#}[\{1, \ldots, q\}]$ guess an n-constraint system $\mathfrak{C}$ with $n \leq f(\varphi)$ and check in polynomial time (in n) whether $\mathfrak{C}$ is both satisfiable and satisfies φ.

We note that it is an open problem whether satisfiability of $\mathcal{MS}^{\#}[\{1, \ldots, q\}]$-formulas in metric spaces is **NExpTime**-hard.

5. SATISFIABILITY IN WEAKER DISTANCE SPACES

Let us now consider the satisfiability problem in the class $\mathcal{D}$ of arbitrary distance spaces and its subclasses $\mathcal{D}_{sym}$ and $\mathcal{D}_{tr}$. For $\mathcal{D}$ and $\mathcal{D}_{sym}$ we can prove decidability even for the language $\mathcal{FM}^2[\mathbb{Q}^{+}]$. For $\mathcal{D}_{tr}$ we will consider the languages $\mathcal{MS}^{\#}[\{1, \ldots, q\}]$ and $\mathcal{MS}^{\#}[\mathbb{Q}^{+}]$.

THEOREM 5.1. *The satisfiability problem for $\mathcal{FM}^2[\mathbb{Q}^{+}]$-formulas in $\mathcal{D}$ and $\mathcal{D}_{sym}$ is decidable. Moreover, both problems are in* **NExpTime** *and in both cases any satisfiable formula is satisfiable in a finite model.*

PROOF. The proof is based on a simple reduction to the satisfiability problem for the two-variable fragment of first-order logic. Recall that atomic formulas $\boldsymbol{\delta}(x, y) < a$ and $\boldsymbol{\delta}(x, y) = a$ can be regarded as binary predicates $P_{<a}(x, y)$ and $P_{=a}(x, y)$.

Denote by φ^+ the result of replacing all subformulas in φ of the form $\boldsymbol{\delta}(x,y) < a$ and $\boldsymbol{\delta}(x,y) = a$ by $P_{<a}(x,y)$ and $P_{=a}(x,y)$, respectively. Let

$$0 = a_0 < a_1 < \cdots < a_n$$

be the list of rational numbers that occur in φ, together with 0, and let Γ be the set of the following formulas, for $i \leq n$:

$$\forall x, y \big(P_{=a_i}(x,y) \to \bigwedge_{0 \leq j < i} \neg P_{<a_j}(x,y) \land \bigwedge_{i \neq j} \neg P_{=a_j}(x,y) \land \bigwedge_{n \geq j > i} P_{<a_j}(x,y)\big),$$
$$\forall x, y \big(P_{<a_i}(x,y) \to \bigwedge_{i < j \leq n} P_{<a_j}(x,y)\big),$$
$$\forall x, y \neg P_{<0}(x,y),$$
$$\forall x, y \big(P_{=0}(x,y) \leftrightarrow x = y\big).$$

We claim that the set $\Gamma \cup \{\varphi^+\}$ is satisfiable in a first-order structure

$$\mathfrak{A} = \big\langle W, P^{\mathfrak{A}}_{=a_0}, \ldots, P^{\mathfrak{A}}_{<a_0}, \ldots, P^{\mathfrak{A}}_1, \ldots, c^{\mathfrak{A}}_1, \ldots \big\rangle$$

iff φ is satisfiable in a distance space model.

The direction ($\Leftarrow$) is clear. So suppose that $\mathfrak{A}$ satisfies $\Gamma \cup \{\varphi^+\}$. Define a distance space structure

$$\mathfrak{B} = \big\langle W, d, P^{\mathfrak{A}}_1, \ldots, c^{\mathfrak{A}}_1, \ldots \big\rangle$$

by taking, for $a, b \in W$,

$$d(a,b) = a_i \text{ iff } \mathfrak{A} \models P_{=a_i}(a,b),$$
$$d(a,b) = \frac{a_i + a_{i+1}}{2} \text{ iff } \mathfrak{A} \models \neg P_{<a_i}(a,b) \land P_{<a_{i+1}}(a,b) \land \neg P_{=a_i}(a,b),$$
$$d(a,b) = 2 \cdot a_n \text{ iff } \mathfrak{A} \models \neg P_{<a_n}(a,b) \land \neg P_{=a_n}(a,b).$$

It is not difficult to see that $\mathfrak{B}$ satisfies φ. Hence, to decide whether φ is satisfiable in a distance space model, it suffices to check whether $\Gamma \cup \{\varphi^+\}$ is satisfiable in a first-order structure. This proves the decidability of satisfiability in $\mathcal{D}$.

For $\mathcal{D}_{sym}$, we take the set Γ_{sym} which is

$$\Gamma \cup \{\forall x, y \big(P_{<a_i}(x,y) \leftrightarrow P_{<a_i}(y,x)\big),\ \forall x, y \big(P_{=a_i}(x,y) \leftrightarrow P_{=a_i}(y,x)\big) : i \leq n\}.$$

It is readily checked that φ is satisfiable in $\mathcal{D}_{sym}$ iff $\Gamma_{sym} \cup \{\varphi^+\}$ is satisfiable.

The remaining claims follow immediately from the **NExpTime**-completeness of the two-variable fragment of first-order logic and its finite model property [Mortimer 1975; Fürer 1984; Grädel et al. 1997]. ☐

Let us now consider the satisfiability problem in $\mathcal{D}_{tr}$.

THEOREM 5.2. (i) *The satisfiability problem for $\mathcal{MS}^{\#}[\mathbb{Q}^+]$-formulas in $\mathcal{D}_{tr}$ is decidable.*

(ii) *Any $\mathcal{MS}^{\#}[\mathbb{Q}^+]$-formula satisfiable in $\mathcal{D}_{tr}$ is satisfiable in a finite member of $\mathcal{D}_{tr}$.*

(iii) *The satisfiability problem for $\mathcal{MS}^{\#}[\{0, \ldots, q\}]$-formulas in $\mathcal{D}_{tr}$ is in* **NExpTime**.

PROOF. The proof is quite similar to that of Theorem 4.11. Steps 1 and 2 of the proof are virtually as before. We start with a formula φ that is satisfied in a distance space model $\mathfrak{B} \in \mathcal{D}_{tr}$ and, using the same terminology as in section 4.2, again define the set Φ and the model $\mathfrak{B}_1 = \left\langle W, d', X_1^{\mathfrak{B}_1}, \dots, c_1^{\mathfrak{B}_1}, \dots \right\rangle$ such that $\mathfrak{B}_1 \models \Phi$. The main difference here is that d' is now not necessarily symmetric.

However, in steps 3 and 4 two important modifications are required: one concerns the filtration, another the copying technique:

Step 3. The closure $cl(\Phi)$ of $t(\Phi)$ is defined in almost the same way as on page 20; the only difference is that the last condition is replaced with the following one:

(5′) if $\mathsf{A}^{>a}t \in T$ and $b > a$, for $b \in M[\Phi]$, then $\mathsf{A}^{>b}t \in T$.

The relational counterpart of $\mathfrak{B}_1$, i.e., the model

$$\mathfrak{S}(\mathfrak{B}_1) = \langle W, (R_a)_{a\in M}, (R_{\overline{a}})_{a\in M}, \mathfrak{b}\rangle,$$

will again be denoted by $\mathfrak{S}$. The filtration of $\mathfrak{B}_1$ through $\Theta = cl(\Phi)$ is modified in the following way.

Define an equivalence relation $\equiv$ on W by taking $u \equiv v$ if for all $t \in \Theta$ we have $u \in t^{\mathfrak{S}}$ iff $v \in t^{\mathfrak{S}}$. Let $[u] = \{v \in W : u \equiv v\}$. Note again that if $(d \sqsubseteq X^d) \in \Phi_3'$ then $[\mathfrak{b}(d)] = \{\mathfrak{b}(d)\}$, since $X^d \in \Theta$.

Construct a filtration $\mathfrak{S}^f = \left\langle W^f, (R_a^f)_{a\in M}, (R_{\overline{a}}^f)_{a\in M}, \mathfrak{b}^f \right\rangle$ of $\mathfrak{S}$ through Θ by taking

—$W^f = \{[u] : u \in W\}$;
—$\mathfrak{b}^f(c) = [\mathfrak{b}(c)]$;
—$\mathfrak{b}^f(X) = \{[u] : u \in \mathfrak{b}(X)\}$;
—$[u]R_a^f[v]$ for $a > 0$ iff for all terms $\mathsf{A}^{\leq a}t \in \Theta$, $u \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$ implies $v \in t^{\mathfrak{S}}$;
—$[u]R_a^f[v]$ for $a = 0$ iff $[u] = [v]$;
—$[u]R_{\overline{a}}^f[v]$ iff for all terms $\mathsf{A}^{>a}t \in \Theta$, $u \in (\mathsf{A}^{>a}t)^{\mathfrak{S}}$ implies $v \in t^{\mathfrak{S}}$.

Since Θ is finite, W^f is finite as well. Note also that we have $\mathfrak{b}^f(X^d) = \{\mathfrak{b}^f(d)\}$ whenever $(d \sqsubseteq X^d) \in \Phi_3'$ and that uR_av implies $[u]R_a^f[v]$, and $uR_{\overline{a}}v$ implies $[u]R_{\overline{a}}^f[v]$.

LEMMA 5.3. (1) *For every* $t \in \Theta$ *and every* $u \in W$, $u \in t^{\mathfrak{S}}$ *iff* $[u] \in t^{\mathfrak{S}^f}$.
(2) *For every* $(\boldsymbol{\delta}(c,d) = a) \in \Phi_3$, $a = \min\{b \in M : \mathfrak{b}^f(c)R_b^f\mathfrak{b}^f(d)\}$.
(3) $\mathfrak{S}^f$ *satisfies* (i), (iii)–(vi) *and* (viii) *from Section 4.1.*

PROOF. (1) is proved by an easy induction; the proof of (2) is the same as in Lemma 4.7.

To prove (3), we have to check conditions (i), (iii)–(vi) and (viii). The first one, i.e., $R_a^f \cup R_{\overline{a}}^f = W^f \times W^f$, is proved as in Lemma 4.7.

(iii): if $[u]R_a^f[v]$ and $a \leq b$ then $[u]R_b^f[v]$. Let $[u]R_a^f[v]$ and $a < b$ for some $b \in M$. Suppose $u \in (\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$. By the definition of Θ, $\mathsf{A}^{\leq a}t \in \Theta$. Thus, since $a < b$, $u \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$. Then $[u]R_a^f[v]$ implies $v \in t^{\mathfrak{S}}$, and $[u]R_b^f[v]$ follows.

(iv): if $[u]R_{\overline{a}}^f[v]$ and $a \geq b$ then $[u]R_{\overline{b}}^f[v]$. The proof is similar to that of (iii).

(v): $[u]R_0^f[v]$ iff $[u] = [v]$ holds by the definition of R_0^f.

(vi): if $[u]R^f_a[v]$ and $[v]R^f_b[w]$, then $[u]R^f_{a+b}[w]$, for $(a+b) \in M$. Suppose $u \in (\mathsf{A}^{\leq a+b}t)^{\mathfrak{S}}$. Then $\mathsf{A}^{\leq a}\mathsf{A}^{\leq b}t \in \Theta$ and $u \in (\mathsf{A}^{\leq a}\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$. So $v \in (\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$, whence $w \in t^{\mathfrak{S}}$.
(viii): if $[u]R^f_a[v]$ and $[u]R^f_{\overline{a+b}}[w]$ then $[v]R^f_{\overline{b}}[w]$, for $(a+b) \in M$. Let $v \in (\mathsf{A}^{>b}t)^{\mathfrak{S}}$ and $\mathsf{A}^{>b}t \in \Theta$. Then we have $\neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}t \in \Theta$ and $u \in (\neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$, for otherwise (since Θ is closed under subterms) $u \in (\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$ together with $[u]R^f_a[v]$ would imply $v \in (\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$, which is a contradiction. Suppose that $uR_{\overline{a+b}}x$ for some point $x \in W$. Since $u \in (\neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$, there is a point $y \in W$ such that uR_ay and $y \in (\mathsf{A}^{>b}t)^{\mathfrak{S}}$. As $\mathfrak{S}$ satisfies (viii), it follows that $yR_{\overline{b}}x$, and so $x \in t^{\mathfrak{S}}$. Hence $u \in (\mathsf{A}^{>(a+b)}t)^{\mathfrak{S}}$, which implies $w \in t^{\mathfrak{S}}$. □

Step 4. We are now again facing the problem that $\mathfrak{S}^f$ may not satisfy condition (ii) which is required for the construction of the model $\mathfrak{B}^*$. To avoid this problematic case—the situation where for some points $[u]$, $[v]$ in W^f and $a \in M$ both $[u]R^f_a[v]$ and $[u]R^f_{\overline{a}}[v]$ hold—we modify the copying technique in the following way. The problematic points form the set

$$D(W^f) = \{v \in W^f : \exists a \in M \exists u \in W^f \ (uR^f_a v \ \& \ uR^f_{\overline{a}}v)\}.$$

Let

$$W^* = \{\langle v, i\rangle : v \in D(W^f),\ i \in \{0,1,2\}\} \cup \{\langle u, 0\rangle : u \in W^f - D(W^f)\}.$$

So for each $v \in D(W^f)$ we have now three copies $\langle v,0\rangle$, $\langle v,1\rangle$ and $\langle v,2\rangle$. Define an assignment $\mathfrak{b}^*$ in W^* by taking

$$\mathfrak{b}^*(c) = \left\langle \mathfrak{b}^f(c), 0\right\rangle,$$
$$\mathfrak{b}^*(X) = \{\langle u, i\rangle \in W^* : u \in \mathfrak{b}^f(X)\}.$$

Finally, we define accessibility relations R^*_a and $R^*_{\overline{a}}$ as follows:

—If $a > 0$, then $\langle u,i\rangle\, R^*_a \,\langle v,j\rangle$ iff either
  —$uR^f_a v$ and $\neg uR^f_{\overline{a}}v$, or
  —$uR^f_a v$ and $j = 0$, or
  —$\langle u,i\rangle = \langle v,j\rangle$ (then also $uR^f_a v$).
—If $a = 0$, then $\langle u,i\rangle\, R^*_a \,\langle v,j\rangle$ iff $\langle u,i\rangle = \langle v,j\rangle$.
—$R^*_{\overline{a}}$ is defined as the complement of R^*_a, i.e.,

$$\langle u,i\rangle\, R^*_{\overline{a}} \,\langle v,j\rangle \quad \text{iff} \quad \neg\, \langle u,i\rangle\, R^*_a \,\langle v,j\rangle\,.$$

Lemma 5.4. *The relational model* $\mathfrak{S}^* = \langle W^*, (R^*_a)_{a\in M}, (R^*_{\overline{a}})_{a\in M}, \mathfrak{b}^*\rangle$ *satisfies conditions* (i)–(vi) *of M-standard models.*

Proof. That $\mathfrak{S}^*$ satisfies (i), (ii), and (v) follows immediately from the definitions of R^*_a and $R^*_{\overline{a}}$. Let us check the remaining conditions.

(iii) Suppose $\langle u,i\rangle\, R^*_a \,\langle v,j\rangle$ and $a < b$, for $b \in M$. If $\langle u,i\rangle = \langle v,j\rangle$, then $\langle u,i\rangle\, R^*_b \,\langle v,j\rangle$ follows immediately from the definition. So assume $\langle u,i\rangle \neq \langle v,j\rangle$. By definition we have $uR^f_a v$, and since $\mathfrak{S}^f$ satisfies (iii), $uR^f_b v$ holds as well. If $\neg uR^f_{\overline{b}}v$, then clearly $\langle u,i\rangle\, R^*_b \,\langle v,j\rangle$. So suppose $uR^f_{\overline{b}}v$. Since $\mathfrak{S}^f$ satisfies (iv), we then have $uR^f_{\overline{a}}v$, whence $j = 0$ and so $\langle u,i\rangle\, R^*_b \,\langle v,j\rangle$.

(iv) Suppose $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$ and $a > b$, for $b \in M$. Assume $\neg \langle u,i\rangle\, R^*_{\overline{b}}\, \langle v,j\rangle$. By (i), we have $\langle u,i\rangle\, R^*_{b}\, \langle v,j\rangle$, whence by (iv), $\langle u,i\rangle\, R^*_{a}\, \langle v,j\rangle$. Now (ii) implies $\neg \langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$, which is a contradiction. Hence $\langle u,i\rangle\, R^*_{\overline{b}}\, \langle v,j\rangle$.

(vi) Suppose $\langle u,i\rangle\, R^*_{a}\, \langle v,j\rangle$ and $\langle v,j\rangle\, R^*_{b}\, \langle w,k\rangle$, for $a,b,a+b \in M$. We have to show that $\langle u,i\rangle\, R^*_{a+b}\, \langle w,k\rangle$. First, if $\langle u,i\rangle = \langle v,j\rangle$ or $\langle v,j\rangle = \langle w,k\rangle$, then $\langle u,i\rangle\, R^*_{a+b}\, \langle w,k\rangle$ follows immediately from (iii), since $a,b \le a+b$. So we may assume that $\langle u,i\rangle \neq \langle v,j\rangle$ and $\langle v,j\rangle \neq \langle w,k\rangle$. Then by definition, $uR^f_a v$ and $vR^f_b w$, whence $uR^f_{a+b} w$, because $\mathfrak{S}^f$ satisfies (vi). If $\neg uR^f_{\overline{a+b}} w$, then $\langle u,i\rangle\, R^*_{a+b}\, \langle w,k\rangle$ follows from the definition. So assume $uR^f_{\overline{a+b}} w$ holds in $\mathfrak{S}^f$ as well. From $uR^f_a v$ and (viii) we obtain $vR^f_{\overline{b}} w$, and so $k = 0$. But then again, $\langle u,i\rangle\, R^*_{a+b}\, \langle w,k\rangle$ follows from the definition. □

Lemma 5.5. *For all* $\langle u,i\rangle \in W^*$, $i \in \{0,1,2\}$ *and all* $t \in \Theta$, *we have*

$$\langle u,i\rangle \in t^{\mathfrak{S}^*} \quad \textit{iff} \quad u \in t^{\mathfrak{S}^f}.$$

Proof. The proof is by induction on t. The basis of induction follows from the definition and the case of Booleans is trivial. The cases of $t = (\mathsf{A}^{\le a} s)$ and $t = (\mathsf{A}^{> a} s)$ are consequences of the following claims.

Claim 1: If $uR^f_a v$ and $\langle u,i\rangle \in W^*$, then there is j such that $\langle u,i\rangle\, R^*_{a}\, \langle v,j\rangle$. Indeed, if $a > 0$, we put $j = 0$, and $\langle u,i\rangle\, R^*_{a}\, \langle v,j\rangle$ follows from the definition. If $a = 0$, then $u = v$; so we can take $i = j$.

Claim 2: If $\langle u,i\rangle\, R^*_{a}\, \langle v,j\rangle$, then $uR^f_a v$. This follows immediately from the definition of R^*_a.

Claim 3: If $uR^f_{\overline{a}} v$ and $\langle u,i\rangle \in W^*$, then there exists j such that $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$. Fix some $uR^f_{\overline{a}} v$ and $\langle u,i\rangle \in W^*$. Suppose first that $a = 0$. If $\neg uR^f_0 v$ we then have $u \neq v$, since R^f_0 satisfies (v), and so we can choose $j = 0$. If $uR^f_0 v$ then v has been copied, so we can choose $j = i + 1(mod\ 2)$ and $\langle u,i\rangle \neq \langle v,j\rangle$, from which $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$.

Suppose now that $a > 0$. Consider two cases.

Case 1: $uR^f_a v$. Then v has been copied, i.e., W^* contains $\langle v,0\rangle$, $\langle v,1\rangle$ and $\langle v,2\rangle$. Then put $j \neq 0, i$ which is always possible, because we have three copies of v. But then all three defining properties of $\langle u,i\rangle\, R^*_{a}\, \langle v,j\rangle$ are violated, which means $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$.

Case 2: $\neg uR^f_a v$. Then $u \neq v$. So we can put $j = 0$, and again all three defining properties are violated.

Claim 4: If $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$ then $uR^f_{\overline{a}} v$. There are again two cases.

Case 1: $a > 0$. Suppose $\neg uR^f_{\overline{a}} v$. Then, since the first defining property of $uR^*_a v$ is violated, we have $\neg uR^f_a v$, contrary to (i). Therefore $uR^f_{\overline{a}} v$.

Case 2: $a = 0$. Then $\langle u,i\rangle \neq \langle v,j\rangle$. If $u \neq v$, then $\neg uR^f_0 v$ and hence $uR^f_{\overline{0}} v$ as required. If $u = v$ and $i \neq j$, then u has been copied. So there are $w \in W^f$ and $b \in M$ such that $wR^f_b u$ and $wR^f_{\overline{b}} u$. Since the latter can be written as $wR^f_{\overline{b+0}} u$, condition (viii) yields $uR^f_{\overline{0}} u$, as required.

Now, consider the induction step for $t = (\mathsf{A}^{\le a} s)$. Suppose $\langle u,i\rangle \in (\mathsf{A}^{\le a} s)^{\mathfrak{S}^*}$

and pick some v such that $uR_a^f v$. By Claim 1, there exists $j \in \{0,1,2\}$ such that $\langle u,i\rangle \, R_a^* \, \langle v,j\rangle$. Then $\langle v,j\rangle \in s^{\mathfrak{S}^*}$ and, by the induction hypotheses, it follows that $v \in s^{\mathfrak{S}^f}$. Hence $u \in (\mathsf{A}^{\leq a} s)^{\mathfrak{S}^f}$. Conversely, if $u \in (\mathsf{A}^{\leq a} s)^{\mathfrak{S}^f}$ and $\langle v,j\rangle$ is such that $\langle u,i\rangle \, R_a^* \, \langle v,j\rangle$, then by Claim 2, $uR_a^f v$ and $v \in s^{\mathfrak{S}^f}$, and so by the induction hypotheses, $\langle v,j\rangle \in s^{\mathfrak{S}^*}$, i.e., $\langle u,i\rangle \in (\mathsf{A}^{\leq a} s)^{\mathfrak{S}^*}$.

The case of $t = (A^{>a} s)$ follows analogously from Claims 3 and 4. □

Step 5. In the same way as in Theorem 4.10, we can now transform $\mathfrak{S}^*$ into a finite distance space model, which is possibly non-symmetric, and prove that this model satisfies Φ. This shows that $\mathcal{MS}^{\#}[\mathbb{Q}^+]$ has the finite model property with respect to $\mathcal{D}_{tr}$.

To complete the proof, we can follow the lines of the proof of Theorem 4.11 and establish both the decidability and complexity claims for $\mathcal{MS}^{\#}$-formulas in non-symmetric distance spaces, thus proving Theorem 5.2. Of course, in the definition of the satisfiability of constraint systems, we now omit the symmetry condition $x_{ij} = x_{ji}$.

6. CONCLUSION

In this paper, we have started an investigation into the expressive power and computational properties of the first-order language $\mathcal{FM}^2[M]$ (with two individual variables) and the 'modal' language $\mathcal{MS}[M]$ both interpreted in metric and 'weaker' distance spaces. We showed that these languages have the same expressive power over the class $\mathcal{M}$ of all metric spaces (in fact, even over the class $\mathcal{D}_{sym}$ of symmetric distance spaces). While both $\mathcal{FM}^2[\mathbb{Q}^+]$-satisfiability and $\mathcal{MS}[\mathbb{Q}^+]$-satisfiability are decidable for the class of all (symmetric) distance spaces, even weaker languages turn out to have an undecidable satisfiability problem for the class of metric spaces and the class $\mathcal{D}_{tr}$ of distance spaces satisfying the triangular inequality. We also discovered a natural fragment $\mathcal{MS}^{\sharp}[M]$ of $\mathcal{MS}[M]$ which has the finite model property and is decidable (both for metric spaces and distance spaces with the triangular inequality). If the parameter set M is of the form $\{1,\dots,q\}$, then in both cases the satisfiability problem is in ***NExpTime***.

The logics we considered in this paper have promising applications in knowledge representation and reasoning by introducing a numerical, quantitative concept of distance into the conventional qualitative KR&R (see the example in Section 1 and [Kutz et al. 2002]). In this connection we would like to attract the readers' attention to the following interesting open problems:

(1) Compare the expressive power of $\mathcal{FM}^2[M]$ and $\mathcal{MS}[M]$ over $\mathcal{D}$ and $\mathcal{D}_{tr}$.

(2) Is $\mathcal{MS}^{\sharp}[\{1,\dots,q\}]$-satisfiability in metric spaces ***NExpTime***-complete? What is the computational complexity of $\mathcal{MS}^{\sharp}[\{1,\dots,q\}]$-satisfiability in other classes of distance spaces?

(3) Is the satisfiability of $\mathcal{MS}^{\sharp}[\mathbb{Q}^+]$-formulas in metric spaces decidable in ***NExpTime***? What about other classes of distance spaces?

(4) We have considered satisfiability in 'abstract' metric and distance spaces. However, from the application point of view, it would be more interesting to analyze

the computational behavior of our logics in n-dimensional (especially, 2D) Euclidean spaces?

(5) The presented decision procedure based on the finite model property does not appear to be 'practical.' An important open problem is to develop tableau or resolution based algorithms for $\mathcal{MS}^\sharp$ or its sublanguages.

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