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New Semantics for Modal Predicate Logics.

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Abstract. The problem of finding adequate semantics for languages of first-order modal logic, both from a mathematical and philosophical point of view, turned out to be rather difficult. The 1990ies have seen a number of attempts to attack this problem from a new angle, by introducing semantics that extend the usual framework of Kripkean possible worlds semantics.
In this paper, I briefly introduce the most important of these semantics and state the main theoretical results that are known so far, concentrating on the (frame-) completeness problem and the role of substitution principles. It is argued that while the mathematical generality of the proposed semantics is a great step forward, a satisfying philosophical interpretation of "Kripke-type" semantics has still to be accomplished.

1 Introduction

The stages to which propositional and first–order modal logic have been developed are quite different. While the former has turned into an established research area with a profound mathematical grounding and many applications in diverse fields such as philosophy, computer science and

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linguistics, the latter is still notoriously confused, from Aristotle's *Prior Analytics* via Quine's 'dictum of incomprehensibility' to the current disagreement on the right syntax and semantics.

In the propositional case, the possible worlds semantics—being developed among others by Hintikka, Kripke and Montague in the early sixties—provides a canonical conception of semantics. Furthermore, the phenomenon of *Kripke incompleteness* usually arises only as a (technical) side issue, because almost all 'popular' logics are complete with respect to Kripke frames.

Now, in the first–order case, the interest focused on particular modal systems (*e.g.*, for analysing issues in metaphysics) which led to a deficit in the general mathematical analysis. This tendency was of course amplified by the high complexity of semantic issues involved, like the infamous notion of a 'modal individual'. Actually, the diverging intuitions concerning this notion and the corresponding conflicting theories of identity are the major impediment to a uniform treatment of modal predicate logic, MPL for short. Whereas there is almost agreement on how to define a first–order version of a propositional modal logic[1], the issue of adding additional axioms that correspond to certain assumptions on the class of models, the most popular of which are the *Barcan and Converse Barcan Formulae* (BF and CBF for short), arises. In the standard semantics they correspond to the assumptions of decreasing and increasing domains, respectively.[2]

Two typical problems are the following. If one assumes standard semantics, even the quantified extensions of very simple propositional modal logics, like, *e.g.*, $\mathbf{Q}.\mathbf{S4}.\mathbf{2} + \mathbf{BF}$ (*cf.* [11]), exhibit Kripke incompleteness.[3] On the other hand, if equality is part of the language and non–rigid constants are preferred—which is quite natural in a number of applications—then an asymmetry between variables and constants appears. While variables denote objects, constants now denote *individual concepts*, that is, functions from the set of possible worlds to their do-

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[1] Roughly, given a propositional modal logic $\mathcal{L}$, a first-order axiomatization $\mathcal{S}$ and a first–order modal language $\mathbb{L}$, take all substitution instances of axioms of $\mathcal{L}$ or $\mathcal{S}$ in the language $\mathbb{L}$ and add the rule of necessitation. For details, *cf.*, *e.g.*, [11] or [4].

[2] By "standard semantics" I mean here standard Kripke frames enriched by an assignment of domains to worlds, meeting some extra conditions. Actually, if the CBF schema is omitted, one has to deal with non–denoting terms and to move to a quantificational base in free logic, *e.g.*, by introducing an *existence predicate*.

[3] For further simple examples of incomplete logics and proof sketches, compare, *e.g.*, [11].

mains.[4] Those questions are intimately linked to whether the Necessity of Identity, $(c \doteq d) \rightarrow \Box(c \doteq d)$ (NoI), or the Necessity of Distinctness, $(c \not\doteq d) \rightarrow \Box(c \not\doteq d)$ (NoD), should be regarded as valid. Many other modifications have to be made if certain assumptions about the denotation of terms across possible worlds or the behaviour of identity are made, *cf.* [5].

Correspondingly, two endeavours may be distinguished. The first, being essentially mathematical, is to single out a general class of MPLs and to give adequate semantics for it. The second, being philosophical in nature, is to give a satisfying analysis of the notion of 'modal individual' and to provide for an appropriate syntax.

In what is to follow, I concentrate on the technical aspects and briefly introduce and compare the different proposed generalized semantics—referred to also as *Kripke–type semantics*—and state what is known about them. I first discuss substitution principles and give a general notion of (normal) modal predicate logic. A discussion of syntax extensions is avoided altogether, but confer [4] for an extensive treatment of the use of term–binding operators that, for instance, enable one to distinguish between a *de dicto* and *de re* usage of *constants* .

2 First- versus Second–Order Closed MPLs

In [13], we introduced the distinction between first- and second–order closed MPLs. These logics are defined by appealing to the following *substitution principles*. By *first–order substitution* we mean the usual substitution of *terms* for *variables*, while second–order substitutions are defined as follows:

Definition 1 (Second–order Substitutions). Let φ be a formula in which the n-place relation symbol P appears and let ψ be some modal formula. Then $(\psi/P)\varphi$ is called a **second–order substitution instance**, if $(\psi/P)\varphi$ is the result of replacing every occurrence of $P(\bar{y})$ in φ by $\psi(\bar{y}/\bar{x})$, possibly renaming some bound variables.

Notice that such a substitution principle is actually derivable in the case of classical first–order logic and more generally for any logic that is

[4] For a treatment of this in a classical setting, *cf.* [4].

axiomatized by *unrestricted* schemata. Nevertheless, by assuming unrestricted second order substitution for a given logic L one automatically extends the underlying modal theory of identity. *E.g.*, given that $(x \doteq y) \rightarrow (P(x,x) \rightarrow P(x,y))$ is an admissible instance of Leibniz' Law, second order substitution yields $(x \doteq y) \rightarrow (\Box(x \doteq x) \rightarrow \Box(x \doteq y))$ and hence $(x \doteq y) \rightarrow \Box(x \doteq y)$. Actually, this situation is one of the reasons for introducing a weaker base logic than the usual $\mathbf{QK}$[5]. In [13] we worked with a system called **FK** which is a combination of propositional modal logic **K** and *positive free logic*, $\mathbf{PFL}$[6].

If equality is introduced, the base logic is enriched by a weak form of *Leibniz' Law*, which we called the *Modal Leibniz' Law*. This basically results from the usual *Leibniz' Law* by restricting the Quinean principle of the "substitutability of identicals" to those instances that do not entail 'transworld-identifications' of individuals of any kind. Briefly, if $x \doteq y$ and the variable x appears free within the scope of a modal operator, then either *all* or *no* occurrence of x may be replaced by y. Hence, $(x \doteq y) \rightarrow (\Box(x \doteq x) \rightarrow \Box(x \doteq y))$ is not admissible, which blocks the provability of the necessity of identity.[7]

We may now define MPLs as follows:

Definition 2 (Modal Predicate Logics). A set of formulae L with $\mathbf{FK} \subseteq L$ is called a **first–order closed modal predicate logic**, if it is closed under the rules *necessitation*, *universal generalization* and *modus ponens* and L is also closed under first–order substitutions. If L is additionally closed under second–order substitutions, it is called a **second–order closed modal predicate logic**. If we speak of a modal predicate logic L simpliciter, L is assumed to be at least first–order closed.

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5 Here, $\mathbf{QK}$ denotes the quantified extension of the smallest normal modal logic—known as $\mathbf{K}$ and being named after Saul Kripke—using standard first–order logic. $\mathbf{K}$ is obtained by taking all substitution instances of axioms of classical propositional logic in the modal language and by adding the axiom schema $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$, referred to as *normality*, and the additional rule $A/\Box A$, known as *necessitation*.

6 Free logic is a family of logics weakening classical first–order logic in such a way that existential presuppositions are avoided. For instance, the formula $\varphi(c) \rightarrow \exists x.\varphi(x)$ is not regarded as valid. Positive free logic is a special flavour of free logic where formulas that contain terms that are not within the scope of the quantifiers are ascribed truth values ('non–existent' versus 'non–denoting'). For an introduction to free logic and detailed definitions compare [2], and for an argumentation why free logic is not only useful but necessary, *cf.* [6] or [15].

7 Thus, the modal operators behave quite similar to what is known as an *unselective binder* in linguistics.

A natural solution to the above problem of generating possibly unintended theorems involving equality is therefore to deal with second–order closed logics without identity and to add a modal theory of identity, or, alternatively, to incorporate the theory of identity into the logic while restricting substitution in an appropriate way.

Note that a second–order closed logic is a second–order logic in the sense that predicate symbols are treated as second–order variables without allowing explicit second–order quantification. Hence, predicate variables are treated as being implicitly, universally quantified.

We have seen that closure under second–order substitutions has a quite different flavour in a propositional setting as opposed to a first–order setting. In particular, unlike the case of classical (non–modal) first–order logic (where this principle is derivable), there are a number of reasons to be interested in first–order closed MPLs and to treat them as *genuine* logics. We list just a few of them. First, if atomic propositions/predicates enjoy a special status—like in certain logics of time—then substitution of complex formulae for atoms may not be admissible. Actually, this was one of the reasons for Robert Goldblatt to introduce a similar distinction in the propositional case and to call it a "significant conceptual change" (compare his [9]). Similarly for the case where basic predicates may be *intensional*. Second, if one works with a weak *logic of identity*, then a restriction of substitution is unavoidable. Last but not least, if generalized semantics are considered, there are naturally defined frame classes whose logic is only first–order closed. However, one can also argue in favour of closure under second–order substitution as a *defining property* of the general concept of a 'logic', which has been attempted for the case of MPL in [1].

3 Kripke- versus Kripke–Type Semantics

Kripke–type semantics differ from the usual Kripke semantics in two essential aspects. First, instead of taking a Kripke frame, that is, a set of possible worlds together with an accessibility relation, and to enrich it by assigning domains to worlds, one starts with a family of first–order domains and adds some set of functions or relations between the domains, which in turn *define* accessibility between worlds. Hence, accessibility is no longer a primitive of the frame but rather depends on the functions/relations being present. This leads to the second fundamental difference, namely that there may indeed be many distinct functions/relations between two given worlds. The following Figure 1 shows some of the different proposed Kripke–type semantics and their interdependencies. An arrow from A to B indicates that the semantics A is a special case of semantics B.

Fig. 1. An overview over Kripke–type semantics

Informally speaking, there may be many 'different ways' to move from one world to another. This distinguishes Kripke–type semantics also significantly from standard counterpart theory (*cf.* [16]) and its derived possible worlds semantics (*cf.* [10]). In fact, the simultaneous quantification over both worlds and individuals in counterpart theory obscures the notion of accessibility between worlds and leads for example to the semantic refutability of certain **K**–theorems (*cf.*, e.g., [11]). But the exact connection between counterpart theory and Kripke–type semantics has yet to be fully analysed.[8] That the feature of multiple functions or relations is not eliminable is due to the fact that there are second–order closed MPLs that are complete only with respect to frames having at least two counterpart relations between worlds, *cf.* [14].

[8] But compare [3] for a variation of counterpart semantics that can be understood as a special case of the counterpart frames to be introduced below.

3.1 Functor and Metaframe Semantics

The functor semantics—mainly developed by Ghilardi—can be defined as follows. Let $\mathcal{C} = \langle Ob_{\mathcal{C}}, Mor_{\mathcal{C}} \rangle$ be a *small* category, i.e., the classes $Ob_{\mathcal{C}}$ and $Mor_{\mathcal{C}}$ of the objects and morphisms are sets (rather than proper classes). Every such small category has a frame representation $\mathcal{F} = \langle \mathcal{W}, \leq \rangle$ by defining the set of possible worlds as $\mathcal{W} := Ob_{\mathcal{C}}$ and by setting for $u, v \in \mathcal{W}$: $u \lhd v$ iff $\mathcal{C}(u, v) \neq \emptyset$, where $\mathcal{C}(u, v)$ is the set of all morphisms in $Mor_{\mathcal{C}}$ from u to v. The notion replacing the usual first–order Kripke frames is the notion of a $\mathcal{C}$–set.

Definition 3 ($\mathcal{C}$–set). Let $\mathcal{C}$ be a small category. A $\mathcal{C}$–**set** is a set–valued functor over $\mathcal{C}$, *i.e.*, a triple $\mathfrak{F} = \langle \mathcal{F}, \mathcal{D}, \mathcal{E} \rangle$, where $\mathcal{F} = Ob_{\mathcal{C}}$, $\mathcal{D} = (D_u)_{u \in \mathcal{F}}$ is a family of non–empty disjoint sets and $\mathcal{E} = (E_\mu)_{\mu \in Mor_{\mathcal{C}}}$ is a family of functions parametrized by morphisms from $\mathcal{C}$, such that $E_\mu : D_u \longrightarrow D_v$, whenever $\mu \in \mathcal{C}(u, v)$, and $E_{\mu \circ \nu} = E_\mu \circ E_\nu$ as well as $E_{id_\mu} = id_{D_\mu}$.

Truth in a $\mathcal{C}$–set is as usual defined at a world $u \in \mathcal{W}$ with D_u being its domain and with respect to an interpretation of relation symbols and a valuation β_u that assigns elements of D_u to the variables. I just give the clause for the modal case:

$$\langle u, \beta_u \rangle \models \Diamond \varphi(\bar{y}) \text{ iff } \exists E_\mu : D_u \longrightarrow D_v : \langle v, E_\mu \circ \beta_u \rangle \models \varphi(\bar{y})$$

One interesting aspect of this semantics is that it enables one to prove general incompleteness results with respect to Kripke semantics like the following theorem which is from [7].

Theorem 1 (Ghilardi 1991). *Let* $\mathcal{L} \supseteq \mathbf{S4}$ *be an extension of the propositional modal logic* $\mathbf{S4}$. *Then, if* $\mathbf{Q}\mathcal{L}$ *is complete with respect to some class of (standard) Kripke frames, then* $\mathcal{L} \supseteq \mathbf{S5}$ *or* $\mathcal{L} \subseteq \mathbf{S4.3}$.

On the other hand, it also provided the first general completeness results for a wide class of (interesting) MPLs, which is illustrated by the following (reformulation of a) theorem from [8].

Theorem 2 (Ghilardi 1992). *Every (standard) quantified extension of a canonical propositional modal logic above* $\mathbf{S4}$ *is functor frame complete.*[9]

[9] A logic is said to be canonical, if the frame underlying its canonical model is a frame for the logic.

The restriction to extensions of **S4** is basically due to the formulation in category theoretic language, namely to the last two conditions in the definition of $\mathcal{C}$–set, $E_{\mu \circ \nu} = E_\mu \circ E_\nu$ and $E_{id_\mu} = id_{D_\mu}$, that correspond to transitivity and reflexivity, respectively. We will see that it can easily be dispensed with when dealing with counterpart frames in the next section. An analogous result to theorem 2 can also be found in [18] for the *metaframe semantics*. These structures were first introduced in [18] and further extended to *general metaframes* in [17]. They may be (roughly) defined thus: Let Σ denote the category of finite ordinals (i.e. natural numbers with their usual ordering) and functions between them.

Definition 4 ((General) Metaframes). A **general metaframe** is a contravariant functor from the category Σ into the category of general frames such that for every $\sigma : m \to n$, $M(\sigma) : M(n) \longrightarrow M(m)$ is a p–morphism. In particular, for every n, $M(n) = \langle F_n, \lhd_n, \mathbb{F}_n \rangle$ is a general frame, *i.e.*, $\mathbb{F}_n$ is a modal algebra based on the underlying Kripke frame $F(n) = \langle F_n, \lhd_n \rangle$. A **metaframe** is a contravariant functor from Σ into the category of Kripke frames such that $F(\sigma) : F(n) \longrightarrow F(m)$ is a p–morphism for every $\sigma : m \to n$. We call the members of F_n *n–points*.[10]

The idea is this. $F(0)$ represents the frame of possible worlds, and $F(n)$ for $n > 0$, represents n–tuples over worlds. The arrows are needed to identify the worlds and the *abstract tuples*. For example, there is a unique map $0_n : 0 \to n$ for each n. Consequently, we have a map $M(0_n) : M(n) \to M(0)$. Thus, for each $a \in F(n)$, the *world* of a is $M(0_n)(a)$. Further, there is a unique map $i_{n,n+1} : n \to n+1 : i \mapsto i$. Hence, we define a *projection* of $a \in F(n+1)$ onto $F(n)$ by $M(i_{n,n+1})(a)$.

Actually, a metaframe has to meet further (algebraic) soundness conditions in order to provide a proper semantics for MPLs, but we have to omit the details. Informally, we now evaluate a formula—depending on the number of its free variables—at an n–point in the metaframe. In fact,

[10] A modal algebra $\mathbb{F}_n$ based on the frame $F(n) = \langle F_n, \lhd_n \rangle$ is a set of subsets of F_n closed under Boolean operations and the operation $\triangle$ defined via $\triangle A := \{x \in F_n \mid \forall y \in F_n : x \lhd_n y \to y \in A\}$. An interpretations into a general frame assigns elements of $\mathbb{F}_n$ to the propositional variables. Further, a p-morphism $M(\sigma) : M(n) \longrightarrow M(m)$ satisfies (i): $x \lhd_n y \to M(\sigma)(x) \lhd_m M(\sigma)(y)$ for all $x, y \in F_n$, (ii): $M(\sigma)(x) \lhd_m u \to \exists y \in F_n(u = M(\sigma)(y) \wedge x \lhd_n y)$ for all $x \in F_n$ and $u \in F_m$, and (iii): $M(\sigma)^{-1}$ is an algebra homomorphism from $\mathbb{F}_m$ to $\mathbb{F}_n$. For further details compare, e.g., [12].

it can be the case that there are more n-points than n–tuples of individuals from $F(1)$, which underlines that the notion of an individual in a metaframes is indeed abstract.[11]

If the n–points correspond exactly to the n–tuples from $M(1)$, we speak of *cartesian metaframes*. It has been shown in [18] that every functor frame corresponds to a cartesian metaframe.

Finally, a completeness result for the class of all second–order closed modal predicate logics based on standard first order logic and laws of equality can be found in [17].

Theorem 3 (Shirasu 1998). *All second–order closed modal predicate logics are complete with respect to general metaframes.*

3.2 Counterpart and Coherence Frames

I now come to the so-called *counterpart frames* which have been defined in [13]. This semantics is actually quite close to the functor semantics from the last section, yet it is formulated without the 'padding' of categorial language—thus skirting around **QS4** as a base logic—and, on the other hand, picks up some ideas from David Lewis' Counterpart Theory (*cf.* [16]) to deal, e.g., with the failure of the principle of the *necessity of identity*. Before we properly define counterpart frames let us fix some notation. We call a 2–place relation $C \subseteq D_i \times D_j$ a *CE–relation* (CE for counterpart existence), if for all $d \in D_i$ there exists some $e \in D_j$ such that $\langle d, e \rangle \in C$. This condition is needed to ensure the *bivalence* of the semantics and also to establish the usual **K**–axioms, *i.e.*, *normality*.

Definition 5 (Counterpart frames). A **counterpart frame** is a pair $\mathfrak{F} = \langle \mathcal{W}, \mathcal{C} \rangle$, where $\mathcal{W} = \{D_i \mid i \in I\}$ is some family of first–order domains and $\mathcal{C}$ is a set of families $\mathcal{C}(D_i, D_j)$ of CE–relations between each pair of domains from $\mathcal{W}$.[12]

Accessibility is then defined by $D_i \lhd D_j$ iff $\mathcal{C}(D_i, D_j) \neq \emptyset$. Again, truth of a modal formula is defined with respect to an interpretation $\mathfrak{I}$, a valuation β and a world $w \in \mathcal{W}$. I give the clause for the modal case only.

[11] For an analysis of this compare [14].

[12] Because we work with free logic, either assume that the language contains an existence predicate, or that the first–order domains consist of an inner and an outer domain—the former being the domain for the actualist quantifiers.

$\langle w, \mathfrak{I}, \beta \rangle \models \Diamond\varphi(\bar{x})$ iff there is some $v \in \mathcal{W}$ and $C \in \mathcal{C}(w, v)$ such that $\langle \beta(x_i), \widetilde{\beta}(x_i) \rangle \in C (i = 1 \ldots n)$ and $\langle v, \mathfrak{I}, \widetilde{\beta} \rangle \models \varphi(\bar{x})$.

It should be obvious that neither the necessity of identity nor the necessity of distinctness are valid in this semantics, because instead of functions (as in the functor semantics) we now use relations to relate the individuals from one world with those of another.

Of course, Theorem 2 generalizes to these semantics because counterpart frames are just a generalization and reformulation of functor frames. Basically, we replace functions by relations and remove the conditions imposed by using the categorial language. But it is open whether the (rather complicated) completeness proof from [8] might be substantially simplified using the techniques from [13] or how far the class of frame complete logics can be extended beyond the class of extensions of canonical propositional logics. Yet, interesting frame classes can be defined by imposing appropriate conditions on the families of counterpart relations, giving a new perspective on a *correspondence theory* in the first–order case, *cf.* [13, **?**].

The counterpart frames as presented above are not complex enough to yield general completeness, quite analogous to propositional modal logic. To arrive at the desired completeness result one has to add modal algebras of 'admissible interpretations' which leads to the concept of *general counterpart frames* and to the following theorem from [13].

Theorem 4 (Kracht & Kutz 2000). *Every first- or second–order closed modal predicate logic is complete with respect to general counterpart frames.*

As a last example of generalized semantics I want to mention the *coherence frames* of [14]. These are close to standard constant domain frames in that they comprise standard Kripke frames plus a global domain of (possibilist) modal individuals. But modal individuals are assumed to have an internal structure, namely, when talking about an object *at a world* we assume that we don't talk about the modal individual per se, but rather its world–bound realization. We call these realizations of individuals *things* and refer to them as the *trace* of an individual. Therefore, what is generalized is the notion of identity at a world and more generally the interplay between interpretations of predicate symbols at a world and identity.

In coherence semantics one thinks of an individual as being world–transcendent which leads for example to a uniform treatment of constants and variables. Here is the exact definition.

Definition 6 (Coherence Frames and Structures). By a **coherence frame** we understand a quintuple $\mathfrak{F} = \langle W, \lhd, U, T, \tau \rangle$, where $\langle W, \lhd \rangle$ is a Kripke frame, $U \neq \emptyset$ the set of **objects**, $T \neq \emptyset$ the set of **things**, and $\tau : U \times W \to T$ a function. We call τ the **trace function** and $\tau(o, w)$ the **trace of** *o in w*. An *interpretation* is a function $\mathfrak{I}$ mapping each n-place predicate symbol P to a function from W to U^n and each constant symbol c to a member of U. $\mathfrak{I}$ is called **equivalential** if for all $\boldsymbol{a}, \boldsymbol{b} \in U^n$ and $w \in W$, if $\tau(a_i, w) = \tau(b_i, w)$ for all $i < n$ then $\boldsymbol{a} \in \mathfrak{I}(P)(w)$ iff $\boldsymbol{b} \in \mathfrak{I}(P)(w)$. A **coherence structure** is a sextuple $\mathcal{S} = \langle W, \lhd, U, T, \tau, \mathfrak{I} \rangle$ where $\langle W, \lhd, U, T, \tau \rangle$ is a coherence frame and $\mathfrak{I}$ an equivalential interpretation.

This notion of equivalence is perhaps a curious one. It says that atomic predicates cannot discriminate between objects of equal trace. So, if Pierre believes that London is beautiful and Londres is not, we have two (intensional) objects which happen to have the same trace in this world. Hence they must share all properties *in this world*. So, London and Londres can only be both beautiful or both ugly. This seems very plausible indeed. From a technical point of view, however, the fact that they cannot simply have different properties is a mere stipulation on our part and reflects the fact that we are dealing with purely extensional rather than intensional properties. An alternative setup for strictly extensional atomic predicates would be to assign not tuples of objects but tuples of things to predicates and to do without equivalential interpretations. Then an object bears a property at a world if and only if its trace does. Yet, technically it amounts to the same.

For definiteness, I should give at least the truth definitions for atomic predicates, identity, the existential quantifier and the modal operator. Note that valuations assign objects and not traces to variables.

$$\begin{array}{lcl}
\langle \mathcal{S}, \beta, w \rangle \vDash P(\boldsymbol{x}) & \Leftrightarrow & \beta(\boldsymbol{x}) \in \mathfrak{I}_w(P) \\
\langle \mathcal{S}, \beta, w \rangle \vDash x \doteq y & \Leftrightarrow & \tau(\beta(x), w) = \tau(\beta(y), w) \\
\langle \mathcal{S}, \beta, w \rangle \vDash \bigvee x.\varphi & \Leftrightarrow & \text{for some } \gamma \text{ with } \gamma \sim_x \beta : \langle \mathcal{S}, \gamma, w \rangle \vDash \varphi \\
\langle \mathcal{S}, \beta, w \rangle \vDash \Diamond \varphi & \Leftrightarrow & \text{there is } w' : w \lhd w' \text{ and } \langle \mathcal{S}, \beta, w' \rangle \vDash \varphi
\end{array}$$

The addition of modal algebras which are similar to those of [13] yields the following theorem, which is from [14].

Theorem 5 (Kracht & Kutz 2001). *Every modal predicate logic is complete with respect to general coherence frames.*

Finally, for information on algebraic semantics that also cover the case of super–intuitionistic predicate logics, consult [19], and for first–order intensional logics that allow quantification over both, objects and individual concepts, confer [3].

4 Final Comments

The diversification of model classes in the case of classical Kripke semantics obstructs the development of a proper model theory of languages of first–order modal logic. Because modal systems that are modelled by incompatible model classes can't be properly, comparatively studied, this is a significant systematic deficit.

Unlike standard Kripke semantics, generalized semantics provide a general framework for the model-theoretic study of MPLs. On the other hand, most of the intuitions that underlie the standard approach are lost or somewhat obscured. For example, the central concept of a modal individual becomes a derived notion in semantics such as metaframes or counterpart frames.

Therefore, a thorough analysis of the concept of 'modal individual' that underlies generalized semantics is required. The major open problem in the area is hence to harmonize the intuitions behind modal predicate logic with the apparatus of generalized semantics and to clarify the connections between the different proposed semantics. The first steps in this direction may be found in [14] and [3] but much has still to be done.

After all, modal predicate logic is not only about propositions, but about (changing) individuals and their (changing) properties—whatever that means.

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