Semantic Technologies

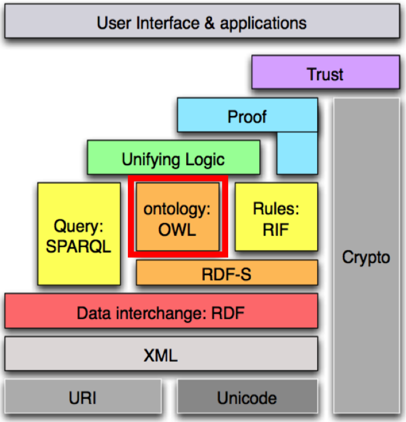
Part 16: Tableaux Procedures

Werner Nutt

Acknowledgment

These slides are based on the Latex version of slides
by Sebastian Rudolph of TU Dresden

Tableau Calculus



Agenda

- **Basic Idea of the Tableau Calculus**
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- Summary

Automation

- by now: ad hoc arguments about satisfiability of DL axioms
- a concept is satisfiable, if it has a model
  $\leadsto$ idea: constructive decision procedure that tries to build models
- analog: truth tables in propositional logic

$$(p \lor q) \rightarrow (\neg p \lor \neg q)$$

negation in front of complex expressions and non-atomic operators difficult to handle, thus reformulate:

$$\begin{aligned} \neg(p \lor q) \lor (\neg p \lor \neg q) \\ (\neg p \land \neg q) \lor (\neg p \lor \neg q) \\ (\neg p \land \neg q) \lor \neg p \lor \neg q \end{aligned}$$

Agenda

- Basic Idea of the Tableau Calculus
- **Propositional Example**
- Transformation into Negation Normal Form
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- Summary

Simple Tableau

$$(\neg p \wedge \neg q) \vee \neg p \vee \neg q$$

$\neg p \wedge \neg q$ | $\neg p$ | $\neg q$

$\neg p$

$\neg q$

- disjunctions lead to *branches* in the tableau
- tableau: finite set of tableau branches
- compare: truth table

$I(p)$	$I(q)$	$I(\neg p)$	$I(\neg q)$	$I(p \vee q)$	$I(\neg p \vee \neg q)$	$I((p \vee q) \rightarrow (\neg p \vee \neg q))$
t	t	f	f	t	f	f
t	f	f	t	t	t	t
f	t	t	f	t	t	t
f	f	t	t	f	t	t

Simple Tableau with Contradiction

$$(\neg p \vee q) \wedge p \wedge \neg q$$

$$\neg p \vee q$$

$$p$$

$$\neg q$$

$$\begin{array}{ccc} & & \\ \neg p & \qquad & q \\ \bot & & \bot \end{array}$$

- if a branch contains an atomic contradiction (clash), we call this branch *closed*
- a tableau is *closed*, if all its branches are
- a complete tableau without open branches shows the formula's unsatisfiability

Constructing a Model from the Tableau

$$(\neg p \wedge \neg q) \vee \neg p \vee \neg q$$

$\neg p \wedge \neg q$	$\neg p$	$\neg q$
$\neg p$		
$\neg q$		

- given an open branch, we can construct a model
- let I(p)=false and let I(q)=false
- let I(p)=false (I(q) is irrelevant since not in the branch, default assignment false)
- let I(q)=false (I(p) is irrelevant since not in the branch, default assignment false)

Propositional Tableau

- not always exponentially many combinations have to be checked (as opposed to truth table method)
- branches can be built one after the other $\leadsto$ only polynomial space needed
- if we care about satisfiability we can stop after constructing the first complete open branch

Construction with only one Branch in Memory

$$(\neg p \lor q) \land p \land q$$
$$\neg p^{1a} \lor q^{1b}$$
$$p$$
$$q$$
$$\cancel{\neg p^{1a}}$$
$$\cancel{\bot^{1a}}$$
$$q^{1b}$$

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
- when encountering a contradiction caused by a choice, remove marked formulae and try next choice

From Propositional Tableau to Tableau for DLs

How can the tableaux be extended for checking satisfiability of $\mathcal{ALC}$ concepts?
NB: initially, we assume no underlying knowledge base, thus unsatisfiability means that the concept is contradictory "by itself".

- tableau represents an element of the domain (plus its "environment")
- tableau branch: finite set of propositions of the form $C(a)$, $r(a, b)$
- for existential quantifiers, new domain elements are introduced
- universal quantifiers propagate formulae (=concept expressions) to neighboring elements

Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- **Transformation into Negation Normal Form**
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- Summary

Propositional Logic – Some Logical Equivalences

- We aim at negations being present only in front of atomic concepts

$$\varphi \wedge \psi \equiv \psi \wedge \varphi$$
$$\varphi \vee \psi \equiv \psi \vee \varphi$$

$$\varphi \wedge (\psi \wedge \omega) \equiv (\varphi \wedge \psi) \wedge \omega$$
$$\varphi \vee (\psi \vee \omega) \equiv (\varphi \vee \psi) \vee \omega$$

$$\varphi \wedge \varphi \equiv \varphi$$
$$\varphi \vee \varphi \equiv \varphi$$

$$\varphi \wedge (\psi \vee \varphi) \equiv \varphi$$
$$\varphi \vee (\psi \wedge \varphi) \equiv \varphi$$

$$\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$$
$$\varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$

$$\neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi$$
$$\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$$

$$\neg\neg\varphi \equiv \varphi$$

$$\varphi \vee (\psi \wedge \omega) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \omega)$$
$$\varphi \wedge (\psi \vee \omega) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \omega)$$

Further Logical Equivalences

$$
\begin{aligned}
\neg(C \sqcap D) &\rightsquigarrow \neg C \sqcup \neg D \\
\neg(D \sqcup D) &\rightsquigarrow \neg C \sqcap \neg D \\
\neg\neg C &\rightsquigarrow C
\end{aligned}
$$

$$
\begin{aligned}
\neg(\forall r.C) &\rightsquigarrow \exists r.(\neg C) \\
\neg(\exists r.C) &\rightsquigarrow \forall r.(\neg C) \\
\neg(\leqslant n\, s.C) &\rightsquigarrow \geqslant n+1\, s.C \\
\neg(\geqslant n\, s.C) &\rightsquigarrow \leqslant n-1\, s.C, \quad n \geq 1 \\
\neg(\geqslant 0\, s.C) &\rightsquigarrow \bot
\end{aligned}
$$

- apply these rules iteratively until none can be applied any more

$\rightsquigarrow$ equivalent concept in negation normal form

NNF Transformation

recursive definition of an NNF transformation:

if C atomic:

$$NNF(C) := C \qquad NNF(\neg C) := \neg C$$

otherwise:

$$NNF(\neg\neg C) := NNF(C)$$

$$NNF(C \sqcap D) := NNF(C) \sqcap NNF(D) \qquad NNF(\neg(C \sqcap D)) := NNF(\neg C) \sqcup NNF(\neg D)$$

$$NNF(C \sqcup D) := NNF(C) \sqcup NNF(D) \qquad NNF(\neg(C \sqcup D)) := NNF(\neg C) \sqcap NNF(\neg D)$$

$$NNF(\forall r.C) := \forall r.(NNF(C)) \qquad NNF(\neg(\forall r.C)) := \exists r.(NNF(\neg C))$$

$$NNF(\exists r.C) := \exists r.(NNF(C)) \qquad NNF(\neg(\exists r.C)) := \forall r.(NNF(\neg C))$$

$$NNF(\leqslant n\, s.C) := \leqslant n\, s.(NNF(C)) \qquad NNF(\neg(\leqslant n\, s.C)) := \geqslant n+1\, s.(NNF(C))$$

$$NNF(\geqslant n\, s.C) := \geqslant n\, s.(NNF(C)) \qquad NNF(\neg(\geqslant n\, s.C)) := \leqslant n-1\, s.(NNF(C)) \quad \text{if } n \geq 1$$

$$NNF(\geqslant 0\, s.C) := \top \qquad NNF(\neg(\geqslant 0\, s.C)) := \bot \quad \text{otherwise}$$

NNF Transformation – Example

$$
\begin{aligned}
& NNF(\neg(\neg C \sqcap (\neg D \sqcup E))) \\
= {} & NNF(\neg\neg C) \sqcup NNF(\neg(\neg D \sqcup E)) \\
= {} & NNF(C) \sqcup NNF(\neg(\neg D \sqcup E)) \\
= {} & C \sqcup NNF(\neg(\neg D \sqcup E)) \\
= {} & C \sqcup (NNF(\neg\neg D) \sqcap NNF(\neg E)) \\
= {} & C \sqcup (NNF(D) \sqcap NNF(\neg E)) \\
= {} & C \sqcup (D \sqcap NNF(\neg E)) \\
= {} & C \sqcup (D \sqcap \neg E)
\end{aligned}
$$

Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- **Satisfiability of $\mathcal{ALC}$ Concepts**
- Correctness and Termination
- Summary

Tableau for $\mathcal{ALC}$ Concepts

- tableau for a propositional formulal α: one element, labeled with subformulae of α
- tableau for an $\mathcal{ALC}$ concept C: graph (more precisely: tree) where the nodes are labeled with subformulae of C
- root labeled with C
- represents model for C (if complete and clash-free)
- non-root nodes are enforced by existential quantifiers

Definition

Let C be an $\mathcal{ALC}$ concept, $\mathsf{SF}(C)$ the set of all subformulae of C and $\mathsf{Rol}(C)$ the set of all roles occurring in C. A *tableau for* C is a tree $G = \langle V, E, L \rangle$, with nodes V, edges $E \subseteq V \times V$ and a labeling function L with $L\colon V \to 2^{\mathsf{SF}(C)}$ and $L\colon V \times V \to 2^{\mathsf{Rol}(C)}$.

Properties of the $\mathcal{ALC}$ Tableau Algorithm

- the algorithm is specified as a set of rules
- every rule breaks down a complex concept into its parts
- rules applicable in any order
- the algorithm is non-deterministic (due to disjunction)
- check for atomic contradictions

tableau algorithm for checking satisfiability of $\mathcal{ALC}$ concepts

Input: an $\mathcal{ALC}$ concept in NNF

Output: `true` if there is a clash-free tableau where no more rules can be applied

`false` otherwise (tableau closed)

Tableau Rules for $\mathcal{ALC}$ Concepts

$\sqcap$-rule: For an **arbitrary** $v \in V$ mit $C \sqcap D \in L(v)$ and $\{C, D\} \not\subseteq L(v)$, let $L(v) := L(v) \cup \{C, D\}$.

$\sqcup$-rule: For an **arbitrary** $v \in V$ with $C \sqcup D \in L(v)$ and $\{C, D\} \cap L(v) = \emptyset$, **choose** $X \in \{C, D\}$ and let $L(v) := L(v) \cup \{X\}$.

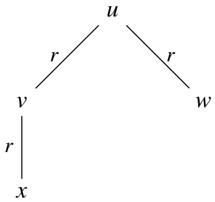
$\exists$-rule: For an **arbitrary** $v \in V$ with $\exists r.C \in L(v)$ such that there is no r-successor v' of v with $C \in L(v')$, let $V = V \cup \{v'\}$, $E = E \cup \{\langle v, v' \rangle\}$, $L(v') := \{C\}$ and $L(v, v') := \{r\}$ for v' a new node.

$\forall$-rule: For **arbitrary** $v, v' \in V$, v' r-neighbor of v, $\forall r.C \in L(v)$ and $C \notin L(v')$, let $L(v') := L(v') \cup \{C\}$.

- a node v' is an **r-neighbor** of a node v if $\langle v, v' \rangle \in E$ and $r \in L(v, v')$
- rule application order: “don’t care” non-determinism
- choice of disjunction: “don’t know” non-determinism

Tableau Algorithmus Example

$$C = \exists r.(A \sqcup \exists r.B) \sqcap \exists r.\neg A \sqcap \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))$$



$$\begin{aligned}
L(u) &= \{C\}, \exists r.(A \sqcup \exists r.B), \\
&\qquad \exists r.\neg A, \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))\} \\
L(v) &= \{A \sqcup \exists r.B\}, \neg A, \forall r.(\neg B \sqcup A)\}, \cancel{A}\}, \exists r.B\} \\
L(w) &= \{\neg A\}, \forall r.(\neg B \sqcup A)\} \\
L(x) &= \{B\}, \neg B \sqcup A\}, \cancel{\neg B}\}, A\}
\end{aligned}$$

Tableau Algorithm Example

the model $\mathcal{I}$ constructed by the algorithm is the following:

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{u, v, w, x\} \\
A^{\mathcal{I}} &= \{x\} \\
B^{\mathcal{I}} &= \{x\} \\
r^{\mathcal{I}} &= \{\langle u, v\rangle, \langle u, w\rangle, \langle v, x\rangle\}
\end{aligned}$$

Check that indeed $C^{\mathcal{I}} = \{u\}$, given the defined semantics of $\mathcal{ALC}$

Tableau Algorithm Properties

1. the model is **finite**: only finitely many elements in the domain
2. the model is **tree-shaped**: the tableau is a labeled tree

the algorithm will *always* construct finite trees

- from a clash-free tableau, we can construct a finite model
- if there is no clash-free tableau, there is no model

Agenda

Tableau Properties

- the depth (number of nested quantifiers) decreases in every node
- every node is labeled only with subformulae of C
- C has only polynomially many subformulae
- if the ouput is `true` we can build a model out of the constructed tableau
- on the other hand, we can use a model of a satisfiable concept to construct a clash-free tableau for this concept

Tableau Algorithm for $\mathcal{ALC}$ Concepts

Theorem

1. the algorithm terminates for every input
2. if the output is `true`, then the input concept is satisfiable
3. if the input concept is satisfiable, then the output is `true`.

Corollary

Every $\mathcal{ALC}$ concept C has the following properties:

1. **finite model property**: If C has a model, then it has a finite one.
2. **tree model property** If C has a model, then it has a tree-shaped one.

Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- **Summary**

Summary

- we now have a constructive method for building model abstractions
- satisfiable $\mathcal{ALC}$ concepts always have a finite model that we can construct
- the algorithm is correct, complete and terminating
- serves as basis for practically implemented algorithms
- next: extension to knowledge bases