Looking Back ... 

We have reviewed three formalisms for expressing queries

- Relational Algebra
- Relational Calculus (with its domain-independent fragment)
- Nice SQL

and seen that they have the same expressivity

However, crucial properties ((un)satisfiability, equivalence, containment) are undecidable

Hence, automatic analysis of such queries is impossible

Can we do some analysis if queries are simpler?
Many Natural Queries Can Be Expressed . . .

... in SQL
- using only a single SELECT-FROM-WHERE block and conjunctions of atomic conditions in the WHERE clause;
- we call these the CSQL queries.

... in Relational Algebra
- using only the operators selection $\sigma_C(E)$, projection $\pi_C(E)$, join $E_1 \Join_C E_2$, renaming $(\rho_{A \leftarrow B}(E))$;
- we call these the SPJR queries (= select-project-join-renaming queries)

... in Relational Calculus
- using only the logical symbols “$\land$” and $\exists$ such that every variable occurs in a relational atom;
- we call these the conjunctive queries
Conjunctive Queries

Theorem

The classes of CSQL queries, SPJR queries, and conjunctive queries have all the same expressivity. Queries can be equivalently translated from one formalism to the other in polynomial time.

Proof.

By specifying translations.

Intuition: By a conjunctive query we define a pattern of what the things we are interested in look like. Evaluating a conjunctive query is matching the pattern against the database instance.
Rule Notation for Conjunctive Queries

By pulling the quantifiers outside, every conjunctive calculus query can be written as

\[ Q = \{ (x_1, \ldots, x_k) \mid \exists y_1, \ldots, \exists y_l (A_1 \land \cdots \land A_m) \}, \]

where \( A_1, \ldots, A_m \) are (relational and built-in) atoms

We say that \( x_1, \ldots, x_k \) are the distinguished variables of \( Q \) and \( y_1, \ldots, y_m \) the nondistinguished variables

We will often write such a query, using a rule in the style of PROLOG, as

\[ Q(\bar{x}) :– A_1, \ldots, A_m \]

We say \( Q(x_1, \ldots, x_k) \) is the head of the query and \( A_1, \ldots, A_m \) the body

Note: Existential quantifiers are implicit, since we list the free variables in the head.
Semantics of Conjunctive Queries

Consider a **conjunctive formula**

\[ \phi = \exists y_1, \ldots, y_l (A_1 \land \cdots \land A_m) \]

such that

- \( A_1, \ldots, A_m \) are atoms, with relational or built-in predicates
- \( \bar{x} = (x_1, \ldots, x_k) \) is the tuple of free variables of \( \phi \)
- every variable occurs in a relational atom

Then \( Q_\phi \) is a conjunctive query

**Proposition (Answer Tuple for a Calculus Query)**

Let \( I \) be an instance. A \( k \)-tuple of constants \( \bar{c} \) is an **answer tuple** for \( Q_\phi \) over \( I \) if and only if there is an assignment \( \alpha \) such that

- \( \bar{c} = \alpha(\bar{x}) \)
- \( I, \alpha \models A_j \) for \( j = 1, \ldots, m \)
Schematic Notation of Conjunctive Queries

\[ Q(\bar{x}) :\leftarrow L, M, \]

where

- \( L = R_1(\bar{t}_1), \ldots, R_n(\bar{t}_n) \) is a conjunction of relational atoms
- \( M = B_1, \ldots, B_p \) is a conjunction of built-in atoms (that is, with predicates “\(<\)”, “\(\leq\)”, “\(\neq\)”),
- every variable occurs in some \( R_j(\bar{t}_j) \) (guarantees safety of \( Q \))

Proposition (Answer Tuple for a Rule)

The tuple \( \bar{c} \) is an answer for \( Q \) over \( \mathbf{I} \) iff there is an assignment \( \alpha \) such that

- \( \bar{c} = \alpha(\bar{x}) \)
- \( \alpha(\bar{t}_j) \in I(R_j), \text{ for } j = 1, \ldots, n \)
- \( \alpha \models M \)
Conjunctive Queries: Logic Programming (LP) Perspective

\( I \) finite set of ground facts (\( \equiv \) instance in LP perspective)

**Proposition (Answer Tuple in LP Perspective)**

The tuple \( \bar{c} \) is an answer for \( Q(\bar{x}) :– L, M \) over \( I \)
iff there is an assignment \( \alpha \) for the variables of \( \phi \) such that

- \( \bar{c} = \alpha(\bar{x}) \)
- \( \alpha(L) \subseteq I \)
- \( \alpha \models M \)

Note that for relational conjunctive queries (i.e., w/o built-ins), satisfaction of \( Q \) by \( \alpha \) over \( I \) boils down to

\[ \alpha(L) \subseteq I \]
Elementary Properties of Conjunctive Queries

Proposition (Properties of Conjunctive Queries)
Let $Q(\bar{x}) :– L, M$ be a conjunctive query. Then

1. the answer set $Q(I)$ is finite for all instances $I$
2. $Q$ is monotonic, that is, $I \subseteq J$ implies $Q(I) \subseteq Q(J)$ for all instances $I$, $J$
3. $Q$ is satisfiable if and only if $M$ is satisfiable

Proof.

1. Holds due to safety condition and finiteness of $I$
2. Follows easily with LP perspective
3. Exercise!
Evaluation of Conjunctive Queries

Consider

\[ Q(x_1, \ldots, x_k) := R_1(t_1), \ldots, R_m(t_m), M \]

How difficult is it to compute \( Q(I) \)?

By definition,

\[ Q(I) = \{ \alpha(\bar{x}) \mid \alpha(\bar{t}_j) \in R_j(I), \text{ for } j = 1, \ldots, n, \text{ and } \alpha \models M \} \]

Naïve algorithm:

\[
\text{for each assignment } \alpha: \text{var}(Q) \rightarrow \text{adom}(I) \text{ do }
\]
\[
\text{for each } R_j(\bar{t}_j), \text{ where } j = 1, \ldots, n, \text{ do }
\]
\[
\text{check whether } \alpha(\bar{x}) \in R_j(I)
\]
\[
\text{check whether } \alpha \models M
\]
\[
\text{if all checks are positive }
\]
\[
\text{return } \alpha(\bar{x})
\]
Evaluation of Conjunctive Queries: Running Time (1)

What is the running time of the naïve algorithm?

We analyze the structure:

- **Outer loop**: executed for each $\alpha: \text{var}(Q) \rightarrow \text{dom}(I)$,
  - consists of
    - inner loop
    - satisfaction check “$\alpha \models M$”
- **Inner loop**: executed for each $R_j(t_j), j = 1, \ldots, n$,
  - consists of looking up $\alpha(t_j)$ in $R_j(I)$
Evaluation of Conjunctive Queries: Running Time (2)

We calculate the running time:

- **Number of assignments** $\alpha : \text{var}(Q) \rightarrow \text{adom}(I)$,
  
  \[
  \#\text{adom}(I) \cdot \#\text{var}(Q) = O(|I|^{|Q|})
  \]

- **Number of relational atoms** $R_j(\vec{t}_j)$ is $n = O(|Q|)$

- **Checking \( \alpha(\vec{t}_j) \) in \( R_j(I) \)** is done by $O(|I|)$ equality checks

- **Checking \( \alpha \models M \)** is done by checking each atom of $M$, which amounts to $O(|Q|)$ checks

This gives an **upper bound** on the total running time:

\[
O(|I|^{|Q|}) \cdot (O(|Q|) \cdot O(|I|) + O(|Q|)) = O(|I|^{|Q|} \cdot |Q| \cdot |I|)
\]

\[
= O(|I|^{|Q|})
\]

**Remark:** With $\#X$ we denote the cardinality of a set $X$, with $|S|$ the size of a syntactic object $S$
Evaluation of Conjunctive Queries: Decision Problems

How difficult is it to compute $Q(I)$?

**Definition (Evaluation problem for a single conjunctive query $Q$)**

Given: instance $I$, tuple $\bar{c}$

Question: is $\bar{c} \in Q(I)$?

**Definition (Evaluation problem for the class of conjunctive queries)**

Given: conjunctive query $Q$, instance $I$, tuple $\bar{c}$

Question: is $\bar{c} \in Q(I)$?

**Note:**

First problem: $Q$ is fixed (Data Complexity)

Second problem: $Q$ is part of the input (Combined Complexity)
Reminder on the Class NP

NP = the class of problems that can be decided by a nondeterministic Turing machine in polynomial time.

We compare problems in terms of reductions:
For two problems $P_1 \in \Sigma_1^*$, $P_2 \in \Sigma_2^*$, a function $f : \Sigma_1^* \rightarrow \Sigma_2^*$ is a polynomial time many-one reduction (or Karp reduction) of $P_1$ to $P_2$ if and only if

- $s_1 \in P_1 \iff f(s_1) \in P_2$ for all $s_1 \in \Sigma_1^*$
- $f$ can be computed in polynomial time

We write $P_1 \leq_m P_2$ if there is a Karp reduction from $P_1$ to $P_2$.
The relation “$\leq_m$” is a preorder (= reflexive, transitive relation)

**Theorem (Cook, Karp)**

There are problems in NP that are maximal wrt “$\leq_m$”.

These problems are called NP-complete.
Evaluation of Conjunctive Queries: Complexity

**Proposition (Data Complexity)**

For every conjunctive query $Q$, there is a polynomial $p$, such that the evaluation problem can be solved in time $O(p(|I|))$.

**Idea:** $Q$ can be written as a selection applied to a cartesian product. What is the width of the cartesian product?

Hence, data complexity is in PTIME. Actually, data complexity of evaluating arbitrary FO (i.e., algebra or calculus) queries is in LOGSPACE (even in $AC_0$).

**Proposition (Combined Complexity)**

Given $Q(\vec{x}) :– L, M, I$ and $\vec{c}$, one can guess in linear time an $\alpha$ such that

- $\alpha$ satisfies $L, M$ over $I$
- $\alpha(\vec{x}) = \vec{c}$

Hence, combined complexity is in NP. Is evaluation also NP-hard?
The 3-Colorability Problem

Definition (3-Colorability of Graphs)

- **Instance**: A graph \( G = (V, E) \)
- **Question**: Can \( G \) be colored with the three colors \( \{r, g, b\} \) in such a way that two adjacent vertices have a distinct color?

The 3-colorability problem is NP-complete

A graph \( G \) is 3-colourable if and only if there is a graph homomorphism from \( G \) to the simplex \( S_3 \), which consists of three vertices that are connected to each other.
Reducing 3-Colorability to Evaluation

**Theorem (Reduction)**

There is a database instance $I_{3col}$ such that for every finite graph $G$ one can compute in linear time a relational conjunctive query $Q_G(\cdot) := L$ such that

$$G \text{ is 3-colorable if and only if } Q_G(I_{3col}) = \{()\}$$

**Remark (Boolean Queries)**

- A query without distinguished variables is called a *boolean* query
- Over an instance, a boolean query returns the empty tuple $(\cdot)$, or nothing

This shows NP-hardness of the combined complexity of conjunctive query evaluation
The Reduction

Given graph $G = (V, E)$, where 

$V = \{v_1, \ldots, v_n\}$ and 

$E = \{(v_{i_l}, v_{j_l}) \mid i_l < j_l, \ 1 \leq l \leq m\}$

We construct $I_{3col}$ and $Q_G$ as follows

$I_{3col} = \{e(r, b), e(b, r), e(r, g), e(g, r), e(b, g), e(g, b)\}$

$Q_G() \leftarrow e(y_{i_1}, y_{j_1}), \ldots, e(y_{i_m}, y_{j_m})$

where $y_1, \ldots, y_n$ are new variables and

there is one atom $e(y_{i_l}, y_{j_l})$ for each edge $(v_{i_l}, v_{j_l}) \in E$

Clearly, there is an $\alpha: \{y_1, \ldots, y_n\} \rightarrow \{r, g, b\}$ satisfying $Q_G$ over $I_{3col}$

iff there is a graph homomorphism from $G$ to $S_3$
Evaluation of Conjunctive Queries in Practice

- To assess the practical difficulty of query evaluation, one usually looks only at **data complexity**: the size of the query is (very!) small compared to the size of the data.

- **Query optimizers** try to find plans that minimize the cost of executing conjunctive queries:
  - Find a good **ordering of joins**
  - Identify the best **access paths** to data (indexes)

The DBMS keeps **statistics** about size of relations and distribution of attribute values to estimate the cost of plans.

- Query optimization is well understood for a single DBMS, but more **difficult if data sources are distributed**
  - often, info about access paths and statistics are missing in data integration scenarios
  - execution plans need to be changed on the fly
The 3-Satisfiability Problem

Ingredients

- Propositions $p_1, \ldots, p_n, \ldots$
- Literals $l$: proposition ($p$) or negated propositions ($\neg p$)
- 3-Clauses $C$: disjunctions of three literals ($l_1 \lor l_2 \lor l_3$)

Definition (3-Satisfiability)

Given: a finite set $C$ of 3-clauses

Question: is $C$ satisfiable, i.e., is there a truth assignment $\alpha$ such that $\alpha$ makes at least one literal true in every $C \in C$?

The 3-Sat Problem is the classical NP-complete problem

Next, we will use a reduction of 3-Satisfiability to Evaluation . . .
Alternate Reduction From 3-Satisfiability

**Theorem**

For every set of 3-clauses \( C \), there is an instance \( I_C \) and a boolean relational query \( Q_C \) such that

\[ C \text{ is satisfiable } \iff Q_C(I_C) \neq \emptyset \]

**Definition of \( I_C \) and \( Q_C \).**

Let \( C = \{C_1, \ldots, C_m\} \) and consider propositions as variables.

- For every clause \( C_i \in C \), choose a relation symbol \( R_i \).
- Let \( p_1^{(i)}, p_2^{(i)}, p_3^{(i)} \) be the propositions in the clause \( C_i \).
- Let \( T_i = \{\bar{t}_1^{(i)}, \ldots, \bar{t}_7^{(i)}\} \) be the seven triples of truth values that satisfy \( C_i \).
  
  E.g., if \( C_i = p_2 \lor \neg p_4 \lor p_7 \), then \( T_i = \{0, 1\}^3 \setminus \{(0, 1, 0)\} \).
- Define \( I_C = \bigcup_i \{R_i(\bar{t}) \mid \bar{t} \in T_i\} \).
- Define \( Q_C() := R_1(p_1^{(1)}, p_2^{(1)}, p_3^{(1)}), \ldots, R_m(p_1^{(m)}, p_2^{(m)}, p_3^{(m)}) \).
Properties of Conjunctive Queries

Satisfiability can be decided in PTIME, since satisfiability of a conjunction of comparisons can be decided in PTIME.

If we can decide containment, then we can also decide equivalence, since

\[ Q_1 \equiv Q_2 \quad \text{if and only if} \quad Q_1 \subseteq Q_2 \quad \text{and} \quad Q_2 \subseteq Q_1 \]

If we can decide equivalence, we can also decide containment, since

\[ Q_1 \subseteq Q_2 \quad \text{if and only if} \quad Q_1 \equiv Q_1 \cap Q_2 \]

Why is \( Q_1 \cap Q_2 \) again a conjunctive query?

We will concentrate on containment.
Find all containments and equivalences among the following conjunctive queries:

\[ Q_1(x, y) :\neg R(x, y), R(y, z), R(z, w) \]

\[ Q_2(x, y) :\neg R(x, y), R(y, z), R(z, u), R(u, w) \]

\[ Q_3(x, y) :\neg R(x, y), R(z, u), R(v, w), R(x, z), R(y, u), R(u, w) \]

\[ Q_4(x, y) :\neg R(x, y), R(y, 3), R(3, z), R(z, w) \]
Idea: Reduce Containment to Evaluation! (1)

\[ Q'(x, y) :– R(x, y), R(y, z), R(y, u) \]
\[ Q(x, y) :– R(x, y), R(y, z), R(w, z) \]

**Step 1** Turn \( Q \) into an instance \( I_Q \) by “freezing” the body of \( Q \), i.e., replace variables \( x, y, z, w \) with constants \( c_x, c_y, c_z, c_w \): \( I_Q = \{ R(c_x, c_y), R(c_y, c_z), R(c_w, c_z) \} \)

Observe that \( (c_x, c_y) \in Q(I_Q) \)

**Idea:** \( I_Q \) is prototypical for any database where \( Q \) returns a result

**Step 2** Evaluate \( Q' \) over \( I_Q \)

**Case 1** If \( (c_x, c_y) \notin Q'(I_Q) \), then we have found a counterexample: \( Q \not\subseteq Q' \)
Case 2  If \((c_x, c_y) \in Q'(I_Q)\), then there is an \(\alpha\) such that

- \(\alpha(x) = c_x\), \(\alpha(y) = c_y\)
- \(\alpha(A) \in I_Q\) for every atom \(A'\) in the body of \(Q'\)

For instance,

\[
\alpha = \{x/c_x, y/c_y, z/c_z, u/c_z\}
\]

does the job.

With \(\alpha\) we can extend every satisfying assignment for \(Q\) to a satisfying assignment for \(Q'\), as follows:

Let \(I\) be an arbitrary db instance and \((d, e) \in Q(I)\) be an answer of \(Q\) over \(I\). Then there is an assignment \(\beta\) such that

- \(\beta(x) = d\), \(\beta(y) = e\)
- \(\beta(B) \in I\) for every atom \(B\) in the body of \(Q\).
Idea: Reduce Containment to Evaluation! (3)

Define the substitution $\alpha'$ (= mapping from terms to terms, not moving constants) by “melting” $\alpha$, that is, replacing every constant $c_v$ with the corresponding variable $v$:

$$\alpha' = \{x/x, y/y, z/z, u/z\}.$$

Define $\beta' = \beta \circ \alpha'$, that is, as composition of first $\alpha'$ and then $\beta$.

Then $\beta'(x) = \beta(\alpha'(x)) = \beta(x) = d$ and, similarly, $\beta'(y) = e$.

Moreover if $A'$ is an atom of $Q'$, then

- $\alpha'(A')$ is an atom of $Q$, since $\alpha(A') \in I_Q$
- $\beta'(A') = \beta(\alpha'(A')) \in I$, since $\beta$ maps every atom of $Q$ to a fact in $I$

Hence, $(d, e) = (\beta'(x), \beta'(y))$ is an answer of $Q'$ over $I$.

This shows, $Q(I) \subseteq Q'(I)$ for an arbitrary $I$ and thus, $Q \sqsubseteq Q'$. 
Query Homomorphisms

Definition

Consider conjunctive queries without built-ins

\[ Q'(\vec{x}) :\neg L' \]
\[ Q(\vec{x}) :\neg L \]

A mapping \( \delta: \text{Terms}(Q') \rightarrow \text{Terms}(Q) \) is a **query homomorphism** (from \( Q' \) to \( Q \)) if

- \( \delta(c) = c \) for every constant \( c \)
- \( \delta(x) = x \) for every distinguished variable \( x \) of \( Q' \)
- \( \delta(L') \subseteq L \)

Intuitively,

- \( \delta \) respects constants and distinguished variables
- \( \delta \) maps conditions of \( Q' \) to conditions in \( Q \) that are no less strict
Finding Homomorphisms

Find all homomorphisms among the following conjunctive queries:

\[ Q_1(x, y) :– R(x, y), R(y, z), R(z, w) \]
\[ Q_2(x, y) :– R(x, y), R(y, z), R(z, u), R(u, w) \]
\[ Q_3(x, y) :– R(x, y), R(z, u), R(v, w), R(x, z), R(y, u), R(u, w) \]
\[ Q_4(x, y) :– R(x, y), R(y, 3), R(3, z), R(z, w) \]

In terms of complexity, how difficult is it to decide whether there exists a homomorphism between two queries?
The Homomorphism Theorem

Theorem (Chandra/Merlin)

Let $Q'(\bar{x}) :– L'$ and $Q(\bar{x}) :– L$ be conjunctive queries (w/o built-in predicates). Then the following are equivalent:

- there exists a homomorphism from $Q'$ to $Q$
- $Q \subseteq Q'$.

Proof.

Straightforward by generalizing the previous example.
Homomorphisms between Queries with Comparisons

Example

\[ Q'(\cdot) :– R(x, y), \]
\[ x \leq 2, \ y \geq 3 \]

\[ Q(\cdot) :– R(u, v), R(v, w) \]
\[ u \geq 3, \ v \geq 0, \ v \leq 1, \ w \geq 4 \]

There are two “relational” homomorphisms:

\[ \delta = \{ x/u, \ y/v \} \]
\[ \eta = \{ x/v, \ y/w \} \]

Which of the two deserves the title of homomorphism?
Query Homomorphisms

**Definition**

Consider conjunctive queries with comparisons

\[
Q'(\bar{x}) :\neg L', M' \\
Q(\bar{x}) :\neg L, M
\]

A mapping \( \delta : \text{Terms}(Q') \to \text{Terms}(Q) \) is a **query homomorphism** if

- \( \delta(c) = c \) for every constant \( c \)
- \( \delta(x) = x \) for every distinguished variable \( x \) of \( Q' \)
- \( \delta(L') \subseteq L \)
- \( M \models \delta(M') \).

**Intuition:** With respect to \( \delta \), the comparisons in \( Q \)
are more restrictive than those in \( Q' \)
Homomorphisms between Queries with Comparisons

Example

\[ Q'(x) \leftarrow P(x, y), R(y, z), \]
\[ y \leq 3 \]
\[ Q(x) \leftarrow P(x, w), P(x, x), R(x, u), \]
\[ w \geq 5, x \leq 2 \]

The substitution

\[ \delta = \{x/x, y/x, z/u\} \]

- is a relational homomorphism
- satisfies \( w \geq 5, x \leq 2 \models \delta(y) \leq 3 \)
Does the Hom Theorem Hold for Queries w/ Comparisons?

\[ Q'(\bar{x}) :– L', M' \quad \quad Q(\bar{x}) :– L, M \]

Let \( \delta : Q' \rightarrow Q \) be an hom, \( I \) an instance. Suppose \( \bar{c} \in Q(I) \). Is \( \bar{c} \in Q'(I) \)?

Since \( \bar{c} \in Q'(I) \), there is \( \alpha \) such that

- \( \alpha(\bar{x}) = \bar{c} \)
- \( \alpha(L) \subseteq I \)
- \( \alpha \models M \).

Define \( \alpha' = \alpha \circ \delta \). Then

- \( \alpha'(\bar{x}) = \alpha(\delta(\bar{x})) = \alpha(\bar{x}) = \bar{c} \)
- \( \alpha'(L') = \alpha(\delta(L')) \subseteq \alpha(L) \subseteq I \)
- \( \alpha \models \delta(M') \), since \( \alpha \models M \) and \( M \models \delta(M') \) \( \Rightarrow \) \( \alpha \circ \delta \models M' \).

Thus, \( \bar{c} \in Q'(I) \).
We have just proved the following theorem:

**Theorem (Homomorphisms Are Sufficient for Containment)**

Let $Q'(\bar{x}) :\leftarrow L', M'$ and $Q(\bar{x}) :\leftarrow L, M$ be conjunctive queries.

If there is an homomorphism from $Q'$ to $Q$, then $Q \sqsubseteq Q'$. 

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Does the Converse Hold as Well?

Intuition:
- Blocks can be either black or white.
- Block 1 is on top of block 2, which is on top of block 3.
- Block 1 is white and block 3 is black.
- Is there a white block on top of a black block?

Example

\[ Q'(\cdot) := S(x, y), \quad x \leq 0, \quad y > 0 \]
\[ Q(\cdot) := S(0, z), \quad S(z, 1) \]
Case Analysis for $Q$

Define

\[
Q\{z<0\}() := S(0, z), S(z, 1), z < 0
\]

\[
Q\{z=0\}() := S(0, 0), S(0, 1)
\]

\[
Q\{0<z<1\}() := S(0, z), S(z, 1), 0 < z, z < 1
\]

\[
Q\{z=1\}() := S(0, 1), S(1, 1)
\]

\[
Q\{1<z\}() := S(0, z), S(z, 1), z > 1
\]

We note

- $Q$ is equivalent to the union of $Q\{z<0\}, \ldots, Q\{1<z\}$
- there is a homomorphism from $Q'$ to $Q\{\ldots\}$ for each ordering $\{\ldots\}$
- $Q\{\ldots\} \subseteq Q'$ for each $\{\ldots\}$

$\Rightarrow Q \subseteq Q'$

Idea: Replace $Q$ with $\bigcup_{\{\ldots\}} Q\{\ldots\}$ when checking "$Q \subseteq Q'$?"
Linearizations

We now make this idea formal.

- We assume that all of $\text{dom}$ is one linearly ordered type. Let
  - $D$ be a set of constants from $\text{dom}$,
  - $W$ be a set of variables,
  - and let $T := D \cup W$ denote their union.

- A **linearization** of $T$ over $\text{dom}$ is a set of comparisons $N$ over the terms in $T$ such that for any $s, t \in T$ exactly one of the following holds:
  - $N \models_{\text{dom}} s < t$
  - $N \models_{\text{dom}} s = t$
  - $N \models_{\text{dom}} s > t$.

- That is, $N$ partitions the terms into classes such that
  - the terms in each class are equal and
  - the classes are arranged in a strict linear order.
Remark: A class of the induced partition contains at most one constant.

Remark: Whether or not $N$ is a linearization may depend on the domain. Consider e.g.,

$$\{1 < x, \ x < 2\}$$

A linearization $N$ of $T$ over $\text{dom}$ is compatible with a set of comparisons $M$ if $M \cup N$ is satisfiable over $\text{dom}$.
Linearizations of Conjunctive Queries

- When checking containment of two queries, we have to consider linearizations that contain the constants of both queries.

- Let

  \[ Q(\bar{x}) := L, M \]

  be a query and

  - \( W \) be the set of variables occurring in \( Q \)
  - \( D \) be a set of constants that comprise the constants of \( Q \)

- Then we denote with \( L_D(Q) \) the set of all linearizations of \( D \cup W \) that are compatible with the comparisons \( M \) of \( Q \).
Linearizations of Conjunctive Queries (cntd)

**Proposition**

Let $Q$, $W$, $D$ and $M$ be as above and let $\alpha : W \rightarrow \text{dom}$ be an assignment. Then the following are equivalent:

- $\alpha \models M$
- $\alpha \models N$ for some $N \in \mathcal{L}_{D}(Q)$

**Proof.**

"$\Leftarrow$" Let $N \in \mathcal{L}_{D}(Q)$. Since $\text{Terms}(M) \subseteq D \cup W$, and $N$ is a linearization of $D \cup W$, we have that $N \models M$:

To see this, let $s \leq t \in M$. Then $N \models s < t$ or $N \models s = t$ or $N \models s > t$.

Since $M \cup \{s > t\}$ is unsatisfiable, we have $N \models s \leq t$.

"$\Rightarrow$" For $\alpha \models M$ let $N_{\alpha} = \{B \mid B$ is a built-in atom with terms from $D \cup W$ and $\alpha \models B\}$.

Then $N_{\alpha}$ is a linearization of $D \cup W$ compatible with $M$ and $\alpha \models N_{\alpha}$. □
Linearizations of Conjunctive Queries (cntd)

Let $Q$ be as above. Let $N$ be a linearization of $T = D \cup W$ compatible with $M$.

- **Note:** $N$ defines an equivalence relation on $T$, where each equivalence class contains at most one constant.

- A substitution $\phi$ is *canonical* for $N$ if
  - it maps all elements in an equivalence class of $N$ to one term of that class
  - if a class contains a constant, then it maps the class to that constant.

- Then $Q_N$ is obtained from $Q$ by means of a canonical substitution $\phi$ for $N$ as
  \[
  Q_N(\phi(\bar{x})) :– \phi L \land \phi N,
  \]
  that is,
  - we first replace $M$ with $N$
  - and then “eliminate” all equalities by applying $\phi$

- **Note:** We must admit also queries with a tuple of terms $\bar{s}$ in the head.
Definition (Linearization)

The queries

\[ Q_N(\phi(\bar{x})) \leftarrow \phi_L \land \phi_N, \]

are called linearizations of \( Q \) w.r.t. \( N \)

- There may be more than one linearization of \( Q \) w.r.t. \( N \), but all linearizations are identical up to renaming of variables

- Note that \( \phi \) is a homomorphism from \( Q \) to \( Q_N \)
Linear Expansions

Definition (Linear Expansion)

A **linear expansion** of $Q$ over $D$ is a family of queries $(Q_N)_{N \in \mathcal{L}_D(Q)}$, where each $Q_N$ is a linearization of $Q$ w.r.t. $N$.

If $Q$ and $D$ are clear from the context we write simply $(Q_N)_N$.

Proposition

Let $(Q_N)_N$ be a linear expansion of $Q$ over $D$. Then $Q$ and the union $\bigcup_{N \in \mathcal{L}_D(Q)} Q_N$ are equivalent.

Proof.

Follows from two facts:

- $M$ and the disjunction $\bigvee_{N \in \mathcal{L}_D(Q)} N$ are equivalent
- If $\phi$ is a canonical substitution for $N$, then $Q_N(\phi(\bar{x})) :- \phi L, \phi N$ and $Q(\bar{x}) :- L, N$ are equivalent
Containment of Queries with Comparisons

Theorem (Klug 88)

If

– \( Q, Q' \) are conjunctive queries with comparisons with set of constants \( D \)

– \( (Q_N)_N \) is a linear expansion of \( Q \) over \( D \),

then:

\[ Q \sqsubseteq Q' \iff \text{for every } Q_N \text{ in } (Q_N)_N, \text{ there is an homomorphism from } Q' \text{ to } Q_N \]

Corollary

Containment of conjunctive queries with comparisons is in \( \Pi^P_2 \).
Proof.

Suppose $Q'$, $Q$, and $(Q_N)_N$ are as in the theorem. Let $W = \text{var}(Q)$.

“$\Leftarrow$” If there is a homomorphism from $Q'$ to $Q_N$, then $Q_N \subseteq Q'$.
Thus, $Q \subseteq Q'$, since $Q \equiv \bigcup_N Q_N$.

“$\Rightarrow$” If $Q \subseteq Q'$, then $Q_N \subseteq Q'$ for every $N \in \mathcal{L}_D(Q)$.
It suffices to show: “$Q_N \subseteq Q' \Rightarrow$ there is a homomorphism from $Q'$ to $Q_N$”

Recall: $Q_N(\phi \bar{x}) := \phi L, \phi N$.

Let $\alpha \models N$. $N$ is a linearization of $W \cup D \Rightarrow \alpha$ is injective on $\text{Terms}(Q_N)$.

Then: (i) $I_\alpha = \alpha \phi L$ is an instance, (ii) $\alpha(\phi \bar{x}) \in Q_N(I_\alpha)$.

Also: $Q_N \subseteq Q' \Rightarrow \alpha(\phi \bar{x}) \in Q'(I_\alpha)$.

Hence, there is an assignment $\beta'$ for $\text{var}(Q')$ such that

(i) $I_\alpha, \beta' \models Q'$ and (ii) $\beta' \bar{x} = \alpha \phi \bar{x}$.

Now, due to the injectivity of $\alpha$ on $\text{Terms}(Q_N)$, and since every constant of $Q'$ occurs in $N$,

$\beta := \alpha^{-1} \beta'$ is well defined and is a homomorphism from $Q'$ to $Q_N$. 

Containment of Queries with Comparisons (cntd)

Proof (Continued).

To show that $\beta$ is a homomorphism, it remains to prove that $N \models \beta M'$.

Let $s' < t' \in M'$. Then $\beta's' < \beta't'$, since $I_{\alpha, \beta'} \models M'$.

Now, $\alpha^{-1}\beta's'$, $\alpha^{-1}\beta't'$ are terms of $Q_N$, thus one of

$\alpha^{-1}\beta's' < \alpha^{-1}\beta't'$, $\alpha^{-1}\beta's' = \alpha^{-1}\beta't'$, or $\alpha^{-1}\beta's' > \alpha^{-1}\beta't'$

is in $N$, since $N$ is a linearization.

Clearly, $\alpha^{-1}\beta's' < \alpha^{-1}\beta't' \in N$, since $\alpha \models N$.

The case of a comparison $s' \leq t' \in M'$ is dealt with analogously.
Reminder on the Class PSPACE

PSPACE = the class of problems that can be decided by a deterministic (or nondeterministic) Turing machine with polynomial space

There are PSPACE-complete problems. The best-known PSPACE-complete problem is the one of validity of quantified Boolean formulas (QBF).

A quantified Boolean formula (qbf) consists of a prefix and a matrix:
- the matrix is a propositional formula $\phi$
- the prefix is a sequence of quantifications $Q_1 x_1, \ldots, Q_n x_n$
  where $x_1, \ldots, x_n$ are the propositions in $\phi$ and $Q_i \in \{\forall, \exists\}$

An example of a qbf is

$$\forall x \exists y \exists z \forall w \left( x \lor \neg y \lor z \right) \land \left( y \lor \neg z \lor w \right)$$
A qbf is **valid** if there is a set of assignments \( A \) such that

- \( Q \) is compatible with the prefix
- every \( \alpha \in A \) satisfies the matrix

**Definition (QBF Problem)**

Given: a quantified Boolean formula

Question: is the formula valid?

**Theorem (PSPACE-Completeness)**

The QBF problem is complete for the class PSPACE

*What is the combined complexity of the evaluation problem for relational calculus queries? And what is the data complexity?*
Reminder on the Polynomial Hierarchy

There are problems in PSPACE that are NP-hard, but have neither been shown to be in NP nor to be PSPACE-complete.

For a problem $P$, a Turing machine with a $P$-oracle is an extension of a regular Turing machine that

- can write strings $s$ on a special tape, the oracle tape
- receive a one-step answer whether $s \in P$ or not.

Let $C$ be a class of problems.

- The class $\text{NP}^C$ consists of all problems that can be solved by a polynomial time nondeterministic Turing machine with an oracle for some $P_0 \in C$.
- The class $\text{coNP}^C$ consists of all problems $P$ whose complements $\Sigma^* \setminus P$ are in $\text{NP}^C$. 
Reminder on the Polynomial Hierarchy (cntd)

Definition (Polynomial Hierarchy)

One defines recursively the classes $\Sigma^P_k$, $\Pi^P_k$ of the polynomial hierarchy as

$$\Sigma^P_0 = \Pi^P_0 = \text{P}$$

$$\Sigma^P_{k+1} = \text{NP} \Sigma^P_k$$

$$\Pi^P_{k+1} = \text{coNP} \Sigma^P_k$$

Note: $\Sigma^P_1 = \text{NP}$ and $\Pi^P_1 = \text{coNP}$
Complete Problems for the Polynomial Hierarchy

A complete problem for $\Sigma_k^P$ is $\exists$QBF$_k$. It consists of all valid qbfs with $k$ alternations of quantifiers, starting with an existential:

$$\exists X_1 \forall X_2 \ldots, Q_k \phi$$

- If $k$ is even, the problem is already complete if $\phi$ consists of a disjunction of conjunctive 3-clauses.
- If $k$ is odd, the problem is already complete if $\phi$ consists of a conjunction of disjunctive 3-clauses.

A complete problem for $\Pi_k^P$ is $\forall\exists$QBF$_k$. It consists of all valid qbfs with $k$ alternations of quantifiers, starting with a universal:

$$\forall X_1 \exists X_2 \ldots, Q_k \phi$$

Analogous subclasses to the ones above are already complete for $\Pi_k^P$. In particular, $\forall\exists$3SAT is complete for $\Pi_2^P$. 
Containment of Queries with Comparisons

**Theorem (van der Meyden 92)**

Containment with comparisons is $\Pi_2^P$-complete.

The proof here is different from the one by van der Meyden. It uses a simple pattern that can be used to prove many more $\Pi_2^P$-hardness results about query containment, for instance, containment of queries

- with the predicate “≠”
- with negated subgoals (like $\neg R(x)$)
- SQL null values.
We show the reduction for a general formula

$$\psi = \forall x_1, \ldots, x_m \exists y_1, \ldots, y_n \gamma_1 \land \ldots \land \gamma_k$$

where $\gamma_1, \ldots, \gamma_k$ are disjunctive 3-clauses, and for the example

$$\psi_0 = \forall x_1 \forall x_2 \exists y_1 \exists y_2 (x_1 \lor \neg x_2 \lor y_1) \land (x_2 \lor \neg y_1 \lor y_2)$$

We define boolean queries $Q'$, $Q$ such that $Q \sqsubseteq Q'$ iff $\psi$ is valid.
Reduction of $\forall \exists 3$SAT to Containment with Comparisons

We model the universal quantifiers $\forall x_i$

by pairs of “generator conditions” $G'_i$, $G_i$,

following the “black and white blocks” example:

$$G'_i = S_i(u_i, v_i, x_i), u_i \leq 4, v_i > 4$$

$$G_i = S_i(4, w_i, 1), S_i(w_i, 5, 0)$$

Idea: For $G'_i$ to be more general than $G_i$

- $x_i$ must be mapped to 1, if $w_i$ is bound to a value $\leq 4$
- $x_i$ must be mapped to 0, otherwise.
Reduction of $\forall \exists 3$SAT to Containment with Comparisons

For every clause $\gamma_i$, we introduce

\[ H'_i = R_i(p_1^{(i)}, p_2^{(i)}, p_3^{(i)}) \]
\[ H_i = R_i(\bar{t}_1^{(i)}), \ldots, R_i(\bar{t}_7^{(i)}) \]

where $p_1^{(i)}, p_2^{(i)}, p_3^{(i)}$ are the three propositions occurring in $\gamma_i$ and $\bar{t}_1^{(i)}, \ldots, \bar{t}_7^{(i)}$ are the seven combinations of truth values that satisfy $\gamma_i$.

In our example

\[ H'_1 = R_1(x_1, x_2, y_1) \]
\[ H_1 = R_1(0, 0, 0), R_1(0, 0, 1), R_1(0, 1, 1), \]
\[ R_1(1, 0, 0), R_1(1, 0, 1), R_1(1, 1, 0), R_1(1, 1, 1) \]
Reduction of $\forall\exists 3$SAT to Containment with Comparisons

The queries for $\psi$ are

$$Q'(\cdot) :– G'_1, \ldots, G'_m, H'_1, \ldots, H'_n$$
$$Q(\cdot) :– G_1, \ldots, G_m, H_1, \ldots, H_n$$

Lemma

$$Q \sqsubseteq Q' \quad \text{iff} \quad \psi \text{ is valid}$$

Sketch.

“$\Leftarrow$” For each binding of the $w_i$ in $Q$ over a db instance, we can map $G'_i$ to one of the atoms in $G_i$. Such a mapping corresponds to a choice of 0 or 1 for $x_i$. If $\psi$ is valid, then for every binding of the $x_i$ we find values for the $y_j$ that satisfy all clauses. These values allows us to map $H'_i$ to one of the atoms in $H_i$.

“$\Rightarrow$” For each assignment of 0, 1 to the $x_i$, we create a db instance by instantiating $w_i$ in $Q$ with 4 or 5. This instance satisfies $Q$. It must also satisfy $Q'$. This tells us that we can instantiate the $y_j$ such that $\psi$ is satisfied.
Minimizing Conjunctive Queries

- A conjunctive query may have atoms that can be dropped without changing the answers.
- Since computing joins is expensive, this has the potential of saving computation cost.

**Goal:** Given a conjunctive query $Q$, find an equivalent conjunctive query $Q'$ with the minimum number of joins.

**Questions:** How many such queries can exist? How different are they? How can we find them?

**Assumption:** We consider only relational CQs.
The “Drop Atoms” Algorithm

Input: \( Q(\bar{x}) :– L \)

\[
L' := L; \\
\textbf{repeat until} \text{ no change} \\
\quad \text{choose an atom } A \in L; \\
\quad \textbf{if} \text{ there is a homomorphism} \\
\quad \quad \text{from } Q(\bar{x}) :– L' \text{ to } Q(\bar{x}) :– L' \setminus \{A\} \\
\quad \quad \textbf{then} \quad L' := L' \setminus \{A\}
\]

\textbf{end}

Output: \( Q'(\bar{x}) :– L' \)
Questions About the Algorithm

- Does it terminate?
- Is \( Q' \) equivalent to \( Q \)?
- Is \( Q' \) of minimal length among the queries equivalent to \( Q \)?
Subqueries

Definition (Subquery)

If $Q$ is a conjunctive query,

$$Q(\bar{x}) :– R_1(\bar{t}_1), \ldots, R_k(\bar{t}_k),$$

then $Q'$ is a subquery of $Q$ if $Q'$ is of the form

$$Q'(\bar{x}) :– R_{i_1}(\bar{t}_{i_1}), \ldots, R_{i_l}(\bar{t}_{i_l})$$

where $1 \leq i_1 < i_2 < \ldots < i_l \leq k$.

Proposition

The Drop-Atoms Algorithm outputs a subquery $Q'$ of $Q$ such that

- $Q'$ and $Q$ are equivalent
- $Q'$ does not have a subquery equivalent to $Q$. 
To Minimize $Q$, It’s Enough To Shorten $Q$

**Proposition**

Consider the relational conjunctive query

$$Q(\bar{x}) :– R_1(\bar{t}_1), \ldots, R_n(\bar{t}_n).$$

If there is an equivalent conjunctive query

$$Q'(\bar{x}) :– S_1(\bar{s}_1), \ldots, S_l(\bar{s}_m), \quad m < k,$$

then $Q$ is equivalent to a subquery

$$Q_0(\bar{x}) :– R_{i_1}(\bar{t}_{i_1}), \ldots, R_{i_l}(\bar{t}_{i_l}), \quad l \leq m.$$

**In other words:** If $Q$ is a relational CQ with $n$ atoms and $Q'$ an equivalent relational CQ with $m$ atoms, where $m < n$, then there exists a subquery $Q_0$ of $Q$ such that $Q_0$ has at most $m$ atoms in the body and $Q_0$ is equivalent to $Q$.

*Proof as exercise!*
Minimization Theorem

Theorem (Minimization)

Let $Q$ and $Q'$ be two equivalent minimal relational CQs. Then $Q$ and $Q'$ are identical up to renaming of variables.

*Proof as exercise!*

**Conclusions:**
- There is essentially one minimal version of each relational CQ $Q$.
- We can obtain it by dropping atoms from $Q$'s body.
- The Drop-Atoms algorithm is sound and complete.
Minimizing SPJ/Conjunctive Queries: Example

Consider relation $R$ with three attributes $A$, $B$, $C$ and the SPJ query

$$Q = \pi_{AB} (\sigma_{B=4} (R)) \land \pi_{BC} (\pi_{AB} (R) \land \pi_{AC} (\sigma_{B=4} (R)))$$

- Translate into relational calculus:

$$\exists z_1 \ R(x, y, z_1) \land y = 4 \land \exists x_1 \ (\exists z_2 \ R(x_1, y, z_2)) \land (\exists y_1 \ R(x_1, y_1, z) \land y_1 = 4)$$

- Simplify, by substituting the constant, and pulling quantifiers outward:

$$\exists x_1, z_1, z_2 \ (R(x, 4, z_1) \land R(x_1, 4, z_2) \land R(x_1, 4, z) \land y = 4)$$

- Conjunctive query:

$$Q(x, y, z) :- R(x, 4, z_1), R(x_1, 4, z_2), R(x_1, 4, z), y = 4$$

Then minimize: Exercise!
Functional Dependencies

Consider the relation

\[ \text{Lect}(\text{name}, \text{office}, \text{course}) \]

For any university instance,

- all tuples with the same “name” have the same “office” value
- tuples may have the same “course”, but different “name” and “office” (if lecturers share courses)
- tuples may have the same “office”, but different “name” and “course” (if lecturers share offices)
Functional Dependencies (Cntd)

The formula

$$\forall n, o_1, c_1, o_2, c_2 (\text{Lect}(n, o_1, c_1) \land \text{Lect}(n, o_2, c_2) \rightarrow o_1 = o_2)$$

is a functional dependency (FD).

Assuming that Lect is clear from the context, we abbreviate it as

$$\text{name} \rightarrow \text{office}$$

and read “name determines office”.

*FDs are a frequent type of integrity constraints (keys are a special case)*
Functional Dependencies (Cntd)

Notation:
- If $R$ is relation with attribute set $Z$, we write FDs as

$$X \rightarrow A \quad \text{or} \quad X \rightarrow Y$$

where $X, Y \subseteq Z$ and $A \in Z$

- $X, Y, Z$ represent sets of attributes; $A, B, C$ represent single attributes
- no set braces in sets of attributes: just $ABC$, rather than \{A, B, C\}

Semantics:
- $X \rightarrow Y$ is satisfied by an instance $I$, that is $I \models X \rightarrow Y$, iff

$$\pi_X(t) = \pi_X(t') \quad \text{implies} \quad \pi_Y(t) = \pi_Y(t'), \quad \text{for all} \ t, t' \in I(R)$$

- Note: $X \rightarrow AB$ is a equivalent to $X \rightarrow A$ and $X \rightarrow B$

$$\Rightarrow \quad \text{it suffices to deal with FDs } X \rightarrow A$$
Consider the queries

\[ Q = \text{Lect} \]
\[ Q' = \pi_{\text{name},\text{course}}(\text{Lect}) \bowtie_{\text{name}} \pi_{\text{name},\text{office}}(\text{Lect}) \]

- In general, is there equivalence/containment among \( Q, Q' \)?
- What if we take into account the FD \( \text{name} \rightarrow \text{office} \)?

Instead of algebra, let’s use rule notation

\[ Q(n, o, c) :\leftarrow \text{Lect}(n, o, c) \]
\[ Q'(n, o, c) :\leftarrow \text{Lect}(n, o', c), \text{Lect}(n, o, c') \]
Chase and Minimize

\[ Q'(n, o, c) :\neg \text{Lect}(n, o', c), \text{Lect}(n, o, c') \]

Using the FD name $\text{name} \rightarrow \text{office}$, we infer $o = o'$:

\[ Q'(n, o, c) :\neg \text{Lect}(n, o, c), \text{Lect}(n, o, c') \]

Minimizing using Drop Atom, we get

\[ Q'(n, o, c) :\neg \text{Lect}(n, o, c) \]

Thus, $Q' \equiv Q$
FD Violations

**Notation:** Instead of \( \pi_X(t) \) and \( \pi_A(t) \), we write \( t.X \) and \( t.A \)

**Definition (Violation)**

The FD \( X \rightarrow A \) over \( R \) is **violated** by the atoms \( R(t), R(t') \) if

- \( t.X = t'.X \) and
- \( t.A \neq t'.A \)
The Chase Algorithm

Input: query \( Q(\vec{s}) :– L \), set of FDs \( \mathcal{F} \)

let \( (\vec{s}', L') = (s, L) \)

while \( L' \) contains atoms \( R(t), R(t') \), violating some \( X \rightarrow A \in \mathcal{F} \) do

\[ \text{case } t.A, t'.A \text{ of} \]

• one is a nondistinguished variable
  \( \Rightarrow \) in \( (\vec{s}', L') \), replace the nondistinguished variable by the other term

• one is a distinguished variable,
  the other one a distinguished variable or constant
  \( \Rightarrow \) in \( (\vec{s}', L') \), replace the distinguished variable by the other term

• both are constants
  \( \Rightarrow \) set \( L' = \bot \) and \textbf{stop} 

end

end

Output: query \( Q'(\vec{s}') :– L' \)
Questions about the Chase Algorithm

- Does the Chase algorithm terminate? What is the running time?
- What is the relation between a query and its Chase'd version?
- Query containment wrt a set of FDs:
  - How can we define this problem?
  - Can we decide this problem?
- Query minimization wrt to a set of FDs:
  - How can we define this problem?
  - How can we solve it?
- Relational CQs:
  - We know that all such queries are satisfiable.
  - Is this still true if we allow only instances that satisfy a given set of FDs?