Ontology and Database Systems: Foundations of Database Systems

Part 5: Datalog

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Motivation

- Relational Calculus and Relational Algebra were considered to be "*the*" database languages for a long time
- Codd: A query language is "complete," if it yields Relational Calculus
- However, Relational Calculus misses an important feature: *recursion*
- Example: A metro database with relation `links:line, station, nextstation`
  - What stations are reachable from station "Odeon"?
  - Can we go from Odeon to Tuileries?
  - etc.
- It can be proved: such queries cannot be expressed in Relational Calculus
- This motivated a logic-programming extension to conjunctive queries: *datalog*

Example: Metro Database Instance

link	line	station	nextstation
	4	St. Germain	Odeon
	4	Odeon	St. Michel
	4	St. Michel	Chatelet
	1	Chatelet	Louvres
	1	Louvres	Palais Royal
	1	Palais-Royal	Tuileries
	1	Tuileries	Concorde

Datalog program for the first query:

$$\begin{array}{rcl}
\mathtt{reach}(\mathtt{X},\mathtt{X}) & \leftarrow & \mathtt{link}(\mathtt{L},\mathtt{X},\mathtt{Y}) \\
\mathtt{reach}(\mathtt{X},\mathtt{X}) & \leftarrow & \mathtt{link}(\mathtt{L},\mathtt{Y},\mathtt{X}) \\
\mathtt{reach}(\mathtt{X},\mathtt{Y}) & \leftarrow & \mathtt{link}(\mathtt{L},\mathtt{X},\mathtt{Z}),\mathtt{reach}(\mathtt{Z},\mathtt{Y}) \\
\mathtt{answer}(\mathtt{X}) & \leftarrow & \mathtt{reach}(\text{`}\mathtt{Odeon}\text{`},\mathtt{X})
\end{array}$$

- Note: this is a recursive definition
- Intuitively, if the part right of "←" is true,
  the rule "fires" and the atom left of "←" is concluded.

Exercise

Write the following queries in datalog:

- Which stations can be reached from Chatelet, using exactly one line? (This excludes staying at Chatelet).
- Which stations can be reached from one another using exactly one line?
- Which stations can be reached from one another? (Check whether the query in the example is correct!)
- Which stations are terminal stops?

The Datalog Language

- Datalog is akin to Logic Programming
- The basic language (considered next) has many extensions
- There exist several approaches to defining the semantics:

  Model-theoretic approach: View rules as logical sentences, which state the query result

  Operational (fixpoint) approach: Obtain query result by applying an inference procedure, until a fixpoint is reached

  Proof-theoretic approach: Obtain proofs of facts in the query result, following a proof calculus (based on resolution)

Datalog vs. Logic Programming

Although datalog is akin to Logic Programming,
there are important differences:

- There are **no functions symbols** in datalog
  $\leadsto$ no unbounded data structures, such as lists, are supported
- Datalog has a **purely declarative semantics**
  $\leadsto$ In a datalog program,
  - the *order of clauses* is irrelevant
  - the *order of atoms* in a rule body is irrelevant
- Datalog distinguishes between
  - database relations ("*extensional database*", *edb*) and
  - derived relations ("*intensional database*", *idb*)

Syntax of “plain datalog”, or “datalog”

Definition

A *datalog rule* r is an expression of the form

$$R_0(\bar{x}_0) \leftarrow R_1(\bar{x}_1), \ldots, R_n(\bar{x}_n) \tag{1}$$

where

- $n \geq 0$,
- $R_0, \ldots, R_n$ are relations names,
- $\bar{x}_0, \ldots, \bar{x}_n$ are tuples of variables and constants (from **dom**), and
- every variable in $\bar{x}_0$ occurs in $\bar{x}_1, \ldots, \bar{x}_n$ (“safety”)

Remark

- The *head* of r, denoted $H(r)$, is $R_0(\bar{x}_0)$
- The *body* of r, denoted $B(r)$, is $\{ R_1(\bar{x}_1), \ldots, R_n(\bar{x}_n) \}$
- The rule symbol “$\leftarrow$” is often also written as “:-”

Datalog Programs

Definition

A *datalog program* is a finite set of datalog rules.

Let P be a datalog program.

- An *extensional relation* of P is a relation occurring only in rule bodies of P
- An *intensional relation* of P is a relation occurring in the head of some rule in P
- The *extensional schema* of P, *edb*(P), consists of all extensional relations of P
- The *intensional schema* of P, *idb*(P), consists of all intensional relations of P
- The *schema* of P, *sch*(P), is the union of *edb*(P) and *idb*(P).

The Metro Example /1

Datalog program P on the metro database schema (w/o integrity constraints)

$$\mathcal{M} = \{\texttt{link}(\texttt{line},\ \texttt{station},\ \texttt{nextstation})\}:$$

$$\begin{aligned}
\texttt{reach}(\texttt{X},\texttt{X}) &\leftarrow \texttt{link}(\texttt{L},\texttt{X},\texttt{Y})\\
\texttt{reach}(\texttt{X},\texttt{X}) &\leftarrow \texttt{link}(\texttt{L},\texttt{Y},\texttt{X})\\
\texttt{reach}(\texttt{X},\texttt{Y}) &\leftarrow \texttt{link}(\texttt{L},\texttt{X},\texttt{Z}),\texttt{reach}(\texttt{Z},\texttt{Y})\\
\texttt{answer}(\texttt{X}) &\leftarrow \texttt{reach('Odeon',X)}
\end{aligned}$$

Here,

$$\begin{aligned}
\mathit{edb}(P) &= \{\texttt{link}\} \quad (= \mathcal{M}),\\
\mathit{idb}(P) &= \{\texttt{reach},\texttt{answer}\},\\
\mathit{sch}(P) &= \{\texttt{link},\texttt{reach},\texttt{answer}\}
\end{aligned}$$

Datalog Syntax (cntd)

- The set of constants occurring in program P is denoted as *adom*(P)
- The *active domain* of P with respect to an instance $\mathbf{I}$ is defined as

$$\mathit{adom}(P, \mathbf{I}) := \mathit{adom}(P) \cup \mathit{adom}(\mathbf{I}),$$

  that is, as the set of constants occurring in P and $\mathbf{I}$

Definition (Rule Instantiation)

Let $\alpha\colon \mathit{var}(r) \cup \mathbf{dom} \to \mathbf{dom}$ be an assignment for the variables in a rule r of form (1). Then the *instantiation* of r with α, denoted $\alpha(r)$, is the rule

$$R_0(\alpha(\bar{x}_0)) \leftarrow R_1(\alpha(\bar{x}_1)), \ldots, R_n(\alpha(\bar{x}_n)),$$

which results from replacing each variable x with $\alpha(x)$.

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The Metro Example/2

- For the datalog program P above, we have that $adom(P) = \{$ Odeon $\}$
- We consider the database instance $\mathbf{I}$:

link	line	station	nextstation
	4	St. Germain	Odeon
	4	Odeon	St. Michel
	4	St. Michel	Chatelet
	1	Chatelet	Louvre
	1	Louvre	Palais-Royal
	1	Palais-Royal	Tuileries
	1	Tuileries	Concorde

Then $adom(\mathbf{I}) = \{$4, 1, St.Germain, Odeon, St.Michel, Chatelet, Louvres, Palais-Royal, Tuileries, Concorde$\}$

- Also $adom(P, \mathbf{I}) = adom(\mathbf{I})$

The Metro Example/3

- The rule

$$\texttt{reach}(\texttt{St.Germain}, \texttt{Odeon}) \quad \leftarrow \quad \texttt{link}(\texttt{Louvre}, \texttt{St.Germain}, \texttt{Concorde}), \texttt{reach}(\texttt{Concorde}, \texttt{Odeon})$$

is an instantiation of the rule

$$\texttt{reach}(\texttt{X}, \texttt{Y}) \quad \leftarrow \quad \texttt{link}(\texttt{L}, \texttt{X}, \texttt{Z}), \texttt{reach}(\texttt{Z}, \texttt{Y})$$

(take $\alpha(\texttt{X}) = \texttt{St.Germain}$, $\alpha(\texttt{L}) = \texttt{Louvre}$, $\alpha(\texttt{Y}) = \texttt{Odeon}$, $\alpha(\texttt{Z}) = \texttt{Concorde}$)

Datalog: Model-Theoretic Semantics

General Idea:

- We view a program as a set of first-order sentences
- Given an instance $\mathbf{I}$ of *edb*(P),
  the result of P is a database instance of *sch*(P)
  that extends $\mathbf{I}$ and satisfies the sentences
  (or, is a *model* of the sentences)
- There can be many models
- The *intended answer* is specified by particular models
- These particular models are selected by "external" conditions

Logical Theory Σ_P

- To every datalog rule r of the form $R_0(\bar{x}_0) \leftarrow R_1(\bar{x}_1), \ldots, R_n(\bar{x}_n)$, with variables $x_1, \ldots, x_m$, we associate the logical sentence $\sigma(r)$:

$$\forall x_1, \cdots \forall x_m \, (R_1(\bar{x}_1) \wedge \cdots \wedge R_n(\bar{x}_n) \rightarrow R_0(\bar{x}_0))$$

- To a program P, we associate the set of sentences $\Sigma_P = \{\sigma(r) \mid r \in P\}$

Definition

Let P be a datalog program and $\mathbf{I}$ an instance of *edb*(P). Then,

- A *model* of P is an instance of *sch*(P) that satisfies Σ_P
- We compare models wrt set inclusion "$\subseteq$"
  (in the Logic Programming perspective)
- The *semantics* of P on input $\mathbf{I}$, denoted $P(\mathbf{I})$,
  is the *least model* of P containing $\mathbf{I}$, if it exists

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Example

For program P and instance $\mathbf{I}$ of the Metro Example, the least model is:

link	line	station	nextstation
	4	St. Germain	Odeon
	4	Odeon	St. Michel
	4	St. Michel	Chatelet
	1	Chatelet	Louvres
	1	Louvres	Palais-Royal
	1	Palais-Royal	Tuileries
	1	Tuileries	Concorde

reach		
	St. Germain	St. Germain
	Odeon	Odeon
	...	
	Concorde	Concorde
	St. Germain	Odeon
	St. Germain	St.Michel
	St. Germain	Chatelet
	St. Germain	Louvre
	...	

answer	
	Odeon
	St. Michel
	Chatelet
	Louvre
	Palais-Royal
	Tuileries
	Concorde

Questions

1. Is the semantics $P(\mathbf{I})$ well-defined for every input instance $\mathbf{I}$?
2. How can one compute $P(\mathbf{I})$?

Observation: For any $\mathbf{I}$, there is a model of P containing $\mathbf{I}$

- Let $\mathbf{B}(P, \mathbf{I})$ be the instance of *sch*(P) such that

$$\mathbf{B}(P, \mathbf{I})(R) = \begin{cases} \mathbf{I}(R) & \text{for each } R \in \mathit{edb}(P) \\ \mathit{adom}(P, \mathbf{I})^{\mathit{ary}(R)} & \text{for each } R \in \mathit{idb}(P) \end{cases}$$

- Then: $\mathbf{B}(P, \mathbf{I})$ is a model of P containing $\mathbf{I}$
  $\Rightarrow$ $P(\mathbf{I})$ is a subset of $\mathbf{B}(P, \mathbf{I})$ *(if it exists)*
- Naive algorithm: explore all subsets of $\mathbf{B}(P, \mathbf{I})$

Elementary Properties of $P(\mathbf{I})$

Let P be a datalog program, $\mathbf{I}$ an instance of *edb*(P), and $\mathcal{M}(\mathbf{I})$ the set of all models of P containing $\mathbf{I}$.

Theorem

The intersection $\bigcap_{M \in \mathcal{M}(\mathbf{I})} M$ is a model of P.

Corollary

1. $P(\mathbf{I}) = \bigcap_{M \in \mathcal{M}(\mathbf{I})} M$
2. *adom*$(P(\mathbf{I})) \subseteq$ *adom*$(P, \mathbf{I})$, that is, no new values appear
3. $P(\mathbf{I})(R) = \mathbf{I}(R)$, for each $R \in$ *edb*(P)

Consequences:

- $P(\mathbf{I})$ is well-defined for every $\mathbf{I}$
- If P and $\mathbf{I}$ are finite, the $P(\mathbf{I})$ is finite

Why Choose the Least Model?

There are two reasons to choose the least model containing $\mathbf{I}$:

1. The *Closed World Assumption*:
   - If a fact $R(\bar{c})$ is not true in all models of a database $\mathbf{I}$, then infer that $R(\bar{c})$ is false
   - This amounts to considering $\mathbf{I}$ as complete
   - ... which is customary in database practice

2. The relationship to Logic Programming:
   - Datalog should desirably match Logic Programming (seamless integration)
   - Logic Programming builds on the minimal model semantics

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Relating Datalog to Logic Programming

- A logic program makes no distinction between *edb* and *idb*
- A datalog program P and an instance $\mathbf{I}$ of *edb*(P) can be mapped to the logic program

$$\mathcal{P}(P, \mathbf{I}) = P \cup \mathbf{I}$$

  (where $\mathbf{I}$ is viewed as a set of atoms in the Logic Programming perspective)
- Correspondingly, we define the logical theory

$$\Sigma_{P,\mathbf{I}} = \Sigma_P \cup \mathbf{I}$$

- The semantics of the logic program $\mathcal{P} = \mathcal{P}(P, \mathbf{I})$ is defined in terms of *Herbrand interpretations* of the language induced by $\mathcal{P}$:
  - The domain of discourse is formed by the constants occurring in $\mathcal{P}$
  - Each constant occurring in $\mathcal{P}$ is interpreted by itself

Herbrand Interpretations of Logic Programs

Given a rule r, we denote by *Const*(r) the set of all constants in r

Definition

For a (function-free) logic program $\mathcal{P}$, we define

- the *Herbrand universe* of $\mathcal{P}$, by

$$\mathbf{HU}(\mathcal{P}) = \bigcup_{r \in \mathcal{P}} \mathit{Const}(r)$$

- the *Herbrand base* of $\mathcal{P}$, by

$$\begin{aligned}\mathbf{HB}(\mathcal{P}) = \{R(c_1, \ldots, c_n) \mid {} & R \text{ is a relation in } \mathcal{P}, \\ & c_1, \ldots, c_n \in \mathbf{HU}(\mathcal{P}), \text{ and } \mathit{ary}(R) = n\}\end{aligned}$$

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Example

$$\mathcal{P} = \{ \quad \texttt{arc}(\texttt{a},\texttt{b}).$$
$$\texttt{arc}(\texttt{b},\texttt{c}).$$
$$\texttt{reachable}(\texttt{a}).$$
$$\texttt{reachable}(\texttt{Y}) \leftarrow \texttt{arc}(\texttt{X},\texttt{Y}), \texttt{reachable}(\texttt{X}). \ \}$$

$$\mathbf{HU}(\mathcal{P}) \quad = \quad \{\texttt{a},\texttt{b},\texttt{c}\}$$

$$\begin{aligned}\mathbf{HB}(\mathcal{P}) \quad = \quad & \{\texttt{arc}(\texttt{a},\texttt{a}),\ \texttt{arc}(\texttt{a},\texttt{b}),\ \texttt{arc}(\texttt{a},\texttt{c}),\\ & \ \texttt{arc}(\texttt{b},\texttt{a}),\ \texttt{arc}(\texttt{b},\texttt{b}),\ \texttt{arc}(\texttt{b},\texttt{c}),\\ & \ \texttt{arc}(\texttt{c},\texttt{a}),\ \texttt{arc}(\texttt{c},\texttt{b}),\ \texttt{arc}(\texttt{c},\texttt{c}),\\ & \ \texttt{reachable}(\texttt{a}),\ \texttt{reachable}(\texttt{b}),\ \texttt{reachable}(\texttt{c})\}\end{aligned}$$

Grounding

- A rule r' is a *ground instance* of a rule r with respect to $\mathbf{HU}(\mathcal{P})$,
  if $r' = \alpha(r)$ for an assignment α
  such that $\alpha(x) \in \mathbf{HU}(\mathcal{P})$ for each $x \in \mathit{var}(r)$
- The *grounding* of a rule r with respect to $\mathbf{HU}(\mathcal{P})$,
  denoted $\mathit{Ground}_{\mathcal{P}}(r)$,
  is the set of all ground instances of r wrt $\mathbf{HU}(\mathcal{P})$
- The *grounding* of a logic program $\mathcal{P}$ is

$$\mathit{Ground}(\mathcal{P}) = \bigcup_{r \in \mathcal{P}} \mathit{Ground}_{\mathcal{P}}(r)$$

Example

$$
\begin{aligned}
\mathit{Ground}(\mathcal{P}) = \{ & \texttt{arc}(\mathrm{a},\mathrm{b}).\ \ \texttt{arc}(\mathrm{b},\mathrm{c}).\ \ \texttt{reachable}(\mathrm{a}). \\
& \texttt{reachable}(\mathrm{a}) \leftarrow \texttt{arc}(\mathrm{a},\mathrm{a}), \texttt{reachable}(\mathrm{a}). \\
& \texttt{reachable}(\mathrm{b}) \leftarrow \texttt{arc}(\mathrm{a},\mathrm{b}), \texttt{reachable}(\mathrm{a}). \\
& \texttt{reachable}(\mathrm{c}) \leftarrow \texttt{arc}(\mathrm{a},\mathrm{c}), \texttt{reachable}(\mathrm{a}). \\
& \texttt{reachable}(\mathrm{a}) \leftarrow \texttt{arc}(\mathrm{b},\mathrm{a}), \texttt{reachable}(\mathrm{b}). \\
& \texttt{reachable}(\mathrm{b}) \leftarrow \texttt{arc}(\mathrm{b},\mathrm{b}), \texttt{reachable}(\mathrm{b}). \\
& \texttt{reachable}(\mathrm{c}) \leftarrow \texttt{arc}(\mathrm{b},\mathrm{c}), \texttt{reachable}(\mathrm{b}). \\
& \texttt{reachable}(\mathrm{a}) \leftarrow \texttt{arc}(\mathrm{c},\mathrm{a}), \texttt{reachable}(\mathrm{c}). \\
& \texttt{reachable}(\mathrm{b}) \leftarrow \texttt{arc}(\mathrm{c},\mathrm{b}), \texttt{reachable}(\mathrm{c}). \\
& \texttt{reachable}(\mathrm{c}) \leftarrow \texttt{arc}(\mathrm{c},\mathrm{c}), \texttt{reachable}(\mathrm{c}).\ \}
\end{aligned}
$$

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Herbrand Models

- A *Herbrand-interpretation* I of $\mathcal{P}$ is any subset $I \subseteq \mathbf{HB}(\mathcal{P})$
- A *Herbrand-model* of $\mathcal{P}$ is a Herbrand-interpretation that satisfies all sentences in $\Sigma_{P,\mathbf{I}}$
- Equivalently, $M \subseteq \mathbf{HB}(\mathcal{P})$ is a Herbrand model if
  for all $r \in$ *Ground*$(\mathcal{P})$ such that $B(r) \subseteq M$
  we have that $H(r) \subseteq M$

Example

The Herbrand models of program $\mathcal{P}$ above are exactly the following:

- $M_1 = \{\ \texttt{arc}(\texttt{a}, \texttt{b}), \texttt{arc}(\texttt{b}, \texttt{c}),$
  $\texttt{reachable}(\texttt{a}), \texttt{reachable}(\texttt{b}), \texttt{reachable}(\texttt{c})\ \}$
- $M_2 = \mathbf{HB}(\mathcal{P})$
- every interpretation M such that $M_1 \subseteq M \subseteq M_2$

and no others.

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Logic Programming Semantics

Proposition

$\mathbf{HB}(\mathcal{P})$ is always a model of $\mathcal{P}$

Theorem

For every logic program there exists a least Herbrand model (wrt "$\subseteq$").

For a program $\mathcal{P}$, this model is denoted $MM(\mathcal{P})$ (for "minimal model").
The model $MM(\mathcal{P})$ is the semantics of $\mathcal{P}$.

Theorem (Datalog $\leftrightarrow$ Logic Programming))

Let P be a datalog program and $\mathbf{I}$ be an instance of $edb(P)$. Then,

$$P(\mathbf{I}) = MM(\mathcal{P}(P, \mathbf{I}))$$

Consequences

Results and techniques for Logic Programming can be exploited for datalog.

For example,

- proof procedures for Logic Programming (e.g., SLD resolution) can be applied to datalog (with some caveats, regarding for instance termination)
- datalog can be reduced by "grounding" to propositional logic programs

Fixpoint Semantics

Another view:

"If all facts in $\mathbf{I}$ hold, which other facts must hold after firing the rules in P?"

Approach:

- Define an *immediate consequence operator* $\mathbf{T}_P(\mathbf{K})$ on db instances $\mathbf{K}$
- Start with $\mathbf{K} = \mathbf{I}$
- Apply $\mathbf{T}_P$ to obtain a new instance: $\mathbf{K}_{\text{new}} := \mathbf{T}_P(\mathbf{K}) = \mathbf{I} \cup \text{new facts}$
- Iterate until nothing new can be produced
- The result yields the semantics

Immediate Consequence Operator

Let P be a datalog program and $\mathbf{K}$ be a database instance of *sch*(P).
A fact $R(\bar{t})$ is an *immediate* consequence for $\mathbf{K}$ and P, if either

- $R \in$ *edb*(P) and $R(\bar{t}) \in \mathbf{K}$, or
- there exists a ground instance r of a rule in P such that $H(r) = R(\bar{t})$ and $B(r) \subseteq \mathbf{K}$.

Definition (Immediate Consequence Operator)

The *immediate consequence operator* of a datalog program P is the mapping

$$\mathbf{T}_P \colon \mathit{inst}(\mathit{sch}(P)) \to \mathit{inst}(\mathit{sch}(P))$$

where

$$\mathbf{T}_P(\mathbf{K}) = \{A \mid A \text{ is an immediate consequence for } \mathbf{K} \text{ and } P\}.$$

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Example

Consider

$$P = \{ \quad \texttt{reachable}(\texttt{a}), \\ \qquad \texttt{reachable}(\texttt{Y}) \leftarrow \texttt{arc}(\texttt{X}, \texttt{Y}), \texttt{reachable}(\texttt{X}) \ \}$$

where *edb*$(P) = \{\texttt{arc}\}$ and *idb*$(P) = \{\texttt{reachable}\}$.

Let

$$\begin{aligned} \mathbf{I} = \mathbf{K}_1 &= \{\texttt{arc}(\texttt{a}, \texttt{b}),\ \texttt{arc}(\texttt{b}, \texttt{c})\} \\ \mathbf{K}_2 &= \{\texttt{arc}(\texttt{a}, \texttt{b}),\ \texttt{arc}(\texttt{b}, \texttt{c}),\ \texttt{reachable}(\texttt{a})\} \\ \mathbf{K}_3 &= \{\texttt{arc}(\texttt{a}, \texttt{b}),\ \texttt{arc}(\texttt{b}, \texttt{c}),\ \texttt{reachable}(\texttt{a}),\ \texttt{reachable}(\texttt{b})\ \} \\ \mathbf{K}_4 &= \{\texttt{arc}(\texttt{a}, \texttt{b}),\ \texttt{arc}(\texttt{b}, \texttt{c}),\ \texttt{reachable}(\texttt{a}),\ \texttt{reachable}(\texttt{b}),\ \texttt{reachable}(\texttt{c})\} \end{aligned}$$

Example (cntd)

Then,

$$\mathbf{T}_P(\mathbf{K}_1) = \{\mathtt{arc}(\mathsf{a},\mathsf{b}),\ \mathtt{arc}(\mathsf{b},\mathsf{c}),\ \mathtt{reachable}(\mathsf{a})\} \ = \ \mathbf{K}_2$$
$$\mathbf{T}_P(\mathbf{K}_2) = \{\mathtt{arc}(\mathsf{a},\mathsf{b}),\ \mathtt{arc}(\mathsf{b},\mathsf{c}),\ \mathtt{reachable}(\mathsf{a}),\ \mathtt{reachable}(\mathsf{b})\} \ = \ \mathbf{K}_3$$
$$\mathbf{T}_P(\mathbf{K}_3) = \{\mathtt{arc}(\mathsf{a},\mathsf{b}),\ \mathtt{arc}(\mathsf{b},\mathsf{c}),\ \mathtt{reachable}(\mathsf{a}),\ \mathtt{reachable}(\mathsf{b}),\ \mathtt{reachable}(\mathsf{c})\} = \mathbf{K}_4$$
$$\mathbf{T}_P(\mathbf{K}_4) = \{\mathtt{arc}(\mathsf{a},\mathsf{b}),\ \mathtt{arc}(\mathsf{b},\mathsf{c}),\ \mathtt{reachable}(\mathsf{a}),\ \mathtt{reachable}(\mathsf{b}),\ \mathtt{reachable}(\mathsf{c})\} = \mathbf{K}_4$$

Thus, $\mathbf{K}_4$ is a *fixpoint* of $\mathbf{T}_P$.

Definition

$\mathbf{K}$ is a *fixpoint* of operator $\mathbf{T}_P$ if $\mathbf{T}_P(\mathbf{K}) = \mathbf{K}$

Properties

Proposition

Let P be a datalog program.

1. The operator $\mathbf{T}_P$ is monotonic, that is,

$$\mathbf{K} \subseteq \mathbf{K}' \text{ implies } \mathbf{T}_P(\mathbf{K}) \subseteq \mathbf{T}_P(\mathbf{K}');$$

2. For all $\mathbf{K} \in \mathit{inst}(\mathit{sch}(P))$, we have:

$$\mathbf{K} \text{ is a model of } \Sigma_P \text{ if and only if } \mathbf{T}_P(\mathbf{K}) \subseteq \mathbf{K};$$

3. If $\mathbf{T}_P(\mathbf{K}) = \mathbf{K}$ (i.e., $\mathbf{K}$ is a fixpoint), then $\mathbf{K}$ is a model of Σ_P.

Note: The converse of 3. does not hold in general.

Datalog Semantics via Least Fixpoint

The semantics of P on a database instance $\mathbf{I}$ of *edb*(P) is a special fixpoint:

Theorem

Let P be a datalog program and $\mathbf{I}$ be a database instance. Then

1. $\mathbf{T}_P$ has a least (wrt "$\subseteq$") fixpoint containing $\mathbf{I}$, denoted *lfp*$(P, \mathbf{I})$.
2. Moreover, *lfp*$(P, \mathbf{I})$ = *MM*$(\mathcal{P}(P, \mathbf{I})) = P(\mathbf{I})$.

Constructive definition of $P(\mathbf{I})$ by *fixpoint iteration*

Proof (of Claim 2, first equality, sketch).

Let M_1 = *lfp*$(P, \mathbf{I})$ and M_2 = *MM*$(\mathcal{P}(P, \mathbf{I}))$.
Since M_1 is a fixpoint of $\mathbf{T}_P$, it is a model of Σ_P, and since it contains $\mathbf{I}$ it is a model of $\mathcal{P}(P, \mathbf{I})$. Hence, $M_2 \subseteq M_1$. Since M_2 is a model of $\mathcal{P}(P, \mathbf{I})$, it holds that $\mathbf{T}_P(M_2) \subseteq M_2$. Note that for every model M of $\mathcal{P}(P, \mathbf{I})$ we have, due to the monotonicity of $\mathbf{T}_P$, that $\mathbf{T}_P(M)$ is model. Hence, $\mathbf{T}_P(M_2) = M_2$, since M_2 is a minimal model. This implies that M_2 is a fixpoint, hence $M_1 \subseteq M_2$. $\square$

Fixpoint Iteration

For a datalog program P and an instance $\mathbf{I}$, we define the sequence $(\mathbf{I}_i)_{i \geq 0}$ by

$$\begin{aligned} \mathbf{I}_0 &= \mathbf{I} \\ \mathbf{I}_i &= \mathbf{T}_P(\mathbf{I}_{i-1}) \qquad \text{for } i > 0. \end{aligned}$$

We observe:

- By monotoncity of $\mathbf{T}_P$, we have $\quad \mathbf{I}_0 \subseteq \mathbf{I}_1 \subseteq \mathbf{I}_2 \subseteq \cdots \subseteq \mathbf{I}_i \subseteq \mathbf{I}_{i+1} \subseteq \cdots$
- For every $i \geq 0$, we have $\quad \mathbf{I}_i \subseteq \mathbf{B}(P, \mathbf{I})$
- Hence, for some integer $n \leq |\mathbf{B}(P, \mathbf{I})|$, we have $\mathbf{I}_{n+1} = \mathbf{I}_n$ $(=: \mathbf{T}_P^\omega(\mathbf{I}))$
- It holds that $\mathbf{T}_P^\omega(\mathbf{I}) = \mathit{lfp}(P, \mathbf{I}) = P(\mathbf{I})$.

This can be readily implemented by an algorithm.

Example

$$P = \{\,\texttt{reachable(a)},$$
$$\texttt{reachable(Y)} \leftarrow \texttt{arc(X,Y)}, \texttt{reachable(X)}\,\}$$
$$\mathbf{I} = \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c})\}$$

Then,

$$\begin{aligned}
\mathbf{I}_0 &= \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c})\}\\
\mathbf{I}_1 = \mathbf{T}_P^1(\mathbf{I}) &= \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}), \texttt{reachable}(\mathsf{a})\}\\
\mathbf{I}_2 = \mathbf{T}_P^2(\mathbf{I}) &= \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}), \texttt{reachable}(\mathsf{a}),\ \texttt{reachable}(\mathsf{b})\}\\
\mathbf{I}_3 = \mathbf{T}_P^3(\mathbf{I}) &= \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}), \texttt{reachable}(\mathsf{a}),\ \texttt{reachable}(\mathsf{b}),\ \texttt{reachable}(\mathsf{c})\}\\
\mathbf{I}_4 = \mathbf{T}_P^4(\mathbf{I}) &= \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}), \texttt{reachable}(\mathsf{a}),\ \texttt{reachable}(\mathsf{b}),\ \texttt{reachable}(\mathsf{c})\}\\
&= \mathbf{T}_P^3(\mathbf{I})
\end{aligned}$$

Thus, $\mathbf{T}_P^\omega(\mathbf{I}) = \mathit{lfp}(P, \mathbf{I}) = \mathbf{I}_4$.

Excursion: Fixpoint Theory

- Evaluating a datalog program P on $\mathbf{I}$ amounts to evaluating the logic program $\mathcal{P}(P, \mathbf{I})$
- For logic programs, fixpoint semantics is defined by appeal to fixpoint theory
- This provides another possibility to define semantics of datalog programs

Excursion: Fixpoint Theory/2

- A *complete lattice* is a partially ordered set $(U, \leq)$ such that each subset $V \subseteq U$ has a least upper bound *sup*(V) and a greatest lower bound *inf*(V), respectively.
- An operator $T\colon U \to U$ is
  - *monotone*, if for every x, $y \in U$ it holds that $x \leq y$ implies $T(x) \leq T(y)$
  - *continuous*, if $T(\mathit{sup}(V)) = \mathit{sup}(\{T(x) \mid x \in V\}$ for every $V \subseteq U$.

Notice: Continuous operators are monotone
Monotone and continuous operators have nice fixpoint properties

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Fixpoint Theorems of Knaster-Tarski and Kleene

Theorem

Every monotone operator T on a complete lattice $(U, \leq)$ has a least fixpoint *lfp*(T), and *lfp*$(T) = \mathit{inf}(\{x \in U \mid T(x) \leq x\})$.

A stronger theorem holds for continuous operators.

Theorem

Every continuous operator T on a complete lattice $(U, \leq)$ has a least fixpoint, and *lfp*$(T) = \mathit{sup}(\{T^i \mid i \geq 0\})$, where $T^0 = \mathit{inf}(U)$ and $T^{i+1} = T(T^i)$, for all $i \geq 0$.

Notation: $T^\infty = sup(\{T^i \mid i \geq 0\})$.

- Finite convergence: $T^k = T^{k-1}$ for some $k \Rightarrow T^\infty = T^k$
- A weaker form of Kleene's theorem holds for all monotone operators (transfinite sequence T^i).

Applying Fixpoint Theory

- For a logic program $\mathcal{P}$, the power set lattice $(P(\mathbf{HB}(\mathcal{P})), \subseteq)$ over the Herbrand base $\mathbf{HB}(\mathcal{P})$ is a complete lattice.
- We can associate with $\mathcal{P}$ an immediate consequence operator $T_{\mathcal{P}}$ on $\mathbf{HB}(\mathcal{P})$ such that $T_{\mathcal{P}}(I) = \{H(r) \mid r \in \mathit{Ground}(\mathcal{P}), B(r) \subseteq I\}$
- $T_{\mathcal{P}}$ is monotonic (in fact, continuous)
- Thus, $T_{\mathcal{P}}$ has the least fixpoint $\mathit{lfp}(T_{\mathcal{P}})$. It coincides with $T_{\mathcal{P}}^{\infty}$ and $\mathit{MM}(\mathcal{P})$

Theorem

Theorem. Given a datalog program P and a database instance $\mathbf{I}$,

$$P(\mathbf{I}) = \mathit{lfp}(T_{\mathcal{P}(P,\mathbf{I})}) = T^{\infty}_{\mathcal{P}(P\mathbf{I})}$$

Remark: Application of fixpoint theory is primarily of interest for infinite sets

Proof-Theoretic Approach

Basic idea: The answer of a datalog program P on $\mathbf{I}$ is given by the set of facts which can be *proved* from P and $\mathbf{I}$.

Definition (Proof tree)

A *proof tree* for a fact A from $\mathbf{I}$ and P is a labeled finite tree T such that

- each vertex of T is labeled by a fact
- the root of T is labeled by A
- each leaf of T is labeled by a fact in $\mathbf{I}$
- if a non-leaf of T is labeled with A_1 and its children are labeled with $A_2, \ldots, A_n$, then there exists a ground instance r of a rule in P such that $H(r) = A_1$ and $B(r) = \{A_2, \ldots, A_n\}$

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Example (Same Generation)

Let

$$P = \{ r_1 \colon \mathtt{sgc}(\mathtt{X}, \mathtt{X}) \ \leftarrow \ \mathtt{person}(\mathtt{X})$$
$$r_2 \colon \mathtt{sgc}(\mathtt{X}, \mathtt{Y}) \ \leftarrow \ \mathtt{par}(\mathtt{X}, \mathtt{X1}), \mathtt{sgc}(\mathtt{X1}, \mathtt{Y1}), \mathtt{par}(\mathtt{Y}, \mathtt{Y1}) \}$$

where *edb*$(P) = \{\mathtt{person}, \mathtt{par}\}$ and *idb*$(P) = \{\mathtt{sgc}\}$

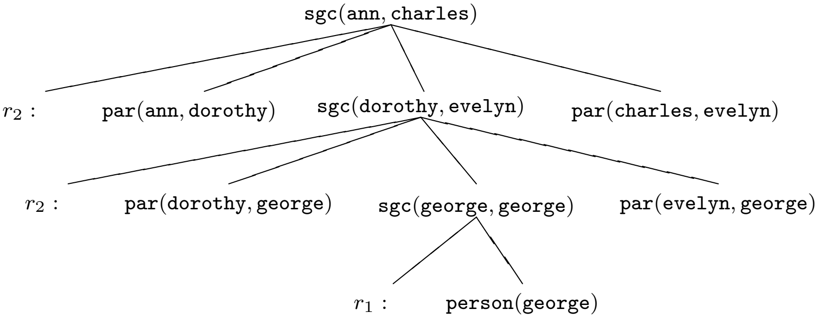
Consider $\mathbf{I}$ as follows:

$$\mathbf{I}(\mathtt{person}) = \{ \langle\mathtt{ann}\rangle, \ \langle\mathtt{bertrand}\rangle, \ \langle\mathtt{charles}\rangle, \langle\mathtt{dorothy}\rangle,$$
$$\langle\mathtt{evelyn}\rangle, \langle\mathtt{fred}\rangle, \ \langle\mathtt{george}\rangle, \ \langle\mathtt{hilary}\rangle \}$$

$$\mathbf{I}(\mathtt{par}) = \{ \langle\mathtt{dorothy}, \mathtt{george}\rangle, \ \langle\mathtt{evelyn}, \mathtt{george}\rangle, \ \langle\mathtt{bertrand}, \mathtt{dorothy}\rangle,$$
$$\langle\mathtt{ann}, \mathtt{dorothy}\rangle, \ \langle\mathtt{hilary}, \mathtt{ann}\rangle, \ \langle\mathtt{charles}, \mathtt{evelyn}\rangle \}.$$

Example (Same Generation)/2

Proof tree for $A = \texttt{sgc}(\texttt{ann}, \texttt{charles})$ from $\mathbf{I}$ and P:



Proof Tree Construction

There are different ways to construct a proof tree for A from P and $\mathbf{I}$:

- *Bottom Up construction:* From leaves to root

  Intimately related to fixpoint approach

  - Define $S \vdash_P B$ to prove fact B from facts S if $B \in S$ or by a rule in P
  - Give $S = \mathbf{I}$ for granted

- *Top Down construction:* From root to leaves

  In Logic Programming view, consider program $\mathcal{P}(P, \mathbf{I})$.

  - This amounts to a set of logical sentences $H_{\mathcal{P}(P,\mathbf{I})}$ of the form

$$\forall x_1 \cdots \forall x_m (R_1(\bar{x}_1) \vee \neg R_2(\bar{x}_2) \vee \neg R_3(\bar{x}_3) \vee \cdots \vee \neg R_n(\bar{x}_n))$$

  - Prove that $A = R(\bar{t})$ is a logical consequence via resolution refutation, that is, that $H_{\mathcal{P}(P,\mathbf{I})} \cup \{\neg A\}$ is unsatisfiable.

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Datalog and SLD Resolution

- Logic Programming uses SLD resolution
- SLD: Selection Rule Driven Linear Resolution for Definite Clauses
- For datalog programs P on $\mathbf{I}$, resp. $\mathcal{P}(P, \mathbf{I})$, things are simpler than for general logic programs (no function symbols, unification is easy)

Let $\mathit{SLD}(\mathcal{P})$ be the set of ground atoms provable with SLD Resolution from $\mathcal{P}$.

Theorem

For any datalog program P and database instance $\mathbf{I}$,

$$\mathit{SLD}(\mathcal{P}(P, \mathbf{I})) = P(\mathbf{I}) = \mathbf{T}^{\infty}_{\mathcal{P}(P,\mathcal{I})} = \mathit{lfp}(\mathbf{T}_{\mathcal{P}(P,\mathcal{I})}) = \mathit{MM}(\mathcal{P}(P, \mathbf{I}))$$

SLD Resolution – Termination

- Notice: Selection rule for next rule/atom to be considered for resolution might affect termination
- Prolog's strategy (leftmost atom/first rule) is problematic

Example:

$$\texttt{child\_of}(\texttt{karl}, \texttt{franz}).$$
$$\texttt{child\_of}(\texttt{franz}, \texttt{frieda}).$$
$$\texttt{child\_of}(\texttt{frieda}, \texttt{pia}).$$
$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \texttt{child\_of}(\texttt{X}, \texttt{Y}).$$
$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \texttt{child\_of}(\texttt{X}, \texttt{Z}), \texttt{descendent\_of}(\texttt{Z}, \texttt{Y}).$$
$$\leftarrow \texttt{descendent\_of}(\texttt{karl}, \texttt{X}).$$

SLD Resolution – Termination/2

Example (cntd.):

$$\texttt{child\_of}(\texttt{karl}, \texttt{franz}).$$

$$\texttt{child\_of}(\texttt{franz}, \texttt{frieda}).$$

$$\texttt{child\_of}(\texttt{frieda}, \texttt{pia}).$$

$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \texttt{child\_of}(\texttt{X}, \texttt{Y}).$$

$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \mathsf{descendent\_of}(\mathsf{X}, \mathsf{Z}), \mathsf{child\_of}(\mathsf{Z}, \mathsf{Y}).$$

$$\leftarrow \texttt{descendent\_of}(\texttt{karl}, \texttt{X}).$$

SLD Resolution – Termination /3

Example (cntd.):

$$\texttt{child\_of}(\texttt{karl}, \texttt{franz}).$$

$$\texttt{child\_of}(\texttt{franz}, \texttt{frieda}).$$

$$\texttt{child\_of}(\texttt{frieda}, \texttt{pia}).$$

$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \texttt{child\_of}(\texttt{X}, \texttt{Y}).$$

$$\begin{aligned}\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow\ & \texttt{descendent\_of}(\texttt{X}, \texttt{Z}),\\ & \texttt{descendent\_of}(\texttt{Z}, \texttt{Y}).\end{aligned}$$

$$\leftarrow \texttt{descendent\_of}(\texttt{karl}, \texttt{X}).$$

Exercise: Metro Reachability

Over the Metro database, consider the predicates `reachableFromOne`$/3$ and `reachableFromBoth`$/3$, with the following meaning for stations a, b, and c:

1. `reachableFromOne`(a, b, c) holds if c is reachable from one of a or b;
2. `reachableFromBoth`(a, b, c) holds if c is reachable from both of a and b.

Write datalog rules that define these predicates.

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