# Ontology and Database Systems: Foundations of Database Systems

## Part 5: Datalog

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# Motivation

- Relational Calculus and Relational Algebra were considered to be "*the*" database languages for a long time
- Codd: A query language is "complete," if it yields Relational Calculus
- However, Relational Calculus misses an important feature: *recursion*
- Example: A metro database with relation `links:line, station, nextstation`
  - What stations are reachable from station "Odeon"?
  - Can we go from Odeon to Tuileries?
  - etc.
- It can be proved: such queries cannot be expressed in Relational Calculus
- This motivated a logic-programming extension to conjunctive queries: *datalog*

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# Example: Metro Database Instance

| link | line | station | nextstation |
|---|---|---|---|
| | 4 | St. Germain | Odeon |
| | 4 | Odeon | St. Michel |
| | 4 | St. Michel | Chatelet |
| | 1 | Chatelet | Louvres |
| | 1 | Louvres | Palais Royal |
| | 1 | Palais-Royal | Tuileries |
| | 1 | Tuileries | Concorde |

Datalog program for the first query:

$$
\begin{array}{rcl}
\mathtt{reach}(\mathtt{X},\mathtt{X}) & \leftarrow & \mathtt{link}(\mathtt{L},\mathtt{X},\mathtt{Y}) \\
\mathtt{reach}(\mathtt{X},\mathtt{X}) & \leftarrow & \mathtt{link}(\mathtt{L},\mathtt{Y},\mathtt{X}) \\
\mathtt{reach}(\mathtt{X},\mathtt{Y}) & \leftarrow & \mathtt{link}(\mathtt{L},\mathtt{X},\mathtt{Z}),\mathtt{reach}(\mathtt{Z},\mathtt{Y}) \\
\mathtt{answer}(\mathtt{X}) & \leftarrow & \mathtt{reach}(\text{`}\mathtt{Odeon}\text{`},\mathtt{X})
\end{array}
$$

- Note: this is a recursive definition
- Intuitively, if the part right of "←" is true, the rule "fires" and the atom left of "←" is concluded.

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# Exercise

Write the following queries in datalog:

- Which stations can be reached from Chatelet, using exactly one line? (This excludes staying at Chatelet).
- Which stations can be reached from one another using exactly one line?
- Which stations can be reached from one another? (Check whether the query in the example is correct!)
- Which stations are terminal stops?

# The Datalog Language

- Datalog is akin to Logic Programming
- The basic language (considered next) has many extensions
- There exist several approaches to defining the semantics:

  Model-theoretic approach: View rules as logical sentences, which state the query result

  Operational (fixpoint) approach: Obtain query result by applying an inference procedure, until a fixpoint is reached

  Proof-theoretic approach: Obtain proofs of facts in the query result, following a proof calculus (based on resolution)

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# Datalog vs. Logic Programming

Although datalog is akin to Logic Programming,
there are important differences:

- There are **no functions symbols** in datalog
  $\rightsquigarrow$ no unbounded data structures, such as lists, are supported
- Datalog has a **purely declarative semantics**
  $\rightsquigarrow$ In a datalog program,
  - the *order of clauses* is irrelevant
  - the *order of atoms* in a rule body is irrelevant
- Datalog distinguishes between
  - database relations ("*extensional database*", *edb*) and
  - derived relations ("*intensional database*", *idb*)

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# Syntax of "plain datalog", or "datalog"

## Definition

A *datalog rule* $r$ is an expression of the form

$$R_0(\bar{x}_0) \leftarrow R_1(\bar{x}_1), \ldots, R_n(\bar{x}_n) \tag{1}$$

where

- $n \geq 0$,
- $R_0, \ldots, R_n$ are relations names,
- $\bar{x}_0, \ldots, \bar{x}_n$ are tuples of variables and constants (from **dom**), and
- every variable in $\bar{x}_0$ occurs in $\bar{x}_1, \ldots, \bar{x}_n$ ("safety")

## Remark

- The *head* of $r$, denoted $H(r)$, is $R_0(\bar{x}_0)$
- The *body* of $r$, denoted $B(r)$, is $\{ R_1(\bar{x}_1), \ldots, R_n(\bar{x}_n) \}$
- The rule symbol "$\leftarrow$" is often also written as ":-"

# Datalog Programs

**Definition**

A *datalog program* is a finite set of datalog rules.

Let $P$ be a datalog program.

- An *extensional relation* of $P$ is a relation occurring only in rule bodies of $P$
- An *intensional relation* of $P$ is a relation occurring in the head of some rule in $P$
- The *extensional schema* of $P$, *edb*$(P)$, consists of all extensional relations of $P$
- The *intensional schema* of $P$, *idb*$(P)$, consists of all intensional relations of $P$
- The *schema* of $P$, *sch*$(P)$, is the union of *edb*$(P)$ and *idb*$(P)$.

# The Metro Example /1

Datalog program $P$ on the metro database schema (w/o integrity constraints)

$$\mathcal{M} = \{\texttt{link}(\texttt{line},\ \texttt{station},\ \texttt{nextstation})\}:$$

$$\begin{aligned}
\texttt{reach}(\texttt{X},\texttt{X}) &\leftarrow \texttt{link}(\texttt{L},\texttt{X},\texttt{Y}) \\
\texttt{reach}(\texttt{X},\texttt{X}) &\leftarrow \texttt{link}(\texttt{L},\texttt{Y},\texttt{X}) \\
\texttt{reach}(\texttt{X},\texttt{Y}) &\leftarrow \texttt{link}(\texttt{L},\texttt{X},\texttt{Z}), \texttt{reach}(\texttt{Z},\texttt{Y}) \\
\texttt{answer}(\texttt{X}) &\leftarrow \texttt{reach('Odeon',X)}
\end{aligned}$$

Here,

$$\begin{aligned}
\mathit{edb}(P) &= \{\texttt{link}\} \quad (= \mathcal{M}), \\
\mathit{idb}(P) &= \{\texttt{reach}, \texttt{answer}\}, \\
\mathit{sch}(P) &= \{\texttt{link}, \texttt{reach}, \texttt{answer}\}
\end{aligned}$$

# Datalog Syntax (cntd)

- The set of constants occurring in program $P$ is denoted as *adom*$(P)$
- The *active domain* of $P$ with respect to an instance $\mathbf{I}$ is defined as

$$\mathit{adom}(P, \mathbf{I}) := \mathit{adom}(P) \cup \mathit{adom}(\mathbf{I}),$$

that is, as the set of constants occurring in $P$ and $\mathbf{I}$

## Definition (Rule Instantiation)

Let $\alpha\colon \mathit{var}(r) \cup \mathbf{dom} \to \mathbf{dom}$ be an assignment for the variables in a rule $r$ of form (1). Then the *instantiation* of $r$ with $\alpha$, denoted $\alpha(r)$, is the rule

$$R_0(\alpha(\bar{x}_0)) \leftarrow R_1(\alpha(\bar{x}_1)), \ldots, R_n(\alpha(\bar{x}_n)),$$

which results from replacing each variable $x$ with $\alpha(x)$.

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# The Metro Example/2

- For the datalog program $P$ above, we have that $\mathit{adom}(P) = \{$ Odeon $\}$
- We consider the database instance $\mathbf{I}$:

| link | line | station | nextstation |
|---|---|---|---|
| | 4 | St. Germain | Odeon |
| | 4 | Odeon | St. Michel |
| | 4 | St. Michel | Chatelet |
| | 1 | Chatelet | Louvre |
| | 1 | Louvre | Palais-Royal |
| | 1 | Palais-Royal | Tuileries |
| | 1 | Tuileries | Concorde |

Then $\mathit{adom}(\mathbf{I}) = \{$4, 1, St.Germain, Odeon, St.Michel, Chatelet, Louvres, Palais-Royal, Tuileries, Concorde$\}$

- Also $\mathit{adom}(P, \mathbf{I}) = \mathit{adom}(\mathbf{I})$

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# The Metro Example/3

- The rule

$$\mathtt{reach}(\mathtt{St.Germain}, \mathtt{Odeon}) \leftarrow \mathtt{link}(\mathtt{Louvre}, \mathtt{St.Germain}, \mathtt{Concorde}), \mathtt{reach}(\mathtt{Concorde}, \mathtt{Odeon})$$

is an instantiation of the rule

$$\mathtt{reach}(\mathtt{X}, \mathtt{Y}) \leftarrow \mathtt{link}(\mathtt{L}, \mathtt{X}, \mathtt{Z}), \mathtt{reach}(\mathtt{Z}, \mathtt{Y})$$

(take $\alpha(\mathtt{X}) = \mathtt{St.Germain}$, $\alpha(\mathtt{L}) = \mathtt{Louvre}$, $\alpha(\mathtt{Y}) = \mathtt{Odeon}$, $\alpha(\mathtt{Z}) = \mathtt{Concorde}$)

# Datalog: Model-Theoretic Semantics

**General Idea:**

- We view a program as a set of first-order sentences
- Given an instance $\mathbf{I}$ of *edb*$(P)$,
  the result of $P$ is a database instance of *sch*$(P)$
  that extends $\mathbf{I}$ and satisfies the sentences
  (or, is a *model* of the sentences)
- There can be many models
- The *intended answer* is specified by particular models
- These particular models are selected by "external" conditions

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# Logical Theory $\Sigma_P$

- To every datalog rule $r$ of the form $R_0(\bar{x}_0) \leftarrow R_1(\bar{x}_1), \ldots, R_n(\bar{x}_n)$, with variables $x_1, \ldots, x_m$, we associate the logical sentence $\sigma(r)$:

$$\forall x_1, \cdots \forall x_m \, (R_1(\bar{x}_1) \wedge \cdots \wedge R_n(\bar{x}_n) \rightarrow R_0(\bar{x}_0))$$

- To a program $P$, we associate the set of sentences $\Sigma_P = \{\sigma(r) \mid r \in P\}$

**Definition**

Let $P$ be a datalog program and $\mathbf{I}$ an instance of *edb*$(P)$. Then,

- A *model* of $P$ is an instance of *sch*$(P)$ that satisfies $\Sigma_P$
- We compare models wrt set inclusion "$\subseteq$"
  (in the Logic Programming perspective)
- The *semantics* of $P$ on input $\mathbf{I}$, denoted $P(\mathbf{I})$,
  is the *least model* of $P$ containing $\mathbf{I}$, if it exists

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# Example

For program $P$ and instance $\mathbf{I}$ of the Metro Example, the least model is:

| link | line | station | nextstation |
|---|---|---|---|
| | 4 | St. Germain | Odeon |
| | 4 | Odeon | St. Michel |
| | 4 | St. Michel | Chatelet |
| | 1 | Chatelet | Louvres |
| | 1 | Louvres | Palais-Royal |
| | 1 | Palais-Royal | Tuileries |
| | 1 | Tuileries | Concorde |

| reach | | |
|---|---|---|
| | St. Germain | St. Germain |
| | Odeon | Odeon |
| | ... | |
| | Concorde | Concorde |
| | St. Germain | Odeon |
| | St. Germain | St.Michel |
| | St. Germain | Chatelet |
| | St. Germain | Louvre |
| | ... | |

| answer | |
|---|---|
| | Odeon |
| | St. Michel |
| | Chatelet |
| | Louvre |
| | Palais-Royal |
| | Tuileries |
| | Concorde |

# Questions

1. Is the semantics $P(\mathbf{I})$ well-defined for every input instance $\mathbf{I}$?
2. How can one compute $P(\mathbf{I})$?

Observation: For any $\mathbf{I}$, there is a model of $P$ containing $\mathbf{I}$

- Let $\mathbf{B}(P, \mathbf{I})$ be the instance of *sch*$(P)$ such that

$$\mathbf{B}(P,\mathbf{I})(R) = \begin{cases} \mathbf{I}(R) & \text{for each } R \in \textit{edb}(P) \\ \textit{adom}(P,\mathbf{I})^{\textit{ary}(R)} & \text{for each } R \in \textit{idb}(P) \end{cases}$$

- Then: $\mathbf{B}(P, \mathbf{I})$ is a model of $P$ containing $\mathbf{I}$
  $\Rightarrow$ $P(\mathbf{I})$ is a subset of $\mathbf{B}(P, \mathbf{I})$ *(if it exists)*
- Naive algorithm: explore all subsets of $\mathbf{B}(P, \mathbf{I})$

# Elementary Properties of $P(\mathbf{I})$

Let $P$ be a datalog program, $\mathbf{I}$ an instance of *edb*$(P)$, and $\mathcal{M}(\mathbf{I})$ the set of all models of $P$ containing $\mathbf{I}$.

**Theorem**

The intersection $\bigcap_{M \in \mathcal{M}(\mathbf{I})} M$ is a model of $P$.

**Corollary**

1. $P(\mathbf{I}) = \bigcap_{M \in \mathcal{M}(\mathbf{I})} M$
2. *adom*$(P(\mathbf{I})) \subseteq$ *adom*$(P, \mathbf{I})$, that is, no new values appear
3. $P(\mathbf{I})(R) = \mathbf{I}(R)$, for each $R \in$ *edb*$(P)$

**Consequences:**

- $P(\mathbf{I})$ is well-defined for every $\mathbf{I}$
- If $P$ and $\mathbf{I}$ are finite, the $P(\mathbf{I})$ is finite

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# Why Choose the Least Model?

There are two reasons to choose the least model containing $\mathbf{I}$:

1. The *Closed World Assumption*:
   - If a fact $R(\bar{c})$ is not true in all models of a database $\mathbf{I}$, then infer that $R(\bar{c})$ is false
   - This amounts to considering $\mathbf{I}$ as complete
   - . . . which is customary in database practice
2. The relationship to Logic Programming:
   - Datalog should desirably match Logic Programming (seamless integration)
   - Logic Programming builds on the minimal model semantics

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# Relating Datalog to Logic Programming

- A logic program makes no distinction between *edb* and *idb*
- A datalog program $P$ and an instance $\mathbf{I}$ of *edb*$(P)$ can be mapped to the logic program

$$\mathcal{P}(P, \mathbf{I}) = P \cup \mathbf{I}$$

  (where $\mathbf{I}$ is viewed as a set of atoms in the Logic Programming perspective)
- Correspondingly, we define the logical theory

$$\Sigma_{P,\mathbf{I}} = \Sigma_P \cup \mathbf{I}$$

- The semantics of the logic program $\mathcal{P} = \mathcal{P}(P, \mathbf{I})$ is defined in terms of *Herbrand interpretations* of the language induced by $\mathcal{P}$:
  - The domain of discourse is formed by the constants occurring in $\mathcal{P}$
  - Each constant occurring in $\mathcal{P}$ is interpreted by itself

# Herbrand Interpretations of Logic Programs

Given a rule $r$, we denote by $\mathit{Const}(r)$ the set of all constants in $r$

**Definition**

For a (function-free) logic program $\mathcal{P}$, we define

- the *Herbrand universe* of $\mathcal{P}$, by

$$\mathbf{HU}(\mathcal{P}) = \bigcup_{r \in \mathcal{P}} \mathit{Const}(r)$$

- the *Herbrand base* of $\mathcal{P}$, by

$$\mathbf{HB}(\mathcal{P}) = \{R(c_1, \ldots, c_n) \mid R \text{ is a relation in } \mathcal{P}, \\ c_1, \ldots, c_n \in \mathbf{HU}(\mathcal{P}), \text{ and } \mathit{ary}(R) = n\}$$

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# Example

$$\mathcal{P} = \{ \quad \texttt{arc}(\texttt{a}, \texttt{b}).$$
$$\texttt{arc}(\texttt{b}, \texttt{c}).$$
$$\texttt{reachable}(\texttt{a}).$$
$$\texttt{reachable}(\texttt{Y}) \leftarrow \texttt{arc}(\texttt{X}, \texttt{Y}), \texttt{reachable}(\texttt{X}). \;\}$$

$$\mathbf{HU}(\mathcal{P}) \quad = \quad \{\texttt{a}, \texttt{b}, \texttt{c}\}$$

$$\begin{aligned}\mathbf{HB}(\mathcal{P}) \quad = \quad \{&\texttt{arc}(\texttt{a}, \texttt{a}),\ \texttt{arc}(\texttt{a}, \texttt{b}),\ \texttt{arc}(\texttt{a}, \texttt{c}),\\ &\texttt{arc}(\texttt{b}, \texttt{a}),\ \texttt{arc}(\texttt{b}, \texttt{b}),\ \texttt{arc}(\texttt{b}, \texttt{c}),\\ &\texttt{arc}(\texttt{c}, \texttt{a}),\ \texttt{arc}(\texttt{c}, \texttt{b}),\ \texttt{arc}(\texttt{c}, \texttt{c}),\\ &\texttt{reachable}(\texttt{a}),\ \texttt{reachable}(\texttt{b}),\ \texttt{reachable}(\texttt{c})\}\end{aligned}$$

# Grounding

- A rule $r'$ is a *ground instance* of a rule $r$ with respect to $\mathbf{HU}(\mathcal{P})$,
  if $r' = \alpha(r)$ for an assignment $\alpha$
  such that $\alpha(x) \in \mathbf{HU}(\mathcal{P})$ for each $x \in \mathit{var}(r)$
- The *grounding* of a rule $r$ with respect to $\mathbf{HU}(\mathcal{P})$,
  denoted $\mathit{Ground}_{\mathcal{P}}(r)$,
  is the set of all ground instances of $r$ wrt $\mathbf{HU}(\mathcal{P})$
- The *grounding* of a logic program $\mathcal{P}$ is

$$\mathit{Ground}(\mathcal{P}) = \bigcup_{r \in \mathcal{P}} \mathit{Ground}_{\mathcal{P}}(r)$$

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# Example

$$
\begin{aligned}
\mathit{Ground}(\mathcal{P}) = \{ & \texttt{arc}(\texttt{a},\texttt{b}).\ \ \texttt{arc}(\texttt{b},\texttt{c}).\ \ \texttt{reachable}(\texttt{a}). \\
& \texttt{reachable}(\texttt{a}) \leftarrow \texttt{arc}(\texttt{a},\texttt{a}), \texttt{reachable}(\texttt{a}). \\
& \texttt{reachable}(\texttt{b}) \leftarrow \texttt{arc}(\texttt{a},\texttt{b}), \texttt{reachable}(\texttt{a}). \\
& \texttt{reachable}(\texttt{c}) \leftarrow \texttt{arc}(\texttt{a},\texttt{c}), \texttt{reachable}(\texttt{a}). \\
& \texttt{reachable}(\texttt{a}) \leftarrow \texttt{arc}(\texttt{b},\texttt{a}), \texttt{reachable}(\texttt{b}). \\
& \texttt{reachable}(\texttt{b}) \leftarrow \texttt{arc}(\texttt{b},\texttt{b}), \texttt{reachable}(\texttt{b}). \\
& \texttt{reachable}(\texttt{c}) \leftarrow \texttt{arc}(\texttt{b},\texttt{c}), \texttt{reachable}(\texttt{b}). \\
& \texttt{reachable}(\texttt{a}) \leftarrow \texttt{arc}(\texttt{c},\texttt{a}), \texttt{reachable}(\texttt{c}). \\
& \texttt{reachable}(\texttt{b}) \leftarrow \texttt{arc}(\texttt{c},\texttt{b}), \texttt{reachable}(\texttt{c}). \\
& \texttt{reachable}(\texttt{c}) \leftarrow \texttt{arc}(\texttt{c},\texttt{c}), \texttt{reachable}(\texttt{c}).\ \}
\end{aligned}
$$

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# Herbrand Models

- A *Herbrand-interpretation* $I$ of $\mathcal{P}$ is any subset $I \subseteq \mathbf{HB}(\mathcal{P})$
- A *Herbrand-model* of $\mathcal{P}$ is a Herbrand-interpretation that satisfies all sentences in $\Sigma_{P,\mathbf{I}}$
- Equivalently, $M \subseteq \mathbf{HB}(\mathcal{P})$ is a Herbrand model if
  for all $r \in$ *Ground*($\mathcal{P}$) such that $B(r) \subseteq M$
  we have that $H(r) \subseteq M$

# Example

The Herbrand models of program $\mathcal{P}$ above are exactly the following:

- $M_1 = \{\ \texttt{arc}(\texttt{a},\texttt{b}), \texttt{arc}(\texttt{b},\texttt{c}),$
  $\texttt{reachable}(\texttt{a}), \texttt{reachable}(\texttt{b}), \texttt{reachable}(\texttt{c})\ \}$
- $M_2 = \mathbf{HB}(\mathcal{P})$
- every interpretation $M$ such that $M_1 \subseteq M \subseteq M_2$

and no others.

# Logic Programming Semantics

**Proposition**

$\mathbf{HB}(\mathcal{P})$ is always a model of $\mathcal{P}$

**Theorem**

For every logic program there exists a least Herbrand model (wrt "$\subseteq$").

For a program $\mathcal{P}$, this model is denoted $MM(\mathcal{P})$ (for "minimal model").
The model $MM(\mathcal{P})$ is the semantics of $\mathcal{P}$.

**Theorem (Datalog $\leftrightarrow$ Logic Programming))**

Let $P$ be a datalog program and $\mathbf{I}$ be an instance of $edb(P)$. Then,

$$P(\mathbf{I}) = MM(\mathcal{P}(P, \mathbf{I}))$$

# Consequences

Results and techniques for Logic Programming can be exploited for datalog.

For example,

- proof procedures for Logic Programming (e.g., SLD resolution) can be applied to datalog (with some caveats, regarding for instance termination)
- datalog can be reduced by "grounding" to propositional logic programs

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# Fixpoint Semantics

Another view:

*"If all facts in $\mathbf{I}$ hold, which other facts must hold after firing the rules in $P$?"*

Approach:

- Define an *immediate consequence operator* $\mathbf{T}_P(\mathbf{K})$ on db instances $\mathbf{K}$
- Start with $\mathbf{K} = \mathbf{I}$
- Apply $\mathbf{T}_P$ to obtain a new instance: $\mathbf{K}_{\text{new}} := \mathbf{T}_P(\mathbf{K}) = \mathbf{I} \cup$ new facts
- Iterate until nothing new can be produced
- The result yields the semantics

# Immediate Consequence Operator

Let $P$ be a datalog program and $\mathbf{K}$ be a database instance of *sch*$(P)$.
A fact $R(\bar{t})$ is an *immediate* consequence for $\mathbf{K}$ and $P$, if either

- $R \in$ *edb*$(P)$ and $R(\bar{t}) \in \mathbf{K}$, or
- there exists a ground instance $r$ of a rule in $P$ such that $H(r) = R(\bar{t})$ and $B(r) \subseteq \mathbf{K}$.

## Definition (Immediate Consequence Operator)

The *immediate consequence operator* of a datalog program $P$ is the mapping

$$\mathbf{T}_P \colon \mathit{inst}(\mathit{sch}(P)) \to \mathit{inst}(\mathit{sch}(P))$$

where

$$\mathbf{T}_P(\mathbf{K}) = \{A \mid A \text{ is an immediate consequence for } \mathbf{K} \text{ and } P\}.$$

# Example

Consider

$$P = \{ \quad \texttt{reachable}(\texttt{a}), \\ \texttt{reachable}(\texttt{Y}) \leftarrow \texttt{arc}(\texttt{X}, \texttt{Y}), \texttt{reachable}(\texttt{X}) \ \}$$

where *edb*$(P) = \{\texttt{arc}\}$ and *idb*$(P) = \{\texttt{reachable}\}$.

Let

$$\begin{aligned}
\mathbf{I} = \mathbf{K}_1 &= \{\texttt{arc}(\texttt{a}, \texttt{b}),\ \texttt{arc}(\texttt{b}, \texttt{c})\} \\
\mathbf{K}_2 &= \{\texttt{arc}(\texttt{a}, \texttt{b}),\ \texttt{arc}(\texttt{b}, \texttt{c}),\ \texttt{reachable}(\texttt{a})\} \\
\mathbf{K}_3 &= \{\texttt{arc}(\texttt{a}, \texttt{b}),\ \texttt{arc}(\texttt{b}, \texttt{c}),\ \texttt{reachable}(\texttt{a}),\ \texttt{reachable}(\texttt{b})\ \} \\
\mathbf{K}_4 &= \{\texttt{arc}(\texttt{a}, \texttt{b}),\ \texttt{arc}(\texttt{b}, \texttt{c}),\ \texttt{reachable}(\texttt{a}),\ \texttt{reachable}(\texttt{b}),\ \texttt{reachable}(\texttt{c})\}
\end{aligned}$$

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# Example (cntd)

Then,

$$\mathbf{T}_P(\mathbf{K}_1) = \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}),\ \texttt{reachable}(\mathsf{a})\} = \mathbf{K}_2$$
$$\mathbf{T}_P(\mathbf{K}_2) = \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}),\ \texttt{reachable}(\mathsf{a}),\ \texttt{reachable}(\mathsf{b})\} = \mathbf{K}_3$$
$$\mathbf{T}_P(\mathbf{K}_3) = \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}),\ \texttt{reachable}(\mathsf{a}),\ \texttt{reachable}(\mathsf{b}),\ \texttt{reachable}(\mathsf{c})\} = \mathbf{K}_4$$
$$\mathbf{T}_P(\mathbf{K}_4) = \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}),\ \texttt{reachable}(\mathsf{a}),\ \texttt{reachable}(\mathsf{b}),\ \texttt{reachable}(\mathsf{c})\} = \mathbf{K}_4$$

Thus, $\mathbf{K}_4$ is a *fixpoint* of $\mathbf{T}_P$.

**Definition**

$\mathbf{K}$ is a *fixpoint* of operator $\mathbf{T}_P$ if $\mathbf{T}_P(\mathbf{K}) = \mathbf{K}$

# Properties

## Proposition

Let $P$ be a datalog program.

1. The operator $\mathbf{T}_P$ is monotonic, that is,

$$\mathbf{K} \subseteq \mathbf{K}' \text{ implies } \mathbf{T}_P(\mathbf{K}) \subseteq \mathbf{T}_P(\mathbf{K}');$$

2. For all $\mathbf{K} \in \mathit{inst}(\mathit{sch}(P))$, we have:

$$\mathbf{K} \text{ is a model of } \Sigma_P \text{ if and only if } \mathbf{T}_P(\mathbf{K}) \subseteq \mathbf{K};$$

3. If $\mathbf{T}_P(\mathbf{K}) = \mathbf{K}$ (i.e., $\mathbf{K}$ is a fixpoint), then $\mathbf{K}$ is a model of $\Sigma_P$.

Note: The converse of 3. does not hold in general.

# Datalog Semantics via Least Fixpoint

The semantics of $P$ on a database instance $\mathbf{I}$ of *edb*$(P)$ is a special fixpoint:

**Theorem**

Let $P$ be a datalog program and $\mathbf{I}$ be a database instance. Then

1. $\mathbf{T}_P$ has a least (wrt "$\subseteq$") fixpoint containing $\mathbf{I}$, denoted *lfp*$(P, \mathbf{I})$.
2. Moreover, *lfp*$(P, \mathbf{I}) =$ *MM*$(\mathcal{P}(P, \mathbf{I})) = P(\mathbf{I})$.

Constructive definition of $P(\mathbf{I})$ by *fixpoint iteration*

**Proof (of Claim 2, first equality, sketch).**

Let $M_1 =$ *lfp*$(P, \mathbf{I})$ and $M_2 =$ *MM*$(\mathcal{P}(P, \mathbf{I}))$.
Since $M_1$ is a fixpoint of $\mathbf{T}_P$, it is a model of $\Sigma_P$, and since it contains $\mathbf{I}$ it is a model of $\mathcal{P}(P, \mathbf{I})$. Hence, $M_2 \subseteq M_1$. Since $M_2$ is a model of $\mathcal{P}(P, \mathbf{I})$, it holds that $\mathbf{T}_P(M_2) \subseteq M_2$. Note that for every model $M$ of $\mathcal{P}(P, \mathbf{I})$ we have, due to the monotonicity of $\mathbf{T}_P$, that $\mathbf{T}_P(M)$ is model. Hence, $\mathbf{T}_P(M_2) = M_2$, since $M_2$ is a minimal model. This implies that $M_2$ is a fixpoint, hence $M_1 \subseteq M_2$. □

# Fixpoint Iteration

For a datalog program $P$ and an instance $\mathbf{I}$, we define the sequence $(\mathbf{I}_i)_{i\geq 0}$ by

$$\begin{aligned} \mathbf{I}_0 &= \mathbf{I} \\ \mathbf{I}_i &= \mathbf{T}_P(\mathbf{I}_{i-1}) \qquad \text{for } i > 0. \end{aligned}$$

We observe:

- By monotoncity of $\mathbf{T}_P$, we have $\mathbf{I}_0 \subseteq \mathbf{I}_1 \subseteq \mathbf{I}_2 \subseteq \cdots \subseteq \mathbf{I}_i \subseteq \mathbf{I}_{i+1} \subseteq \cdots$
- For every $i \geq 0$, we have $\mathbf{I}_i \subseteq \mathbf{B}(P, \mathbf{I})$
- Hence, for some integer $n \leq |\mathbf{B}(P, \mathbf{I})|$, we have $\mathbf{I}_{n+1} = \mathbf{I}_n$ $(=: \mathbf{T}_P^\omega(\mathbf{I}))$
- It holds that $\mathbf{T}_P^\omega(\mathbf{I}) = \mathit{lfp}(P, \mathbf{I}) = P(\mathbf{I})$.

This can be readily implemented by an algorithm.

# Example

$$P = \{\, \texttt{reachable(a)},$$
$$\texttt{reachable(Y)} \leftarrow \texttt{arc(X,Y)}, \texttt{reachable(X)} \,\}$$
$$\mathbf{I} = \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c})\}$$

Then,

$$
\begin{aligned}
\mathbf{I}_0 &= \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c})\}\\
\mathbf{I}_1 = \mathbf{T}_P^1(\mathbf{I}) &= \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}), \texttt{reachable}(\mathsf{a})\}\\
\mathbf{I}_2 = \mathbf{T}_P^2(\mathbf{I}) &= \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}), \texttt{reachable}(\mathsf{a}),\ \texttt{reachable}(\mathsf{b})\}\\
\mathbf{I}_3 = \mathbf{T}_P^3(\mathbf{I}) &= \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}), \texttt{reachable}(\mathsf{a}),\ \texttt{reachable}(\mathsf{b}),\ \texttt{reachable}(\mathsf{c})\}\\
\mathbf{I}_4 = \mathbf{T}_P^4(\mathbf{I}) &= \{\texttt{arc}(\mathsf{a},\mathsf{b}),\ \texttt{arc}(\mathsf{b},\mathsf{c}), \texttt{reachable}(\mathsf{a}),\ \texttt{reachable}(\mathsf{b}),\ \texttt{reachable}(\mathsf{c})\}\\
&= \mathbf{T}_P^3(\mathbf{I})
\end{aligned}
$$

Thus, $\mathbf{T}_P^\omega(\mathbf{I}) = \mathit{lfp}(P, \mathbf{I}) = \mathbf{I}_4$.

# Excursion: Fixpoint Theory

- Evaluating a datalog program $P$ on $\mathbf{I}$ amounts to evaluating the logic program $\mathcal{P}(P, \mathbf{I})$
- For logic programs, fixpoint semantics is defined by appeal to fixpoint theory
- This provides another possibility to define semantics of datalog programs

# Excursion: Fixpoint Theory/2

- A *complete lattice* is a partially ordered set $(U, \leq)$ such that each subset $V \subseteq U$ has a least upper bound *sup*$(V)$ and a greatest lower bound *inf*$(V)$, respectively.
- An operator $T\colon U \to U$ is
  - *monotone*, if for every $x$, $y \in U$ it holds that $x \leq y$ implies $T(x) \leq T(y)$
  - *continuous*, if $T(\mathit{sup}(V)) = \mathit{sup}(\{T(x) \mid x \in V\}$ for every $V \subseteq U$.

Notice: Continuous operators are monotone
Monotone and continuous operators have nice fixpoint properties

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# Fixpoint Theorems of Knaster-Tarski and Kleene

**Theorem**

Every monotone operator $T$ on a complete lattice $(U, \leq)$ has a least fixpoint *lfp*$(T)$, and *lfp*$(T) =$ *inf*$(\{x \in U \mid T(x) \leq x\})$.

A stronger theorem holds for continuous operators.

**Theorem**

Every continuous operator $T$ on a complete lattice $(U, \leq)$ has a least fixpoint, and *lfp*$(T) =$ *sup*$(\{T^i \mid i \geq 0\})$, where $T^0 =$ *inf*$(U)$ and $T^{i+1} = T(T^i)$, for all $i \geq 0$.

Notation: $T^\infty = sup(\{T^i \mid i \geq 0\})$.

- Finite convergence: $T^k = T^{k-1}$ for some $k \Rightarrow T^\infty = T^k$
- A weaker form of Kleene's theorem holds for all monotone operators (transfinite sequence $T^i$).

# Applying Fixpoint Theory

- For a logic program $\mathcal{P}$, the power set lattice $(P(\mathbf{HB}(\mathcal{P})), \subseteq)$ over the Herbrand base $\mathbf{HB}(\mathcal{P})$ is a complete lattice.
- We can associate with $\mathcal{P}$ an immediate consequence operator $T_{\mathcal{P}}$ on $\mathbf{HB}(\mathcal{P})$ such that $T_{\mathcal{P}}(I) = \{H(r) \mid r \in \mathit{Ground}(\mathcal{P}), B(r) \subseteq I\}$
- $T_{\mathcal{P}}$ is monotonic (in fact, continuous)
- Thus, $T_{\mathcal{P}}$ has the least fixpoint *lfp*$(T_{\mathcal{P}})$. It coincides with $T_{\mathcal{P}}^{\infty}$ and *MM*$(\mathcal{P})$

### Theorem

**Theorem.** Given a datalog program $P$ and a database instance $\mathbf{I}$,

$$P(\mathbf{I}) = \mathit{lfp}(T_{\mathcal{P}(P,\mathbf{I})}) = T^{\infty}_{\mathcal{P}(P\mathbf{I})}$$

Remark: Application of fixpoint theory is primarily of interest for infinite sets

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# Proof-Theoretic Approach

Basic idea: The answer of a datalog program $P$ on $\mathbf{I}$ is given by the set of facts which can be *proved* from $P$ and $\mathbf{I}$.

## Definition (Proof tree)

A *proof tree* for a fact $A$ from $\mathbf{I}$ and $P$ is a labeled finite tree $T$ such that

- each vertex of $T$ is labeled by a fact
- the root of $T$ is labeled by $A$
- each leaf of $T$ is labeled by a fact in $\mathbf{I}$
- if a non-leaf of $T$ is labeled with $A_1$ and its children are labeled with $A_2, \ldots, A_n$, then there exists a ground instance $r$ of a rule in $P$ such that $H(r) = A_1$ and $B(r) = \{A_2, \ldots, A_n\}$

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# Example (Same Generation)

Let

$$P = \{ r_1 : \mathtt{sgc}(\mathtt{X},\mathtt{X}) \leftarrow \mathtt{person}(\mathtt{X})$$
$$r_2 : \mathtt{sgc}(\mathtt{X},\mathtt{Y}) \leftarrow \mathtt{par}(\mathtt{X},\mathtt{X1}), \mathtt{sgc}(\mathtt{X1},\mathtt{Y1}), \mathtt{par}(\mathtt{Y},\mathtt{Y1}) \}$$

where *edb*$(P) = \{\mathtt{person}, \mathtt{par}\}$ and *idb*$(P) = \{\mathtt{sgc}\}$

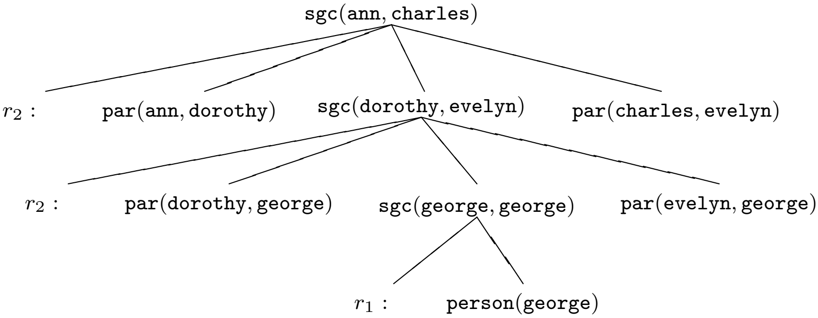
Consider $\mathbf{I}$ as follows:

$$\mathbf{I}(\mathtt{person}) = \{ \langle \mathtt{ann} \rangle, \langle \mathtt{bertrand} \rangle, \langle \mathtt{charles} \rangle, \langle \mathtt{dorothy} \rangle, \langle \mathtt{evelyn} \rangle, \langle \mathtt{fred} \rangle, \langle \mathtt{george} \rangle, \langle \mathtt{hilary} \rangle \}$$

$$\mathbf{I}(\mathtt{par}) = \{ \langle \mathtt{dorothy}, \mathtt{george} \rangle, \langle \mathtt{evelyn}, \mathtt{george} \rangle, \langle \mathtt{bertrand}, \mathtt{dorothy} \rangle, \langle \mathtt{ann}, \mathtt{dorothy} \rangle, \langle \mathtt{hilary}, \mathtt{ann} \rangle, \langle \mathtt{charles}, \mathtt{evelyn} \rangle \}.$$

# Example (Same Generation)/2

Proof tree for $A = \mathtt{sgc}(\mathtt{ann}, \mathtt{charles})$ from $\mathbf{I}$ and $P$:



# Proof Tree Construction

There are different ways to construct a proof tree for $A$ from $P$ and $\mathbf{I}$:

- *Bottom Up construction:* From leaves to root

  Intimately related to fixpoint approach

  - Define $S \vdash_P B$ to prove fact $B$ from facts $S$ if $B \in S$ or by a rule in $P$
  - Give $S = \mathbf{I}$ for granted

- *Top Down construction:* From root to leaves

  In Logic Programming view, consider program $\mathcal{P}(P, \mathbf{I})$.

  - This amounts to a set of logical sentences $H_{\mathcal{P}(P,\mathbf{I})}$ of the form

    $$\forall x_1 \cdots \forall x_m (R_1(\bar{x}_1) \vee \neg R_2(\bar{x}_2) \vee \neg R_3(\bar{x}_3) \vee \cdots \vee \neg R_n(\bar{x}_n))$$

  - Prove that $A = R(\bar{t})$ is a logical consequence via resolution refutation, that is, that $H_{\mathcal{P}(P,\mathbf{I})} \cup \{\neg A\}$ is unsatisfiable.

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# Datalog and SLD Resolution

- Logic Programming uses SLD resolution
- SLD: Selection Rule Driven Linear Resolution for Definite Clauses
- For datalog programs $P$ on $\mathbf{I}$, resp. $\mathcal{P}(P, \mathbf{I})$, things are simpler than for general logic programs (no function symbols, unification is easy)

Let $\mathit{SLD}(\mathcal{P})$ be the set of ground atoms provable with SLD Resolution from $\mathcal{P}$.

**Theorem**

For any datalog program $P$ and database instance $\mathbf{I}$,

$$\mathit{SLD}(\mathcal{P}(P, \mathbf{I})) = P(\mathbf{I}) = \mathbf{T}^{\infty}_{\mathcal{P}(P,\mathcal{I})} = \mathit{lfp}(\mathbf{T}_{\mathcal{P}(P,\mathcal{I})}) = \mathit{MM}(\mathcal{P}(P, \mathbf{I}))$$

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# SLD Resolution – Termination

- Notice: Selection rule for next rule/atom to be considered for resolution might affect termination
- Prolog's strategy (leftmost atom/first rule) is problematic

Example:

$$\texttt{child\_of}(\texttt{karl}, \texttt{franz}).$$

$$\texttt{child\_of}(\texttt{franz}, \texttt{frieda}).$$

$$\texttt{child\_of}(\texttt{frieda}, \texttt{pia}).$$

$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \texttt{child\_of}(\texttt{X}, \texttt{Y}).$$

$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \texttt{child\_of}(\texttt{X}, \texttt{Z}), \texttt{descendent\_of}(\texttt{Z}, \texttt{Y}).$$

$$\leftarrow \texttt{descendent\_of}(\texttt{karl}, \texttt{X}).$$

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# SLD Resolution – Termination/2

Example (cntd.):

$$\texttt{child\_of}(\texttt{karl}, \texttt{franz}).$$

$$\texttt{child\_of}(\texttt{franz}, \texttt{frieda}).$$

$$\texttt{child\_of}(\texttt{frieda}, \texttt{pia}).$$

$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \texttt{child\_of}(\texttt{X}, \texttt{Y}).$$

$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \mathsf{descendent\_of}(\mathsf{X}, \mathsf{Z}), \mathsf{child\_of}(\mathsf{Z}, \mathsf{Y}).$$

$$\leftarrow \texttt{descendent\_of}(\texttt{karl}, \texttt{X}).$$

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# SLD Resolution – Termination /3

Example (cntd.):

$$\texttt{child\_of}(\texttt{karl}, \texttt{franz}).$$
$$\texttt{child\_of}(\texttt{franz}, \texttt{frieda}).$$
$$\texttt{child\_of}(\texttt{frieda}, \texttt{pia}).$$
$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \texttt{child\_of}(\texttt{X}, \texttt{Y}).$$
$$\texttt{descendent\_of}(\texttt{X}, \texttt{Y}) \leftarrow \texttt{descendent\_of}(\texttt{X}, \texttt{Z}), \texttt{descendent\_of}(\texttt{Z}, \texttt{Y}).$$
$$\leftarrow \texttt{descendent\_of}(\texttt{karl}, \texttt{X}).$$

# Exercise: Metro Reachability

Over the Metro database, consider the predicates `reachableFromOne`$/3$ and `reachableFromBoth`$/3$, with the following meaning for stations $a$, $b$, and $c$:

1. `reachableFromOne`$(a, b, c)$ holds if $c$ is reachable from one of $a$ or $b$;
2. `reachableFromBoth`$(a, b, c)$ holds if $c$ is reachable from both of $a$ and $b$.

Write datalog rules that define these predicates.