Data Structures and Algorithms

Chapter 4

Heapsort and Quicksort

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Acknowledgments

- The course follows the book “Introduction to Algorithms‘”, by **Cormen, Leiserson, Rivest and Stein**, MIT Press [CLRST]. Many examples displayed in these slides are taken from their book.
- These slides are based on those developed by Michael Böhlen for this course.

(See http://www.inf.unibz.it/dis/teaching/DSA/)

- The slides also include a number of additions made by Roberto Sebastiani and Kurt Ranalter when they taught later editions of this course

(See http://disi.unitn.it/~rseba/DIDATTICA/dsa2011_BZ//)

DSA, Chapter 4: Overview

- About sorting algorithms
- Heapsort
  - complete binary trees
  - heap data structure
- Quicksort
  - a popular algorithm
  - very fast on average

DSA, Chapter 4: Overview

- About sorting algorithms
- Heapsort
- Quicksort

Why Sorting?

- "When in doubt, sort" –
  one of the principles of algorithm design

- Sorting is used as a subroutine in many algorithms:
  - Searching in databases:
    we can do binary search on sorted data
  - Element uniqueness, duplicate elimination
  - A large number of computer graphics and computational geometry problems

Why Sorting?/2

- Sorting algorithms represent different algorithm design techniques

- One can prove that any sorting algorithm on arrays needs at least $n \log n$ steps

  ➔ Sorting has a lower bound of $\Omega(n \log n)$

- This lower bound of $\Omega(n \log n)$ is used to prove lower bounds of other problems

Sorting Algorithms So Far

- Insertion sort, selection sort, bubble sort
  - worst-case running time $\Theta(n^2)$
  - in-place
- Merge sort
  - worst-case running time $\Theta(n \log n)$
  - requires additional memory $\Theta(n)$

DSA, Chapter 4: Overview

- About sorting algorithms
- Heapsort
- Quicksort

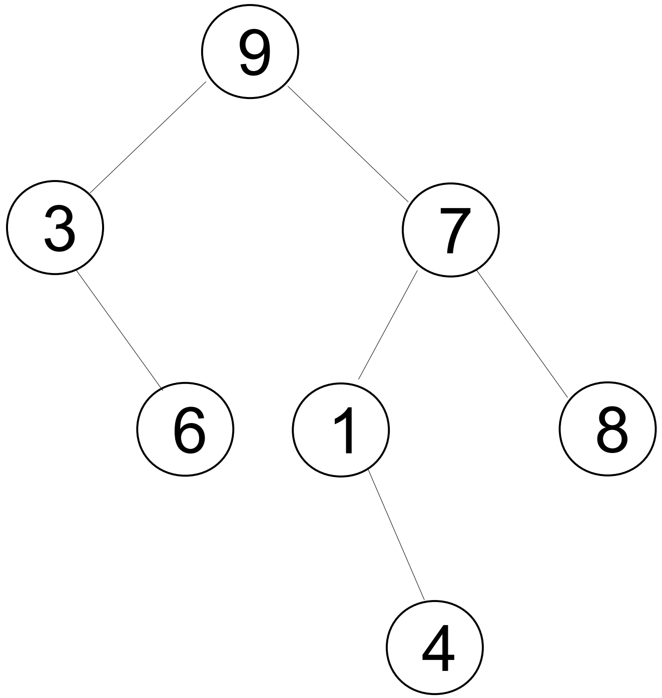
Selection Sort

```
SelectionSort(A[1..n]):
   for i := 1 to n-1
A:     Find the smallest element among A[i..n]
B:     Exchange it with A[i]
```

- A takes $\Theta(n)$ and B takes $\Theta(1)$: $\Theta(n^2)$ in total
- Idea for improvement: smart data structure to
  - do A and B in $\Theta(1)$
  - spend $O(\log n)$ time per iteration to maintain the data structure
  - get a total running time of $O(n \log n)$

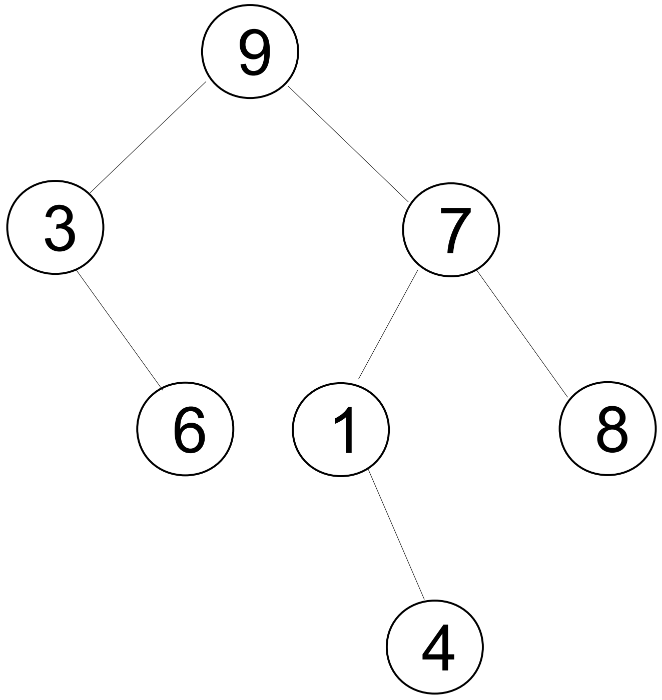
Binary Trees

- Each node may have a left and right child
  - The left child of 7 is 1
  - The right child of 7 is 8
  - 3 has no left child
  - 6 has no children
- Each node has at most one parent
  - 1 is the parent of 4
- The root has no parent
  - 9 is the root
- A leaf has no children
  - 6, 4 and 8 are leafs



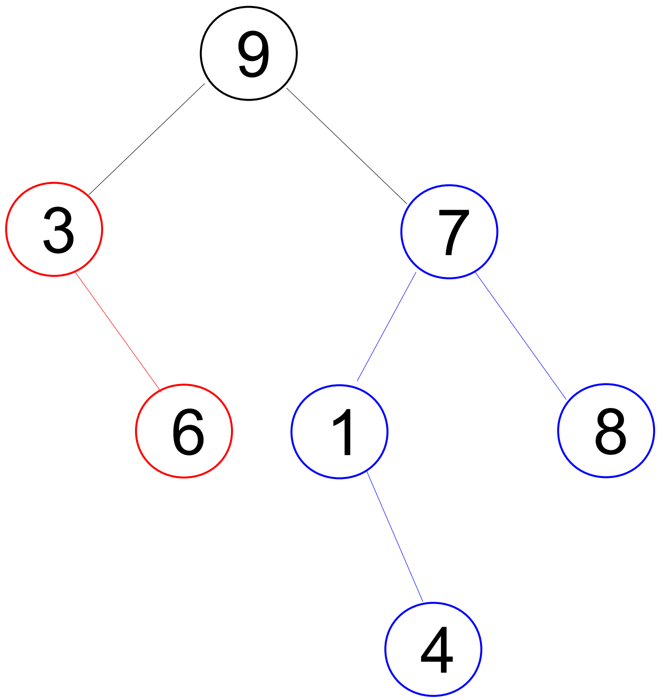
Binary Trees/2

- The depth (or level) of a node x is the length of the path from the root to x
  - The depth of 1 is 2
  - The depth of 9 is 0
- The height of a node x is the length of the longest path from x to a leaf
  - The height of 7 is 2
- The height of a tree is the height of its root
  - The height of the tree is 3



Binary Trees/3

- The right subtree of a node x is the tree rooted at the right child of x
  - The right subtree of 9 is the tree shown in blue
- The left subtree of a node x is the tree rooted at the left child of x
  - The left subtree of 9 is the tree shown in red



Complete Binary Trees

- A complete binary tree is a binary tree where
  - all leaves have the same depth
  - all internal (non-leaf) nodes have two children

What is the number of nodes
in a complete binary tree of height h?

- A nearly complete binary tree is a binary tree where
  - the depth of two leaves differs by at most 1
  - all leaves with the maximal depth are as far left as possible

Binary Heaps

- A binary tree is a binary heap iff
  - it is a nearly complete binary tree
  - each node is greater than or equal to all its children

- The properties of a binary heap allow for
  - efficient storage in an array
    (because it is a nearly complete binary tree)

    fast sorting
    (because of the organization of the values)

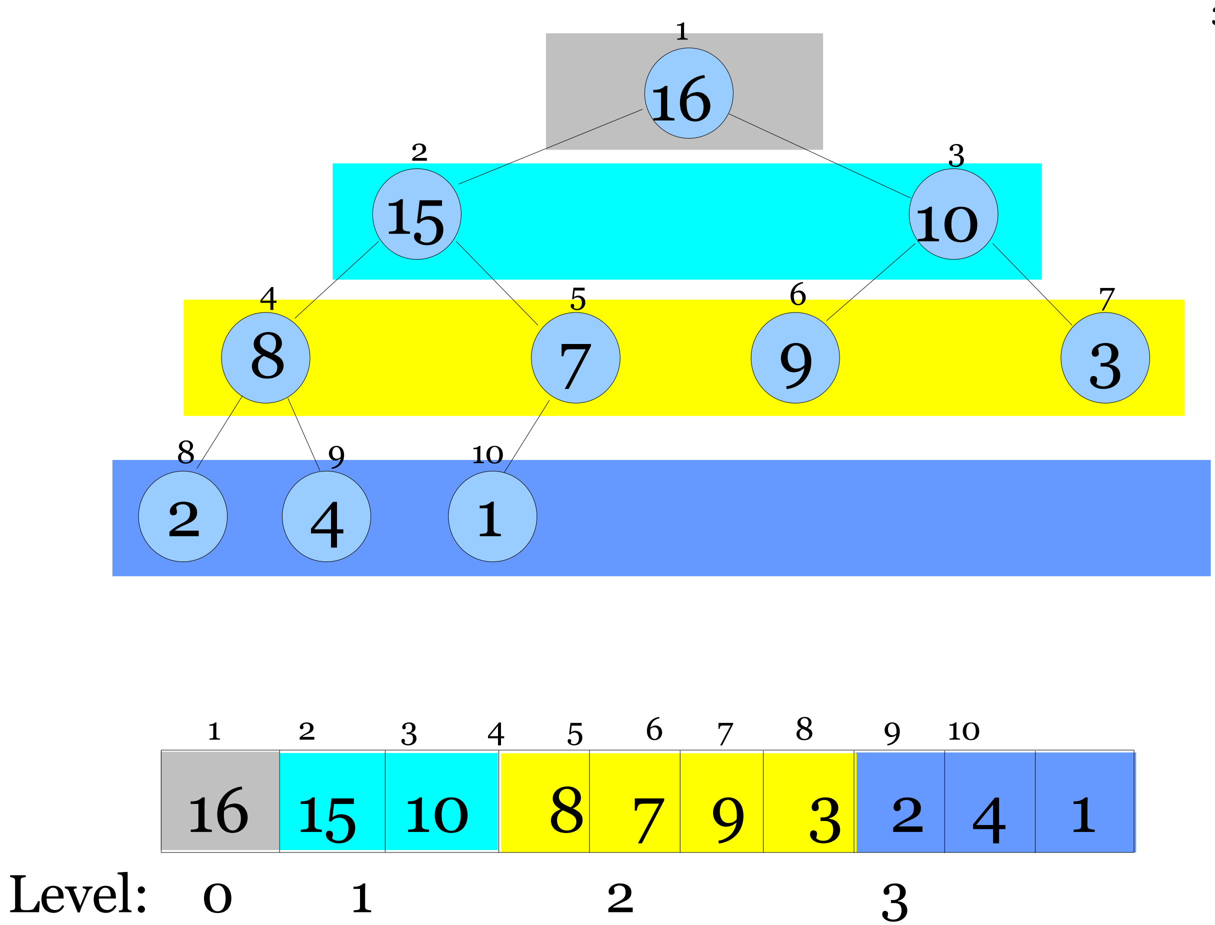
Heaps/2

Heap property
  A[Parent(i)] ≥ A[i]

Parent(i)
  return $\lfloor i/2 \rfloor$

Left(i)
  return 2i

Right(i)
  return 2i+1



Heaps/3

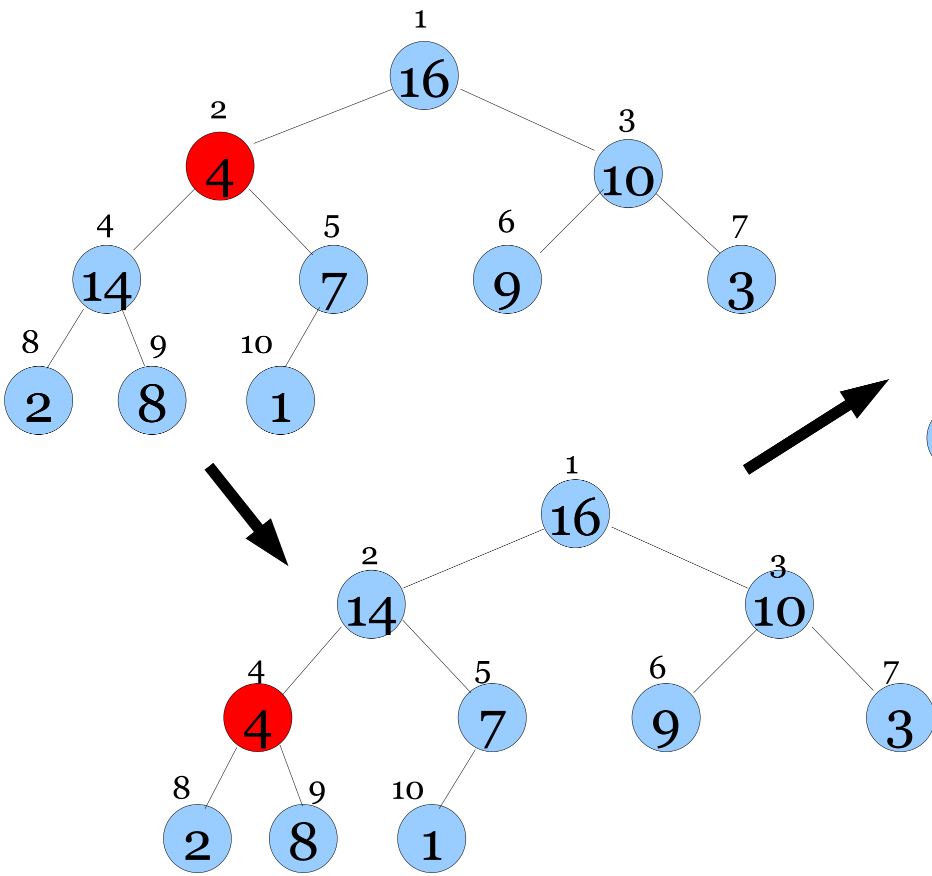
- Notice the implicit tree links in the array: children of node i are $2i$ and $2i+1$
- The heap data structure can be used to implement a fast sorting algorithm
- The basic elements are 3 procedures:
  - Heapify: reconstructs a heap after an element was modified
  - BuildHeap: constructs a heap from an array
  - HeapSort: the sorting algorithm

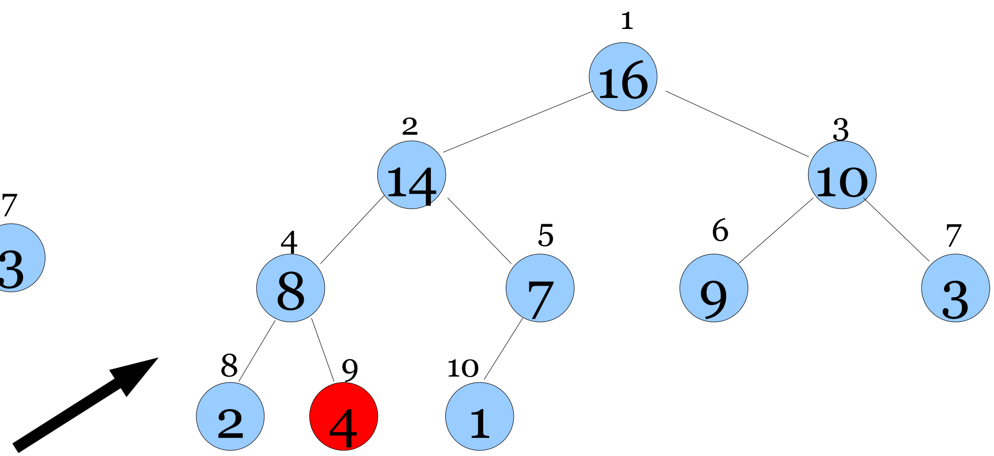
Heapify

- Input:

  index *i* in array *A, number n of elements*
- Precondition:
  - binary trees rooted at *Left(i)* and *Right(i)* are heaps
  - *A[i]* might be smaller than its children,
    thus violating the heap property
- Postcondition:
  - binary tree rooted at *i* is a heap
- How it works: Heapify turns *A* into a heap
  - by moving *A[i]* down the heap
    until the heap property is satisfied again

Heapify Example





1. Call Heapify(A,2)
2. Exchange A[2] with A[4] and recursively call Heapify(A,4)
3. Exchange A[4] with A[9] and recursively call Heapify(A,9)
4. Node 9 has no children, so we are done

Heapify Algorithm

```
Heapify(A,i,m)// m is the length of the heap
  l := 2*i;    // l := Left(i)
  r := 2*i+1; // r := Right(i)
  maxpos := i
  if l <= m and A[l] > A[maxpos]
    then maxpos := l
  if r <= m and A[r] > A[maxpos]
    then maxpos := r
  if maxpos != i then
    swap(A,i,maxpos)
    Heapify(A,maxpos,m)
```

Correctness of Heapify

Induction on the height of Subtree(i),
the tree rooted at position i:

height=0 ➔ l > n (and r > n)
➔ maxpos = i
➔ Heapify does nothing

Not doing anything is fine,
since Subtree(i) is a singleton tree
(and therefore a heap)

Correctness of Heapify/2

height=h+1

Assume Subtree(i) is not a heap

➔ A[i] < A[l] or A[i] < A[r]

Wlog, assume A[r] = max {A[i], A[l], A[r]} and

A[r] > A[i], A[r] > A[l]

➔ maxpos = r

After the return of Heapify(A,maxpos,n),

- Subtree(r) is a heap (by induction hypothesis)
- Subtree(l) is a heap (by assumption)
- A[i] ≥ A[l], A[i] ≥ A[r] (by code of Heapify)

➔ A[i] ≥ all elements in Subtree(l), Subtree(r)

➔ Subtree(i) is a heap

Heapify: Running Time

The running time of Heapify
on a subtree of size *n* rooted at *i*
includes the time to

- determine relationship between elements: $\Theta(1)$
- run Heapify on a subtree rooted at one of the children of *i*
  - 2n/3 is the worst-case size of this subtree (half filled bottom level)
  - $T(n) \le T(2n/3) + \Theta(1)$ implies $T(n) = O(\log n)$
- alternatively
  - running time on a node of height *h is O(h) = O(log n)*

Build a Heap

- Convert an array *A*[1...*n*] into a heap
- Notice that the elements in the array segment

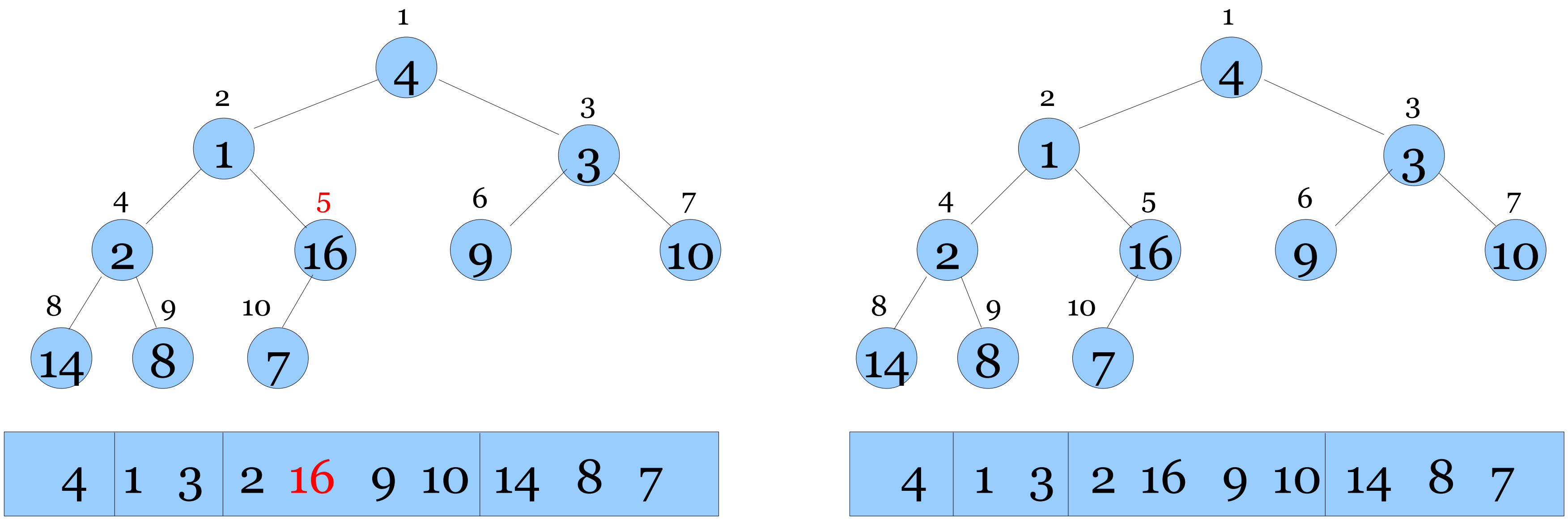
$$A[(\lfloor n/2 \rfloor + 1)..n]$$

are 1-element heaps to begin with

➔ only the first half of indices may need corrections

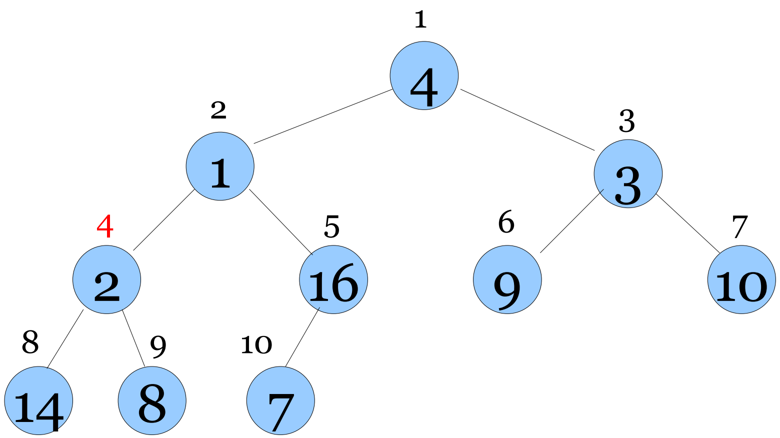
```
BuildHeap(A)
  n := A.length;
  for i := ⌊n/2⌋ downto 1 do
    Heapify(A, i, n)
```

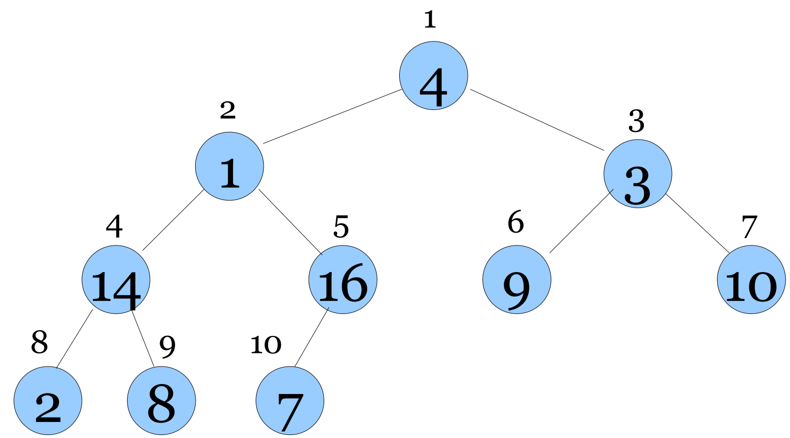
Building a Heap/2



- `Heapify(A, 7, 10)`
- `Heapify(A, 6, 10)`
- `Heapify(A, 5, 10)`

Building a Heap/3



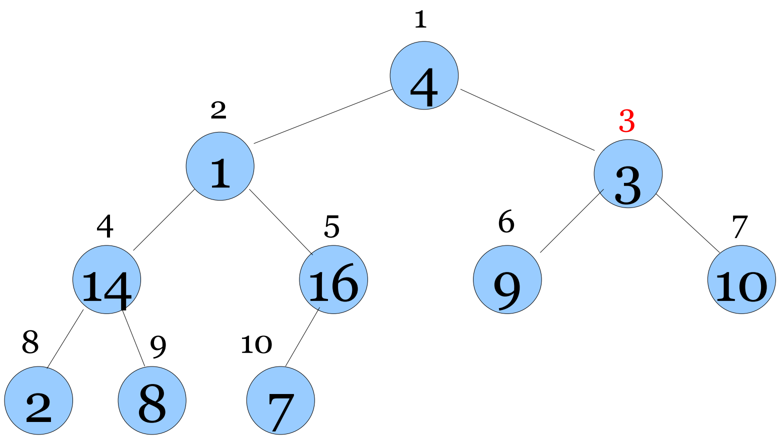


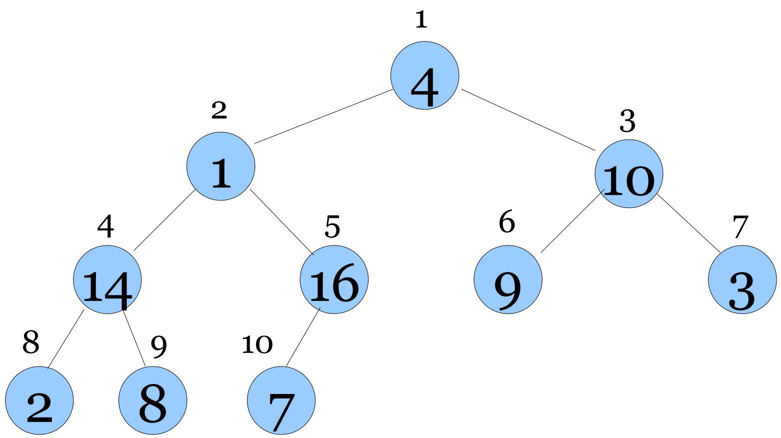
4	1	3	2	16	9	10	14	8	7

4	1	3	14	16	9	10	2	8	7

- `Heapify(A, 4, 10)`

Building a Heap/4



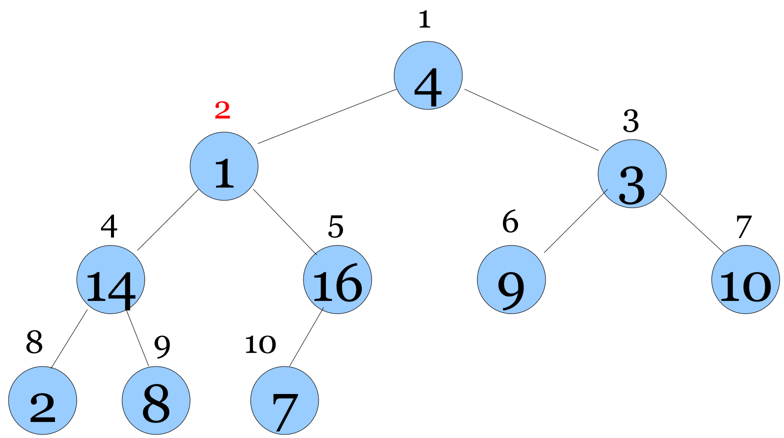


4	1	3	14	16	9	10	2	8	7

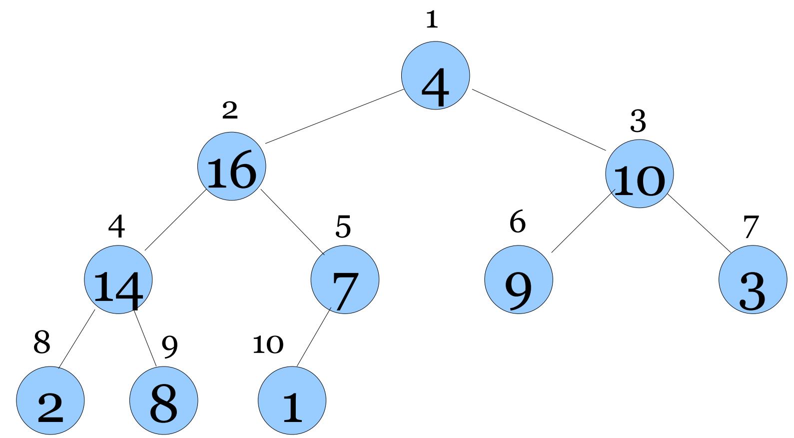
4	1	10	14	16	9	3	2	8	7

- `Heapify(A, 3, 10)`

Building a Heap/5



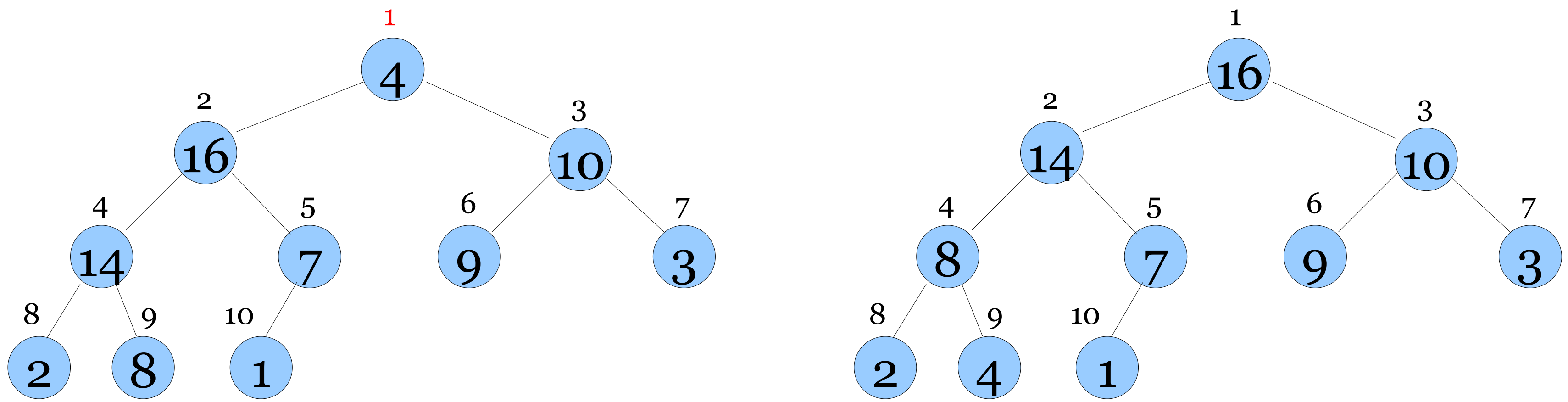
4	1	10	14	16	9	3	2	8	7



4	16	10	14	7	9	3	2	8	1

- `Heapify(A, 2, 10)`

Building a Heap/6



4	16	10	14	7	9	3	2	8	1

16	14	10	8	7	9	3	2	4	1

- `Heapify(A, 1, 10)`

Building a Heap: Analysis

- Correctness:

  Loop invariant:

  When Heapify(A,i,n) is called,
  then Subtree(j) is a heap, for all j > i

- Running time:

  n calls to Heapify = *n O(log n)* = *O(n log n)*

  (non-tight bound, but good enough for an overall *O(n log n)* bound for Heapsort)

- Intuition for a tight bound of *O(n)*

  most of the time Heapify works on heaps that have a very low height

Building a Heap: Analysis/2

- Tight bound:
  - an n-element heap has height log n
  - the heap has $n/2^{h+1}$ nodes of height h
  - cost for one call of Heapify is $O(h)$

$$T(n) = \sum_{h=0}^{\log n} \frac{n}{2^{h+1}} O(h) = O(n \sum_{h=0}^{\log n} \frac{h}{2^h})$$

- Math:

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \qquad \sum_{k=0}^{\infty} \frac{k}{x^k} = \sum_{k=0}^{\infty} k(1/x)^k = \frac{1/x}{(1-1/x)^2}$$

$$T(n) = O(n \sum_{h=0}^{\log n} \frac{h}{2^h}) = O(n \frac{1/2}{(1-1/2)^2}) = O(n)$$

HeapSort

```
Heapsort(A)
  BuildHeap(A)
  for heapsize := A.length downto 2 do
   swap(A,1,heapsize)
   Heapify(A,1,heapsize-1)
```

The total running time of Heapsort is:

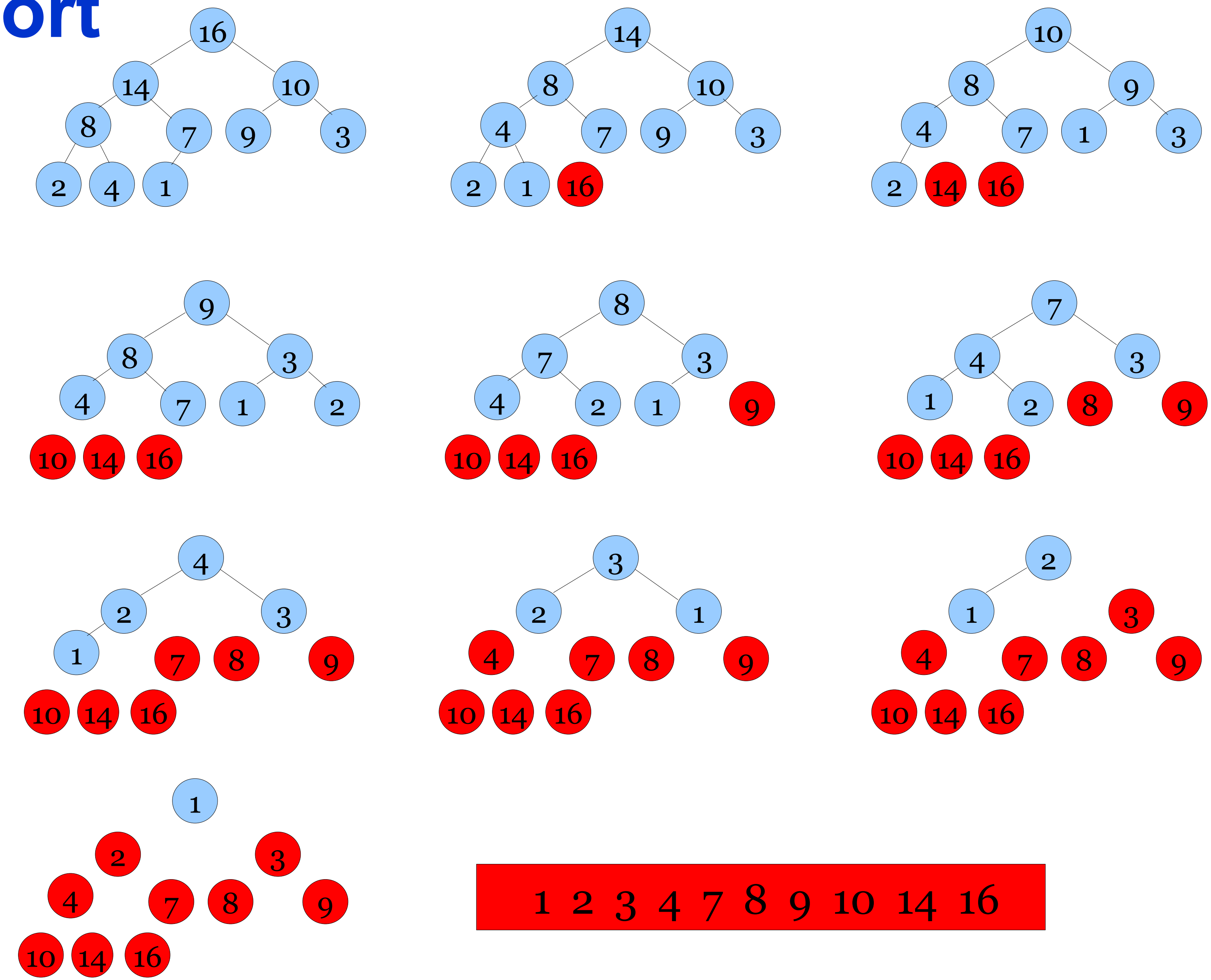
HeapSort

```
Heapsort(A)
  BuildHeap(A)                                   O(n)
  for heapsize := A.length downto 2 do           n times
   swap(A,1,heapsize)                            O(1)
   Heapify(A,1,heapsize-1)                       O(log n)
```

The total running time of Heapsort is:

$O(n) + n * O(\log n) = O(n \log n)$

Heapsort



Correctness of Heapsort

Loop invariant

- `A[1..heapsize]` is a heap
  containing the `heapsize` least elements of `A`
- `A[heapsize+1..(A.length)]` is sorted
  containing the `A.length-heapsize`
  greatest elements of `A`

That is how Heapsort was designed!

Heapsort: Summary

- Heapsort uses a heap data structure to improve selection sort and make the running time asymptotically optimal
- Running time is $O(n \log n)$ – like Merge Sort, but unlike selection, insertion, or bubble sorts
- Sorts in-place – like insertion, selection or bubble sort, but unlike merge sort
- The heap data structure can also be used for other things than sorting

DSA, Chapter 4: Overview

- About sorting algorithms
- Heapsort
- Quicksort

Quicksort

Characteristics

- sorts in place
  (like insertion sort, but unlike merge sort)
  i.e., does not require an additional array
- very practical, average sort performance $O(n \log n)$
  (with small constant factors), but worst case $O(n^2)$

Quicksort: The Principle

When applying the Divide&Conquer principle to sorting, we obtain the following schema for an algorithm:

.

- Divide array segment A[l..r] into two subsegments, say A[l..m] and A[m+1,r]
- Conquer: sort each subsegment by a recursive call
- Combine the sorted subsegments into a sorted version of the original segment A[l..r]

Quicksort: The Principle/2

Merge Sort takes an extreme approach in that

- no work is spent on the division
- a lot of work is spent on the combination

What does an algorithm look like
where no work is spent on the combination?

Quicksort: The Principle/3

If no work is spent on the combination of the sorted segments, then, after the recursive call,

all elements in the left subsegment A[l..m] must be ≤ all elements in the right subsegment A[m+1..r]

However, the recursive call can only have sorted the segments!

We conclude that the division must have partitioned A[l..r] into

- a subsegment with small elements A[l..m]
- a subsegment with big elements A[m+1..r]

Quicksort: The Principle/4

In summary:

A divide-and-conquer algorithm where

- Divide = partition array into 2 subarrays such that elements in the lower part $\leq$ elements in the higher part
- Conquer = recursively sort the 2 subarrays
- Combine = trivial since sorting has been done in place

Quick Sort Algorithm: Overview

INPUT: A[1..n] – an array of integers
l,r – integers satisfying $1 \leq l \leq r \leq n$
OUTPUT: permutation of the segment A[l..r] s.t.
A[l]≤ A[l+1]≤ …≤ A[r]

```
Quicksort(A,l,r)
     if l < r then
        m := Partition(A,l,r)
        Quicksort(A,l,m-1)
        Quicksort(A,m+1,r)
```

Partition divides the segment A[l..r] into
– a segment of "little elements" A[l..m-1]
– a segment of "big elements" A[m+1..r],
with A[m] in the middle between the two

Partition (Version by Lomuto)

INPUT: A[1..n] – an array of integers
l,r – integers satisfying 1 ≤ l < r ≤ n
OUTPUT: m – an integer with l ≤ m ≤ r
a permutation of A[l..r] such that
A[i]< A[m] for all i with l ≤ i < m
A[m]≤ A[i] for all i with m < i ≤ r

```
int Partition(A,l,r)
     p  := A[r]; // pivot, used for the split
     el := l-1;  // end of the little ones
     for bu := l to r-1 do
              // bu is the beginning of the unknown area
          if A[bu] < p
              then swap(A,el+1,bu); el++;
              // all elements < p are little ones
     swap(A,el+1,r)
              // move the pivot into the middle position
     return el+1
```

Partition: Loop Invariant

This version of `Partition` has the following loop invariant:

- **`A[i] < p`**, for all **`i`** with **`l ≤ i ≤ el`**
  (all little ones are `< p`)
- **`A[i]`** $\geq$ **`p`** for all **`i`** with **`el < i < bu`**
  (all big ones are $\geq$ `p`).

Clearly,

- this holds at the beginning of the execution
- this is maintained during the loop
- the loop terminates.

At the end of the loop, **`A[l..el]`** comprises the little ones, and **`A[el+1..r-1]`** comprises the big ones.
Since **`p = A[r]`** is a big one, the postcondition holds after the swap of **`A[el+1]`** and **`A[p]`**.

Partitioning from the Endpoints

There is another approach to partitioning, due to Tony Hoare, the inventor of Quicksort.

As before, we choose p:=A[r] as the pivot.

Then repeatedly, we

- walk from right to left until we find an element $\leq$ p
- walk from left to right until we find an element $\geq$ p
- swap those elements.

Note that in this approach, we have no control where p ends up. Therefore, *Partition* returns an index m such that

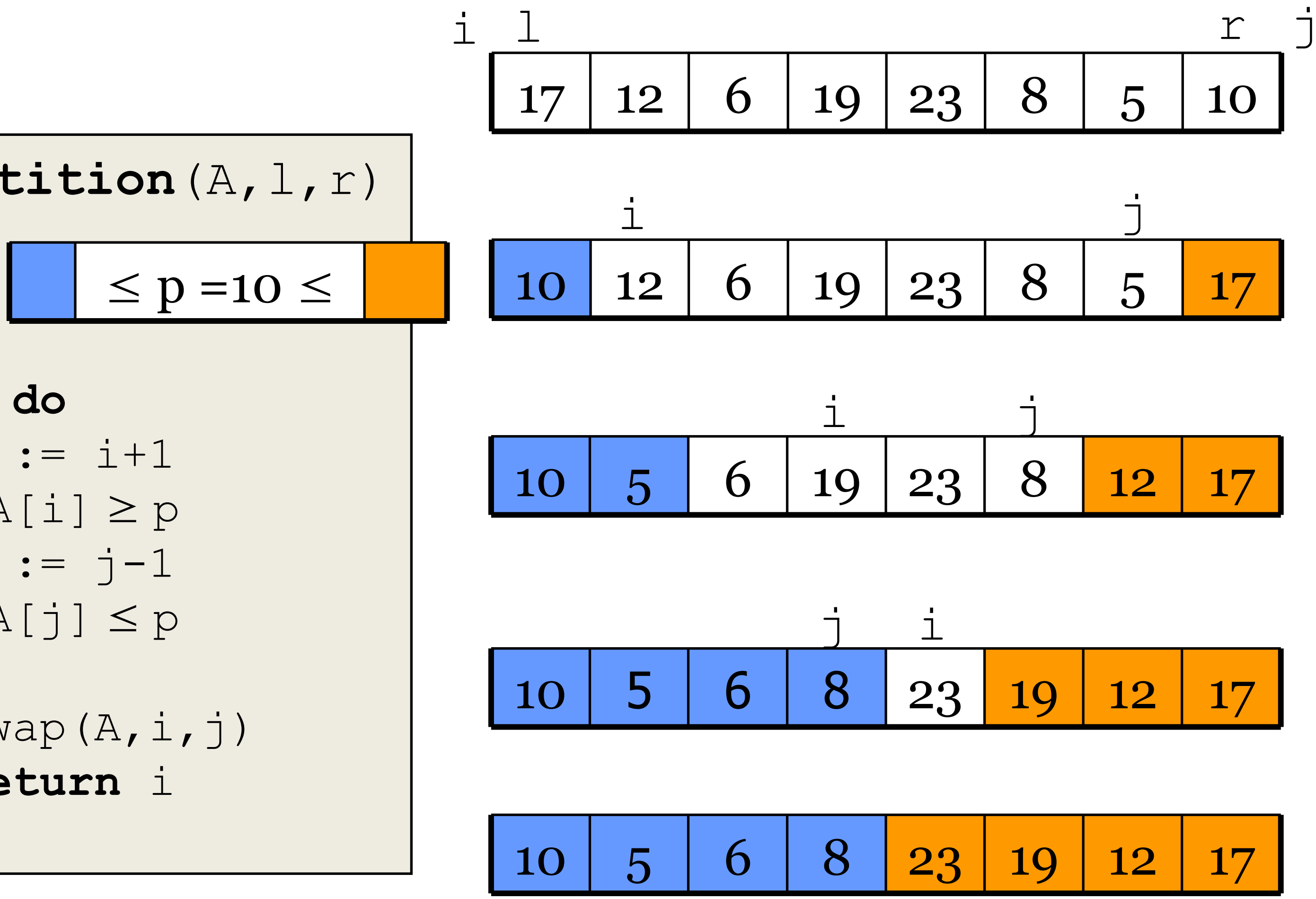
$$A[i] \leq A[j], \quad \text{for all } i, j \text{ with } l \leq i \leq m \text{ and } m+1 \leq j \leq r$$

Consequently, Quicksort(A,l,r) launches two recursive calls

Quicksort(A,l,m-1) and Quicksort(A,m,r)

Partitioning from the Endpoints/2

```
int HoarePartition(A,l,r)
   p := A[r]
   i := l-1
   j := r+1
   while TRUE do
     repeat i := i+1
       until A[i] ≥ p
     repeat j := j-1
       until A[j] ≤ p
     if i<j
       then swap(A,i,j)
       else return i
```



Partitioning from the Endpoints: Correctness

Relies on 3 observations (to be proven!):

- The indices i and j are such that we never access an element of A outside the subarray A[l..r].
- When *HoarePartition* terminates,
  it returns a value i such that $l < i \leq r$.
- When *HoarePartition* terminates, every element of A[l..i-1] is less than or equal to every element of A[i..r] .

Note: *Partition* separates the pivot p from the two partitions,

HoarePartition places it into one of the two partitions (and we don't know which)

Quicksort with Partitioning from the Endpoints

INPUT: A[1..n] – an array of integers
l,r – integers satisfying $1 \le l \le r \le n$
OUTPUT: permutation of the segment A[l..r] s.t.
A[l]≤ A[l+1]≤ …≤ A[r]

```
Quicksort(A,l,r)
    if l < r then
        m := HoarePartition(A,l,r)
        Quicksort(A,l,m-1)
        Quicksort(A,m,r)
```

- Note the different parameters of the second recursive call!

Partitioning: Lomuto vs Hoare

Which one is better?

- Lomuto partitioning is easier to understand and implement
- Hoare partitioning is faster, e.g.,
  - Lomuto swaps whenever it finds one misplaced element
  - Hoare swaps whenever it finds two misplaced elements

Analysis of Quicksort

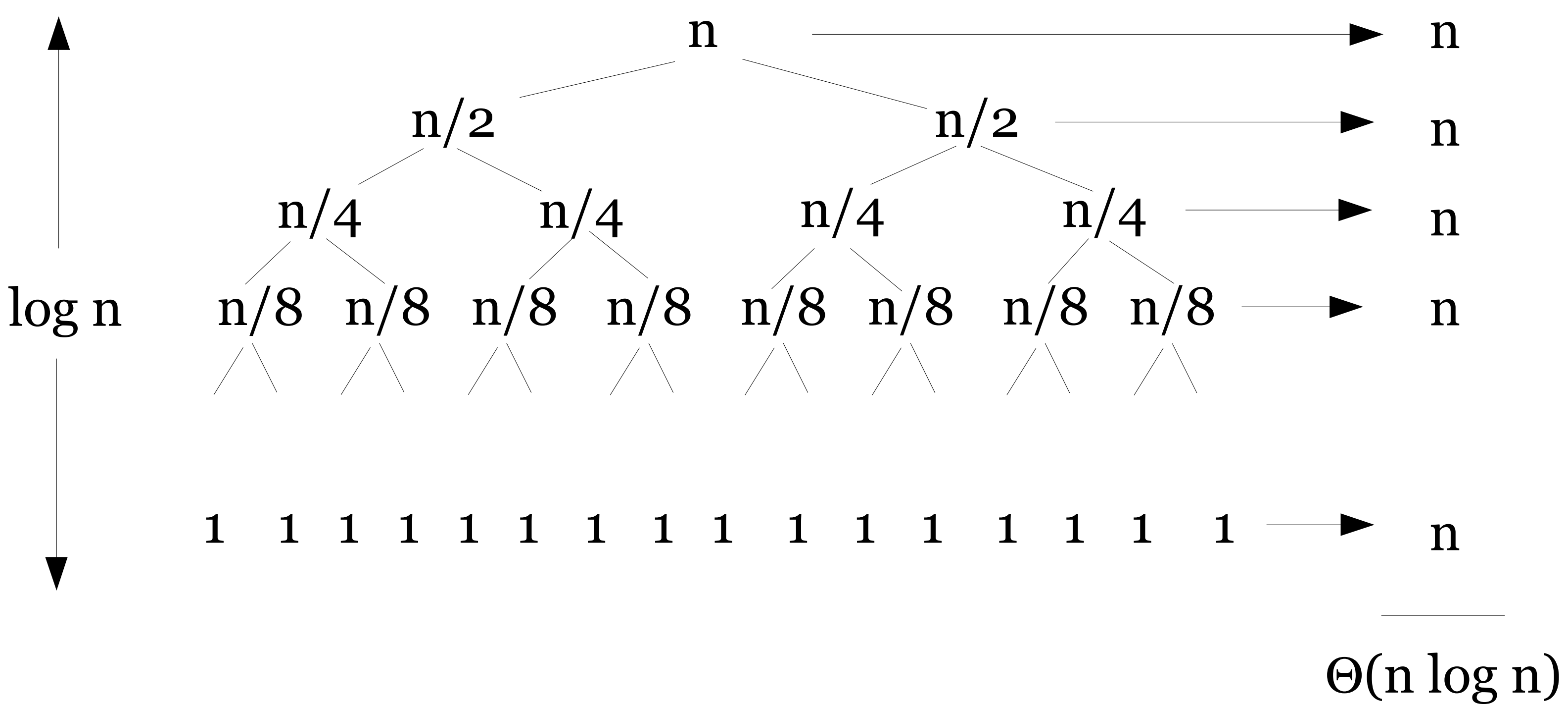
The overall analysis does not depend on the variant

- Assume that all input elements are distinct
- The running time depends on the distribution of splits

Best Case

If we are lucky, Partition splits the array evenly:

$T(n) = 2\ T(n/2) + \Theta(n)$



Worst Case

What is the worst case?

- One side of the partition has one element|
- $T(n) = T(n-1) + T(1) + \Theta(n)$

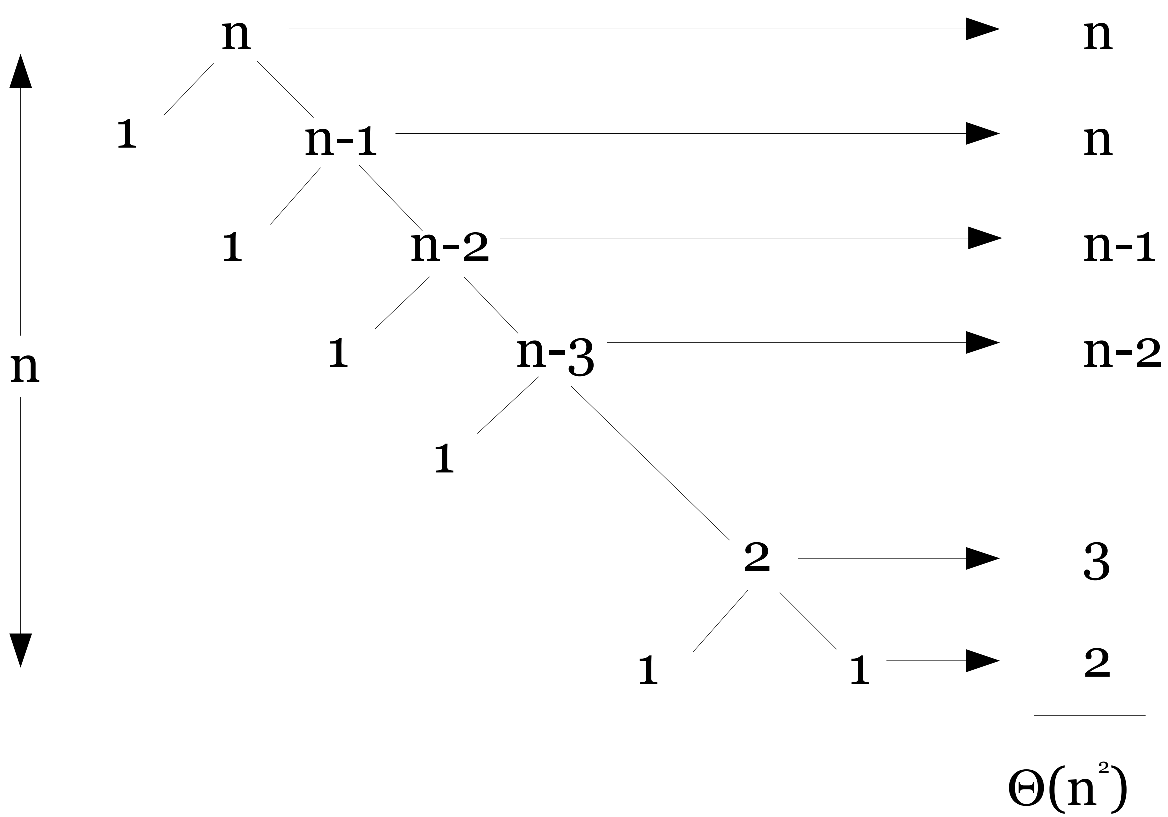
$$= T(n-1) + 0 + \Theta(n)$$

$$= \sum_{k=1}^{n} \Theta(k)$$

$$= \Theta(\sum_{k=1}^{n} k)$$

$$= \Theta(n^2)$$

Worst Case/2



Worst Case/3

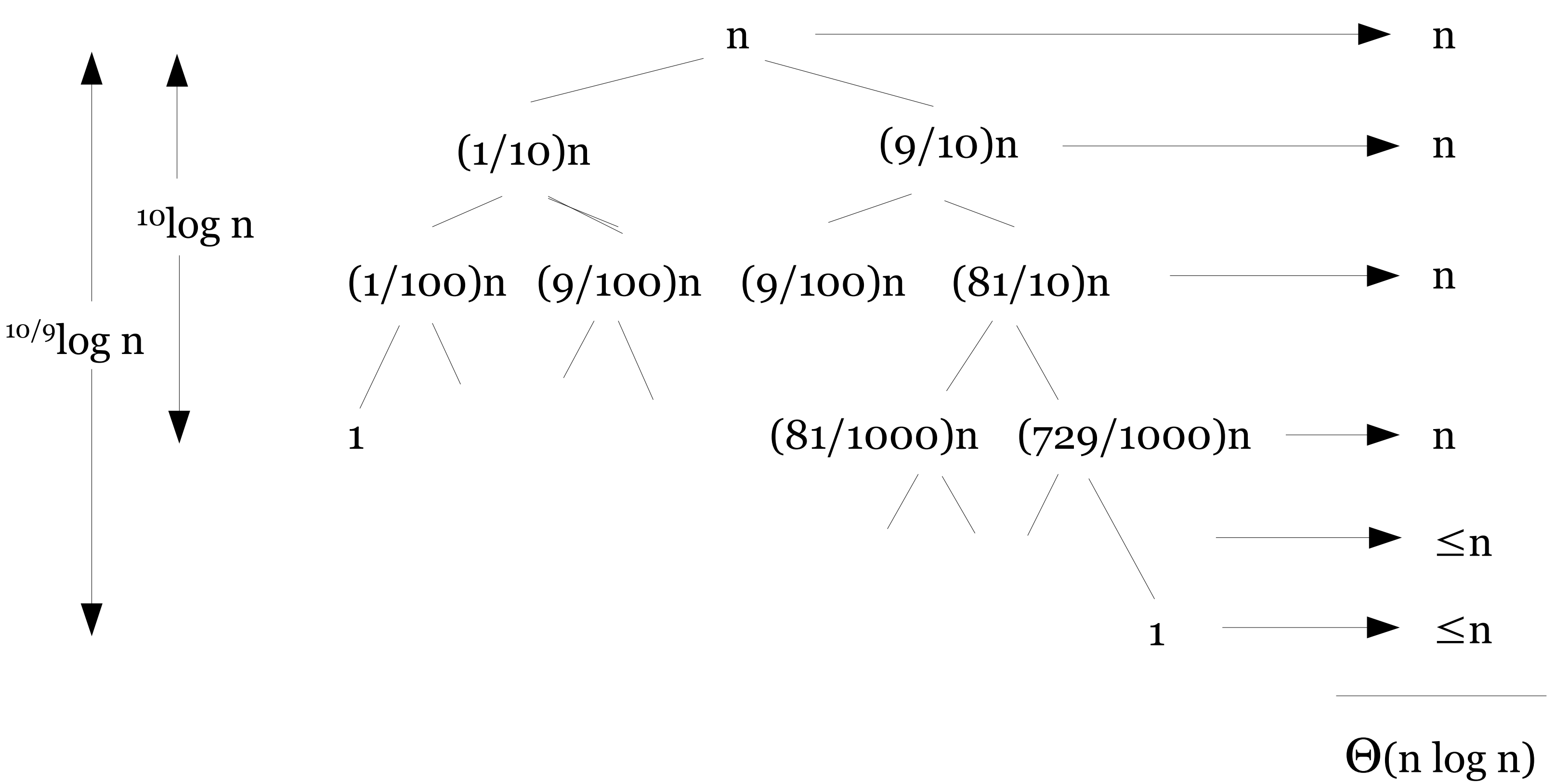
- When does the worst case appear?
  ➔ one of the partition segments is empty
  - input is sorted
  - input is reversely sorted

- Similar to the worst case of Insertion Sort
  (reverse order, all elements have to be moved)

- But sorted input yields the best case for insertion sort

Analysis of Quicksort

Suppose the split is 1/10 : 9/10



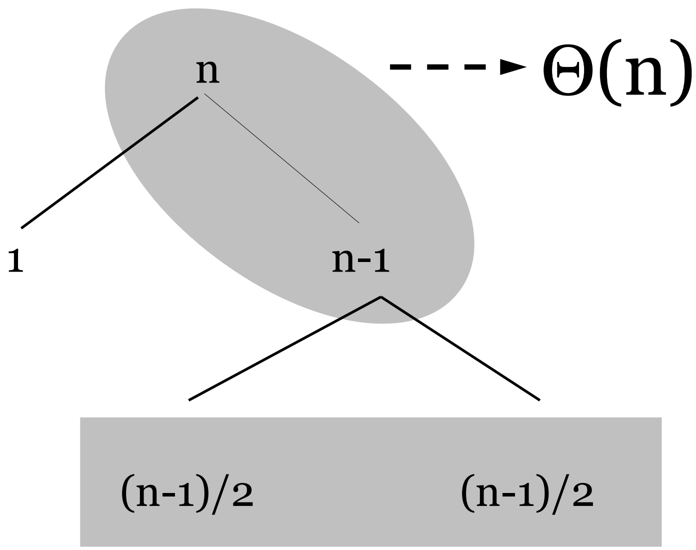
An Average Case Scenario

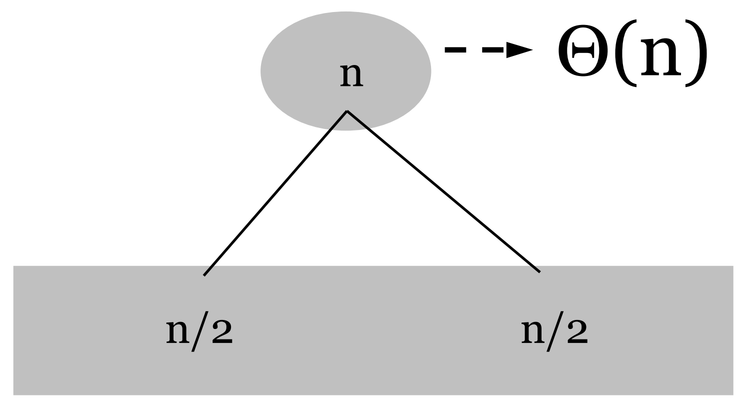
Suppose, we alternate lucky and unlucky cases to get an average behavior

$L(n) = 2U(n/2) + \Theta(n)$ lucky
$U(n) = L(n-1) + \Theta(n)$ unlucky

we consequently get

$$
\begin{aligned}
L(n) &= 2(L(n/2 - 1) + \Theta(n)) + \Theta(n) \\
&= 2L(n/2 - 1) + \Theta(n) \\
&= \Theta(n \log n)
\end{aligned}
$$





An Average Case Scenario/2

- How can we make sure that we are usually lucky?
  - Partition around the “middle” (n/2th) element?
  - Partition around a random element (works well in practice)
- Randomized algorithm
  - running time is independent of the input ordering
  - no specific input triggers worst-case behavior
  - the worst-case is only determined by the output of the random-number generator

Randomized Quicksort

- Assume all elements are distinct
- Partition around a random element
- Consequently, all splits

  1:n-1,
  2:n-2,
  ...,
  n-1:1

  are equally likely with probability 1/n.

- Randomization is a general tool to improve algorithms with bad worst-case but good average-case complexity.

Randomized Quicksort/2

```
int RandomizedPartition(A,l,r)
    i := Random(l,r)
    swap(A,i,r)
    return Partition(A,l,r)

RandomizedQuicksort(A,l,r)
    if l < r then
        m := RandomizedPartition(A,l,r)
        RandomizedQuicksort(A,l,m-1)
        RandomizedQuicksort(A,m+1,r)
```

Summary

- Heapsort
  - same idea as Max sort, but heap data structure helps to find the maximum quickly
  - a heap is a nearly complete binary tree, which here is implemented in an array
  - worst case is n log n
- Quicksort
  - partition-based: extreme case of D&C, no work is spent on combining results
  - popular, behind Unix ”sort” command
  - very fast on average
  - worst case performance is quadratic

Comparison of Sorting Algorithms

- Running time in seconds, n=2048
- Absolute values are not important; compare values with each other
- Relate values to asymptotic running time (n log n, n^2)

	ordered	random	inverse
Insertion	0.22	50.74	103.8
Selection	58.18	58.34	73.46
Bubble	80.18	128.84	178.66
Heap	2.32	2.22	2.12
Quick	0.72	1.22	0.76

Next Chapter

- Dynamic data structures
  - Pointers
  - Lists, trees
- Abstract data types (ADTs)
  - Definition of ADTs
  - Common ADTs