Comparing the Growth of Functions:
Toolbox to Determine Big-Oh, Omega, and Theta

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In the lecture we have defined the three central concepts we use in comparing the growth of functions from the naturals to the nonnegative reals. The functions we are interested in are typically the worst-case running time of and algorithm, sometimes the average-case running time, and sometimes the amount of storage required.

Definition 1. *Let $f,\ g\colon \mathbb{N} \to \mathbb{R}^+$ be functions from the natural numbers to the non-negative real numbers. Then we say:*

1. $f(n) = O(g(n))$ *(or short, $f = O(g)$) if there exists a real number $C > 0$ and a natural number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that*

$$f(n) \leq C \cdot g(n);$$

2. $f(n) = \Omega(g(n))$ *(or short, $f = \Omega(g)$) if there exists a real number $C > 0$ and a natural number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that*

$$f(n) \geq C \cdot g(n);$$

3. $f(n) = \Theta(g(n))$ *(or short, $f = \Theta(g)$) if*

$$f(n) = O(g(n)) \qquad \textit{and} \qquad f(n) = \Omega(g(n)).$$

The first definition says that in the long run, f is bounded from above my a multiple of g, while the second definition says that in the long run, f is bounded from below my a multiple of g. The last definition says that f is bounded multiples of g both from above and from below, and therefore in the long run, f and g behave similarly up to constant multiples.

For instance, if $f(n) = O(n)$, we say that the growth of f is *at most* linear, if $f(n) = O(n^2)$, we say that f's growth is *at most* quadratic, and so on. If the anologous statements about f hold where the big O $O(\cdot)$ is replaced with a big theta $\Theta(\cdot)$, then we say that the growth of f *is* linear or quadratic, respectively.

The next proposition says *(1)* that the relation "$f = O(g)$" is reflexive, *(2)* that the relation "$f = O(g)$" is also transitive, *(3)* that the relation "$f = O(g)$" is the converse of "$f = \Omega(g)$", and *(4)* that the relation "$f = \Theta(g)$" is symmetric. Together, *(1)*, *(1)*, and *(4)* say that the relation "$f = \Theta(g)$" is an equivalence relation.

Proposition 2. *1. For all $f\colon \mathbb{N} \to \mathbb{R}^+$, it holds that $f = O(f)$.*

2. If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.

3. We have $f = O(g)$ iff $g = \Omega(f)$.

4. We have $f = \Theta(g)$ iff $g = \Theta(f)$.

Proof. (1.) Choose $C = 1$ and $n_0 = 1$.

(2.) Suppose $f = O(g)$ and $g = O(h)$, then $f = O(h)$. Then there exist C_1, $C_2 > 0$ and $n_1, n_2 \in \mathbb{N}$ such that

– $f(n) \leq C_1 g(n)$ for all $n \geq n_1$,

– $g(n) \leq C_2 h(n)$ for all $n \geq n_2$.

Let $C := C_1 C_2$ and $n_0 := \max\{n_1, n_2\}$. Then $f(n) \leq C_1 g(n) \leq C_1 C_2 h(n) = Ch(n)$ for all $n \geq n_0$, which means that $f = O(h)$.

(3.) Suppose $f = O(g)$. Then there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that $f(n) \leq Cg(n)$ for all $n \geq n_0$. Letting $C := 1/C$, it follows that $g(n) \geq C' f(n)$ for all $n \geq n_0$ and thus $g = \Omega(f)$.

(4.) Suppose $f = \Theta(g)$. Then $f = O(g)$ and $f = \Omega(g)$. By Claim 2 of this proposition, it follows first that $g = \Omega(f)$ and then, by reversing the role of f and g in Claim 2, it follows that $g = O(f)$. □

The next lemma justifies a common approach to comparing the growth of functions. It says that if we want to compare to functions f_1 and g_1, then we can first simplify these functions by taking Θ-equivalent functions f_2 and g_2 and then compare f_2 and g_2.

Lemma 3 (Simplification Lemma)**.** *Let $f_1 = \Theta(f_2)$ and $g_1 = \Theta(g_2)$. Then*

$$f_1 = O(g_1) \qquad \textit{if and only if} \qquad f_2 = O(g_2).$$

Proof. It is enough to show one direction of the claim because the other one is symmetric, since $f_1 = \Theta(f_2)$ and $g_1 = \Theta(g_2)$ holds if and only if $f_2 = \Theta(f_1)$ and $g_2 = \Theta(g_1)$ holds.

"$\Rightarrow$" Consider f_1, f_2, g_1, g_2 that satisfy the assumptions of the lemma. Moreover, let $f_1 = O(g_1)$. Then we can conclude that three relationships hold between the four functions:

1. Since $f_1 = \Theta(f_2)$, we have $f_1 = \Omega(f_2)$, which implies $f_2 = O(f_1)$.
2. Since $g_1 = \Theta(g_2)$, we have $g_1 = O(g_2)$.

3. By assumption, $f_1 = O(g_1)$.

Listing all three relationships in (1) and (3) and (2), we see that $f_2 = O(f_1)$ and $f_1 = O(g_1)$ and $g_1 = O(g_2)$. Thus, by transitivity we conclude that $f_2 = O(g_2)$. $\square$

Now the questions arises, how we can actually perform growth comparisons. Many functions we consider are actually restrictions of more general functions, namely, functions from (intervals of) the real numbers to (intervals of) the real numbers. For instance, the growth function

$$f\colon \mathbb{N} \to \mathbb{R}^+, \qquad f(n) = 2n^2 + 3n + 5$$

is the restriction of

$$\bar{f}\colon]0, \infty[\to \mathbb{R}, \qquad \bar{f}(x) = 2x^2 + 3x + 5.$$

In the following, we will not distinguish between f and $\bar{f}$ and drop the $\bar{\cdot}$-sign.

To compare two functions f, g, we will often look at the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$. To determine this limit, the main tool is the L'Hôpital's Rule, of which we will only need a special case.

Theorem 4 (Special Case of L'Hôpital's Rule). *Let $f,\ g > 0$ be differentiable functions on some interval $]c, \infty[$ such that*

- $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$
- $g'(x) \neq 0$ *for all $x \in]c, \infty[$*
- $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ *exists.*

Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

You find a proof of this statement in every textbook on Analysis.

Now, we come to our first criterion. It says that if the limit towards infinite of the quotient of two functions is strictly greater than 0, then the two functions are equivalent regarding their long-term growth.

Since this is about limits of quotients, we can often use L'Hôpital's Rule to apply this lemma.

Lemma 5 (Limit Criterion for Θ). *Let $f,\ g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be real-valued functions. If*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = c > 0,$$

that is, the limit exists and is equal to some number c greater 0, then

$$f(n) = \Theta(g(n)).$$

Proof. Consider some $\epsilon > 0$ such that $c > 0$ (and hence $c - \epsilon > 0$). From the definition of the limit, we conclude that there is some $x_0 > 0$ such that for all $x > x_0$ we have

$$\left| \frac{f(x)}{g(x)} - c \right| < \epsilon. \tag{1}$$

From Equation (1) we conclude

$$|f(x) - cg(x)| < \epsilon g(x),$$

which implies $f(x) - cg(x) < \epsilon g(x)$, and hence, $f(x) < (c + \epsilon)g(x)$. This shows that $f = O(g)$.

Similarly, we conclude

$$c - \frac{f(x)}{g(x)} < \epsilon,$$

which implies $c - \epsilon < \frac{f(x)}{g(x)}$, hence, $(c - \epsilon)g(x) < f(x)$, and finally $g(x) < \frac{1}{c-\epsilon} f(x)$. This shows that $g = O(f)$ and concludes the proof. □

A simple application of this criterion gives us that $f(n) = 2n^2 + 3n + 5$ from above is Θ-equivalent to $g(n) = n^2$, that is, we can drop all the lower order terms and the coefficient of the highest-order term. We show that by considering the limit $\lim_{x\to\infty} \frac{2x^2+3x+5}{x^2}$. Applying L'Hôpital's Rule twice, we end up with $\lim_{x\to\infty} \frac{4}{2}$, which equals $2 > 0$.

Sometimes, it can be interesting to know that one function grows strictly more strongly than another one. By applying the lemma below, one can show that one function is dominated by another one, but not vice versa.

Lemma 6 (Criterion for Strictly Stronger Growth)**.** *Let $f, g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be real-valued functions. If*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0, \tag{2}$$

then

1. $f(n) = O(g(n))$
2. $g(n) \neq O(f(n))$.

Proof. Equation (2) implies that there is a number $x_0 > 0$ such that $f(x)/g(x) < 1$ for all $x > x_0$. If we choose n_0 as the first natural number greater x_0, then we have $f(n) < 1 \cdot g(n)$ for all $n > n_0$, and thus, $f(n) = O(g(n))$. This proves the first claim.

To prove the second claim, let us write up formally what it means. By definition, "$g(n) = O(f(n))$" means:

> There exist some $C > 0$ and some $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $g(n) < C \cdot f(n)$.

The negation of this statement is:

> For all $C > 0$ and all $n_0 \in \mathbb{N}$ there exists some $n_1 > n_0$ such that $g(n_1) \geq C \cdot f(n_1)$.

Now, consider some $C > 0$ and some $n_0 \in \mathbb{N}$. We want to find an n_1 as specified in the negated statement above. Because of the limit condition in Equation (2) we know that for C there is a number $n_C \in \mathbb{N}$ such that for all natural numbers $n > n_C$ we have

$$\frac{f(n)}{g(n)} < 1/C, \qquad \text{that is,} \qquad g(n) > C \cdot f(n).$$

To satisfy the negated statement above, we define $n_1 := \max\{n_0, n_c\}$ and thus have found a number with the required properties. $\square$

As an application, let us show that n grows strictly more strongly than $\log n$. To check this, we need to remember the derivatives

$$\frac{d}{dx} x = 1 \qquad \text{and} \qquad \frac{d}{dx} \log x = 1/x.$$

Then calculating the limit

$$\lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$$

tells us that $\log n$ grows strictly more slowly than n.

As another, simpler application let us compare $f(n) = 2^n$ and $g(n) = n^n$. Then considering quotients, we observe that

$$\lim_{x \to \infty} \frac{2}{x} = 0,$$

and that for all $x > 1$ we have $\frac{2}{x} < 1$ and therefore $(\frac{2}{x})^x < \frac{2}{x}$, which implies that

$$\lim_{x \to \infty} \frac{2^x}{x^x} = \lim_{x \to \infty} \left(\frac{2}{x}\right)^x = 0.$$

Lemma 7. *Let $f(n)$, $g(n) > 1$. Then the following are equivalent:*

1. $f(n) = O(g(n))$*;*
2. *There exist numbers $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have*

$$\log(f(n)) \leq c + \log(g(n)). \tag{3}$$

Proof. "*(1) ⇒ (2)*" Let $f(n) = O(g(n))$. Then there exist numbers $C > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $f(n) \leq Cg(n)$. If we apply the logarithm to both sides of this inequality, then we obtain, due to the monotonicity of the logarithm, that $\log(f(n)) \leq \log C + \log(g(n))$, which yields the claim with $c = \log C$.

"*(2) ⇒ (1)*" If we apply the exponential function to both side of the inequality (3), we obtain that $f(n) = e^{\log(f(n))} \leq e^c \cdot e^{\log(g(n))} = C \cdot g(n)$, where we have chosen $C = e^c$. Then the second statement implies that $f(n) = O(g(n))$. □

As an application, let us compare $f(n) = n^n$ and $g(n) = 2^{n^2}$. After applying the logarithm, we have $\log(f(n)) = n \log(n)$ and $\log(g(n)) = n^2 \log 2$. Since $\lim_{x\to\infty} \frac{x \log x}{x^2 \log 2} = 0$, we conclude that $\log(f(n)) \leq \log(g(n))$ for sufficiently large n, and therefore the $f(n) = O(g(n))$.

Finally, we state an almost obvious criterion. For instance, since $\log n$ grows beyond any bound, we can conclude that $n \log n$ outgrows n beyond any bound, that is, $n \log n \neq O(n)$. This kind of conclusion is then generalized in the following proposition.

Proposition 8. *Let $f(n)$, $g(n)$, $h(n) > 0$ such that*

$$f(n) \geq g(n)h(n) \qquad \textit{for all } n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} h(n) = \infty.$$

Then $f(n) \neq O(g(n))$.

Proof. Suppose there are functions f, g, h as above. We want to prove that $f(n) \neq O(g(n))$. To this end we have to show that for all $C > 0$ and all $n_0 \in \mathbb{N}$ there exists some $n_1 > n_0$ such that $g(n_1) \geq C \cdot f(n_1)$.

Therefore, let $C > 0$ and $n_0 \in \mathbb{N}$. Since $\lim_{n\to\infty} h(n) = \infty$, there is a number $n_0 \in \mathbb{N}$ such that $h(n) > C$ for all $n \geq n_0$. Together with inequality 8, this yields the claim. □