

# Data Structures and Algorithms

Werner Nutt

# Acknowledgments

- The course follows the book “Introduction to Algorithms”, by **Cormen, Leiserson, Rivest and Stein**, MIT Press [CLRST]. Many examples displayed on these slides are taken from their book.
- These slides are based on those developed by Michael Böhlen for his course.

(See <http://www.inf.unibz.it/dis/teaching/DSA/>)

- The slides also include a number of additions made by Roberto Sebastiani and Kurt Ranalter when they taught later editions of this course

(See [http://disi.unitn.it/~rseba/DIDATTICA/dsa2011\\_BZ//](http://disi.unitn.it/~rseba/DIDATTICA/dsa2011_BZ//))

# DSA, Chapter 1: Overview

- Introduction, syllabus, organisation
- Algorithms
- Recursion (principle, trace, factorial, Fibonacci)
- Sorting (bubble, insertion, selection)

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# Learning Outcomes

The main things we will learn in this course:

- To *distinguish* between a *problem* and an *algorithm* that solves it
- To get to know a *toolbox* of *classical* algorithms
- To *think algorithmically* and get the spirit of how algorithms are designed
- To learn a number of algorithm design *techniques* (such as divide-and-conquer)
- To analyze (in a precise and formal way) the *efficiency* and the *correctness* of algorithms.

# Syllabus

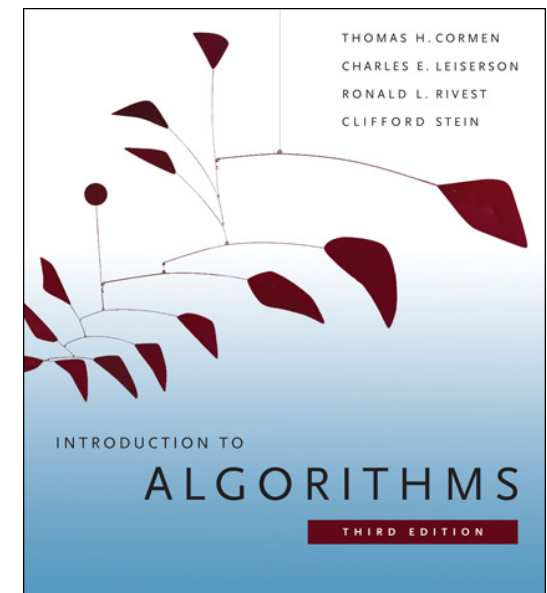
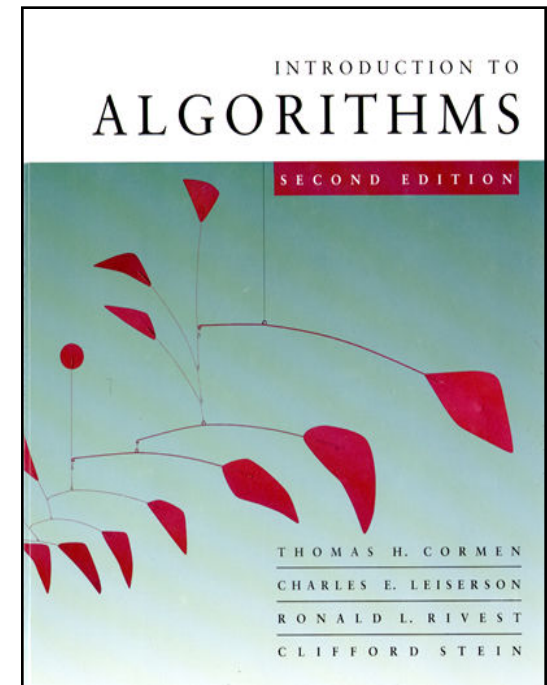
1. Introduction, recursion (chap 1 in CLRS)
2. Correctness and complexity of algorithms (2, 3)
3. Divide and conquer, recurrences (4)
4. Heapsort, Quicksort (6, 7)
5. Dynamic data structures, abstract data types (10)
6. Binary search trees, red-black trees (12, 13)
7. Hash tables (11)
8. Dynamic programming (15)
9. Graphs: Principles and graph traversal (22)
10. Minimum spanning tree and shortest path (23, 24)

# Literature

Cormen, Leiserson, Rivest  
and Stein (CLRS),  
*Introduction to Algorithms*,  
Second Edition, MIT Press and  
McGraw-Hill, 2001  
and  
Third Edition, MIT Press, 2009

(See <http://mitpress.mit.edu/algorithms/>)

*Course is based on this book*



# Other Literature

Kurt Mehlhorn and Peter Sanders

Algorithms and Data Structures - The Basic Toolbox

Offers alternate presentation of topics of the course

Free download from

<http://www.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>



# Course Organization

- Lectures: Wed 10:45-12:45, Fri 8:30-10:30  
Office hours: Wed 14:00-16:00 (but let me know if you want to come)
- Labs (starting next week): Tue 16:00-18:00
- Teaching Assistants
  - Radityo Eko Prasoyo, [RPrasojo@unibz.it](mailto:RPrasojo@unibz.it)
  - Rafael Penaloza, [Rafael.Penaloza@unibz.it](mailto:Rafael.Penaloza@unibz.it)
  - Guohui Xiao, [guohui.xiao@unibz.it](mailto:guohui.xiao@unibz.it)
- Home page:  
<http://www.inf.unibz.it/~nutt/Teaching/DSA1516/>

# Assignments

The assignments are a crucial part of the course

- **Each week** an assignment has to be solved
- The schedule for the publication and the handing in of the assignments will be announced at the next lecture.
- A number of assignments include **programming tasks**. It is strongly recommended that you implement and run all programming exercises.
- Assignments will be **marked**. The assignment mark will count towards the course mark.
- Any attempt at **plagiarism** (copying from the web or copying from other students) leads to a **0 mark** for **all assignments**.

# Assignments, Midterm Exam, Final Exam, and Course Mark

- There will be
  - one written exam at the end of the course
  - one midterm exam around the middle of the course
  - assignments
- To pass the course, one has to pass the written exam.
- Students who do not submit exercises and do not take part in the midterm (or fail the midterm) will be marked on the final exam alone.
- For students who submit all assignments, and take part in the midterm, the final mark will be a weighted average
  - 40% exam mark + 20% midterm
  - + 40% assignment mark

# Assignments, Midterm Exam, Final Exam, and Course Mark

- If students submit fewer assignments, or do not take part in the midterm, the percentage will be lower.
- Assignments for which the mark is lower than the mark of the written exam will not be considered.
- Similarly, the midterm will not be considered if the mark is lower than the mark of the final exam.
- The midterm and assignment marks apply to three exam sessions.

# Organisation of Labs

- You will attend always the lab of the same teaching assistant (TA) during the course
- The TA will mark your assignments and be your first contact person for questions on the assignments
- To help us organize the labs, send an email to [Rafael.Penaloz@unibz.it](mailto:Rafael.Penaloz@unibz.it) containing a group of students that would like to attend the same lab
- One mail per group is enough
- The mail should contain for each student of the group
  - name
  - email address
  - student number

# General Remarks

- Algorithms are first designed on paper  
... and later keyed in on the computer.
- The most important thing is to be **simple** and **precise** .
- During lectures:
  - Interaction is welcome; ask questions  
(I will ask you anyway 😊 )
  - Additional explanations and examples if desired
  - Speed up/slow down the progress

# DSA, Chapter 1:

- Introduction, syllabus, organisation
- **Algorithms**
- Recursion (principle, trace, factorial, Fibonacci)
- Sorting (bubble, insertion, selection)

# What are Algorithms About?

There are problems we solve in everyday life

- **Travel** from Bolzano to Berlin
- **Cook** Spaghetti alla Bolognese *(I know, not in Italy,...)*
- **Register** for a Bachelor thesis at FUB

For all these problems, there are

- **instructions**
- **recipes**
- **procedures,**

which describe a complex operation in terms of

- elementary **operations** *(“beat well ...”)*
- **control** structures and **conditions** *(“... until fluffy”)*



# Algorithms

Problems involving numbers, strings, mathematical objects:

- for **two numbers**, determine their **sum**, **product**, ...
- for **two numbers**, compute their **greatest common divisor**
- for a **sequence** of strings,  
find an alphabetically **sorted permutation** of the sequence
- for two **arithmetic expressions**, find out if they are **equivalent**
- for a **program** in Java,  
create an equivalent program in **byte code**
- on a **map**, find for a given **house** the **closest bus stop**

We call instructions, recipes, for such problems *algorithms*

*What have algorithms in common with recipes?  
How are they different?*

# History

- *First algorithm:* **Euclidean Algorithm**, greatest common divisor, 400-300 B.C.
- *Name:* Persian mathematician **Mohammed al-Khowarizmi**, in Latin became “Algorismus”  
كتاب الجمع و التفریق بحساب الهند  
*Kitāb al-Dscham‘ wa-l-tafrīq bi-ḥisāb al-Hind =*  
*= Book on connecting and taking apart in the calculation of India*
- *19th century*
  - **Charles Babbage**: Difference and Analytical Engine
- *20th century*
  - **Alan Turing, Alonzo Church**: formal models computation
  - **John von Neumann**: architecture of modern computers

# Data Structures, Algorithms, and Programs

- Data structure
  - Organization of data to solve the problem at hand
- Algorithm
  - Outline, the essence of a computational procedure, step-by-step instructions
- Program
  - implementation of an algorithm in some programming language

# Overall Picture

Using a computer to help solve problems:

- Precisely specifying the problem
- Designing programs
  - architecture
  - algorithms
- Writing programs
- Verifying (testing) programs

## Data Structure and Algorithm Design Goals

Correctness



Efficiency



## Implementation Goals

Robustness



Reusability



Adaptability

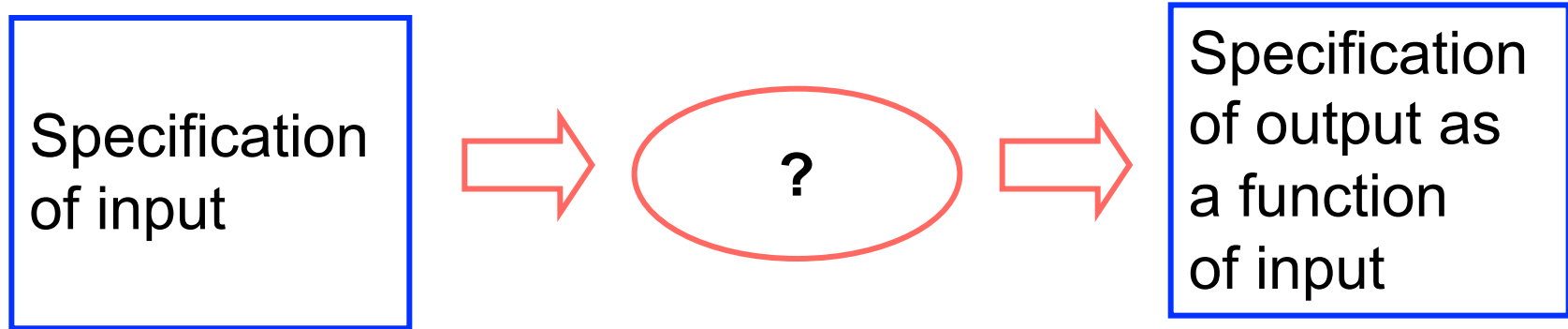


# This course is **not** about:

- Programming languages
- Computer architecture
- Software architecture
- SW design and implementation principles

We will only touch upon  
the theory of complexity  
and computability.

# Algorithmic Problem



There is an infinite number of possible input *instances* satisfying the specification.

For example: An array of distinct integer numbers:

`[-909, -1, -20, 908, 1000000000, 100000]`

# Question from a Google Interview

You are given an **array of distinct numbers**.

You need to return an index to a **“local minimum”** element, which is defined as an element that is smaller than both its adjacent elements.

In the case of the array edges, the condition is reduced to one adjacent element.

If there are multiple **“local minima”**,  
returning any **one** of them **is fine**.

# Question from a Google Interview/2

You are given an **unsorted sequence of integers**  $A$ .

Find the **longest subsequence**  $B$  such that elements of this subsequence are **strictly increasing numbers**.

Elements in the subsequence  $B$  must appear in the **same relative order** as in the sequence  $A$ .

Example:

input:  $A = [-1, 2, 100, 100, 101, 3, 4, 5, -7]$

output:  $B = [-1, 2, 3, 4, 5]$



# Question from a Google Interview/3

You have a **sorted array** containing the **age of every person on Earth**

[0, 0, 0, 0, ..., 1, 1, ..., 28, 28, ..., 110, ...].

Find out **how many people have each age**.

# Question from a Google Interview/4

You are given a text file that has **list of dependencies** between (any) two **projects** in a source code repository.

Write an algorithm to determine the **build order**, i.e., which project needs to be **built first**, followed by which project, **based on the dependencies**.

You get a bonus point, if you can **detect any circular dependencies** and throw an exception if found.

Example: `ProjectDependencies.txt`

$a \rightarrow b$  (means “ $a$  depends on  $b$ ”, so  $b$  needs to be built first and then  $a$ )

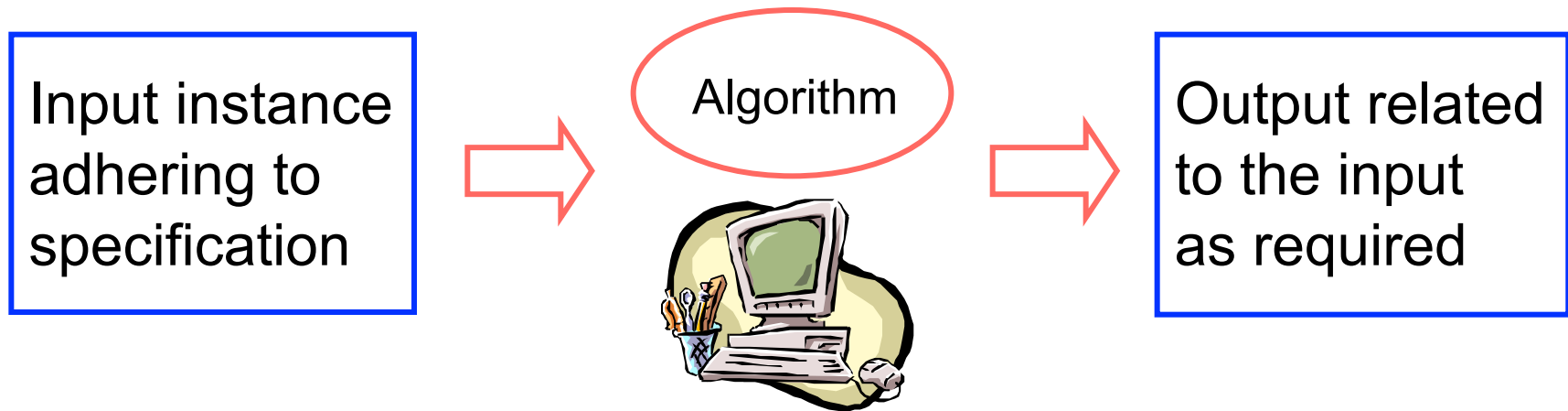
$b \rightarrow c$

$b \rightarrow d$

$c \rightarrow d$

Then the build order can be  $d, c, b, a$ , in that order.

# Algorithmic Solution



- Algorithm describes actions on the input instance
- There may be many correct algorithms for the same algorithmic problem.

# Definition

An **algorithm** is a sequence of *unambiguous* instructions for solving a problem, i.e.,

- for obtaining a *required output*
- for any *legitimate input*

in a finite amount of time.

➔ This presumes a mechanism to execute the algorithm

Properties of algorithms:

- Termination, Correctness, (Non-)Determinism, RunningTime, ...

# How to Develop an Algorithm

- **Precisely define** the problem.  
Precisely specify the **input** and **output**.  
Consider all cases.
- Come up with a **simple (= abstract ?) plan** to solve the problem at hand.
  - The plan is **independent** of a (programming) **language**
  - The precise problem **specification** influences the plan.
- Turn the plan into an implementation
  - The problem representation (data structure) influences the implementation

# Example 1: Searching

## INPUT

- $A$  - (un)sorted sequence/array of  $n$  numbers, ( $n > 0$ )
- $q$  - a single number

$a_1, a_2, a_3, \dots, a_n; q$

2 5 6 10 11; 5

2 5 6 10 11; 9

## OUTPUT

- index of number  $q$  in sequence/array  $A$ , or  $-1$

$j$

2

-1

# Searching/2, search1

```
search1
```

```
INPUT:  A[1..n] (un)sorted array of integers, q an integer.
```

```
OUTPUT: index j such that A[j]=q or -1 if A[j] ≠ q for all j (1 ≤ j ≤ n)
```

```
j := 1
```

```
while j ≤ n and A[j] ≠ q do
```

```
    j++
```

```
if j ≤ n
```

```
    then return j
```

```
    else return -1
```

- The code is written in *pseudo-code* and INPUT and OUTPUT of the algorithm are specified.
- The algorithm uses a *brute-force* technique, i.e., scans the input sequentially.

# Preconditions, Postconditions

Precondition:

- what does the algorithm get as input?

Postcondition:

- what does the algorithm produce as output?
- ... how does this relate to the input?

Make sure you have considered the special cases:

- empty set, number 0, empty reference *NULL*, ...



# Pseudo-code

Like Java, Pascal, C, or any other imperative language

- **Control structures:**

(if then else, while, and for loops)

- **Assignment:** :=

- **Array** element access:  $A[i]$

- Access to **element** of composite type (**record** or object):

$A.b$

*CLRS uses  $b[A]$*

# Control Structures in Pseudo-code: Examples

- **for-to:**  
for  $i := 1$  to  $n$  do  
     $A[i] := A[i] + 1$
- **for-downto:**  
for  $i := n$  downto  $1$  do  
     $A[i] := A[i] + 2$
- **while:**  
while  $A[i] > 0$  do  
     $A[i] := A[i] - i$
- **if-then-else:**  
if  $A[i] > 0$   
    then  $pos := true$   
    else  $pos := false$

# Searching, Java Solution

```
import java.io.*;

class search {
    static final int n = 5;
    static int j, q;
    static int a[] = { 11, 1, 4, -3, 22 };

    public static void main(String args[]) {
        j = 0; q = 22;
        while (j < n && a[j] != q) { j++; }
        if (j < n) { System.out.println(j); }
        else { System.out.println(-1); }
    }
}
```

# Searching, Java Solution

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        while (j < n && a[j] != q) { j++; }
        if (j < n) { System.out.println(j); }
        else { System.out.println(-1); }
    }
}
```

# Searching/3, search2

Another idea:

Run through the array  
and set a pointer if the value is found.

```
search2
```

```
INPUT: A[1..n] (un)sorted array of integers, q an integer.
```

```
OUTPUT: index j such that A[j]=q or -1 if A[j] ≠ q for all j (1 ≤ j ≤ n)
```

```
ptr := -1;
for j := 1 to n do
  if a[j] = q then ptr := j
return ptr;
```

*Does it work?*

# search1 vs search2

Are the solutions equivalent?

Can one construct an example such that, say,

- search1 returns 3
- search2 returns 7 ?

But both solutions satisfy the specification (or don't they?)

# Searching/4, search3

A third idea:

Run through the array and  
**return** the index of the value in the array.

```
search3
```

```
INPUT: A[1..n] (un)sorted array of integers, q an integer.
```

```
OUTPUT: index j such that A[j]=q or -1 if A[j] ≠ q for all j (1 ≤ j ≤ n)
```

```
for j := 1 to n do  
    if a[j] = q then return j  
return -1
```

# Comparison of Solutions

Metaphor: shopping behavior when buying a beer:

- **search1**: scan products;  
stop as soon as a beer is found and go to the exit.
- **search2**: scan products until you get to the exit;  
if during the process you find a beer,  
put it into the basket  
(instead of the previous one, if any).
- **search3**: scan products;  
stop as soon as a beer is found  
and exit through next window.



# Comparison of Solutions/2

- `search1` and `search3` return *the same result* (index of the **first occurrence** of the search value)
- `search2` returns the index of the **last occurrence** of the search value
- `search3` **does not finish the loop** (as a general rule, you better avoid this)

# Beware: Array Indexes in Java/C/C++

- In pseudo-code, array indexes range from **1 to length**
- In Java/C/C++, array indexes range from **0 to length-1**
- Examples:

- Pseudo-code

```
for j := 1 to n do
```

Java:

```
for (j=0; j < a.length; j++) { ...
```

- Pseudo-code

```
for j := n downto 2 do
```

Java:

```
for (j=a.length-1; j >= 1; j--) { ...
```

# Suggested Exercises

- Implement the three variants of search  
(with input and output of arrays)
  - Create random arrays for different lengths
  - Compare the results
  - Add a counter for the number of cycles and return it, compare the result
- Implement them to scan the array in reverse order

# DSA, Chapter 1:

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- Recursion (principle, trace, factorial, Fibonacci)
- Sorting (bubble, insertion, selection)

# Recursion

An object is **recursive** if

- a part of the object **refers to the entire object**, or
- one **part refers to another part** and **vice versa**

(mutual recursion)

recursion - Google Search

+Werner Search Images Maps YouTube News Gmail Documents Calendar More ▾

Google recursion

Search About 2,700,000 results (0.10 seconds)

---

Everything

- Images
- Maps
- Videos
- News
- Shopping
- More

---

Search the web

Search English and German pages

---

Any time


- Past hour
- Past 24 hours
- ...

Did you mean: [recursion](#)

[Recursion - Wikipedia, the free encyclopedia](#)  
[en.wikipedia.org/wiki/Recursion](https://en.wikipedia.org/wiki/Recursion)  
**Recursion** is the process of repeating items in a self-similar way. For instance, when the surfaces of two mirrors are exactly parallel with each other the nested ...  
 ↳ [Formal definitions of recursion - Recursion in language](#)

[Recursion \(computer science\) - Wikipedia, the free encyclopedia](#)  
[en.wikipedia.org/wiki/Recursion\\_\(computer\\_science\)](https://en.wikipedia.org/wiki/Recursion_(computer_science))  
**Recursion** in computer science is a method where the solution to a problem depends on solutions to smaller instances of the same problem. The approach can ...

[Recursion in C and C++ - Cprogramming.com](#)  
[www.cprogramming.com/tutorial/lesson16.html](http://www.cprogramming.com/tutorial/lesson16.html)  
 by Alex Allain · [More by Alex Allain](#)  
 Learn how to use **recursion** in C and C++, with example **recursive** programs.



[Recursion -- from Wolfram MathWorld](#)



Source: <http://bluehawk.monmouth.edu/~rclayton/web-pages/s11-503/recursion.jpg>

# Recursion/2

- A **recursive definition**: a concept is defined by referring to itself.

E.g., arithmetical expressions (like  $(3 * 7) - (9 / 3)$ ):

$$\text{EXPR} := \text{VALUE} \mid (\text{EXPR OPERATOR EXPR})$$

- A **recursive procedure**: a procedure that calls itself

Classical example: **factorial**, that is  $n! = 1 * 2 * 3 * \dots * n$

$$n! = n * (n-1)!$$

*... or is there something missing?*



# The Factorial Function

Pseudocode of factorial:

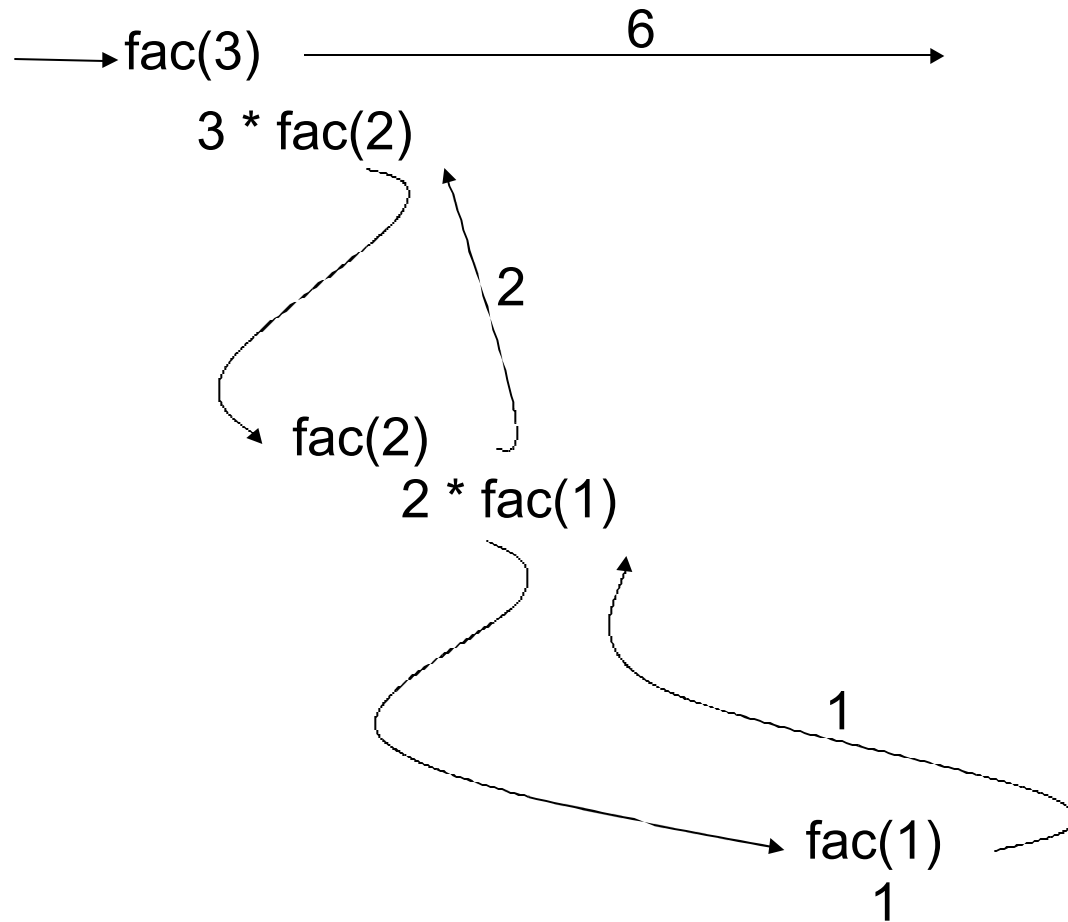
```
fac1
INPUT: n – a natural number.
OUTPUT: n! (factorial of n)

fac1(n)
  if n < 2 then return 1
  else return n * fac1(n-1)
```

This is a recursive procedure. A recursive procedure has

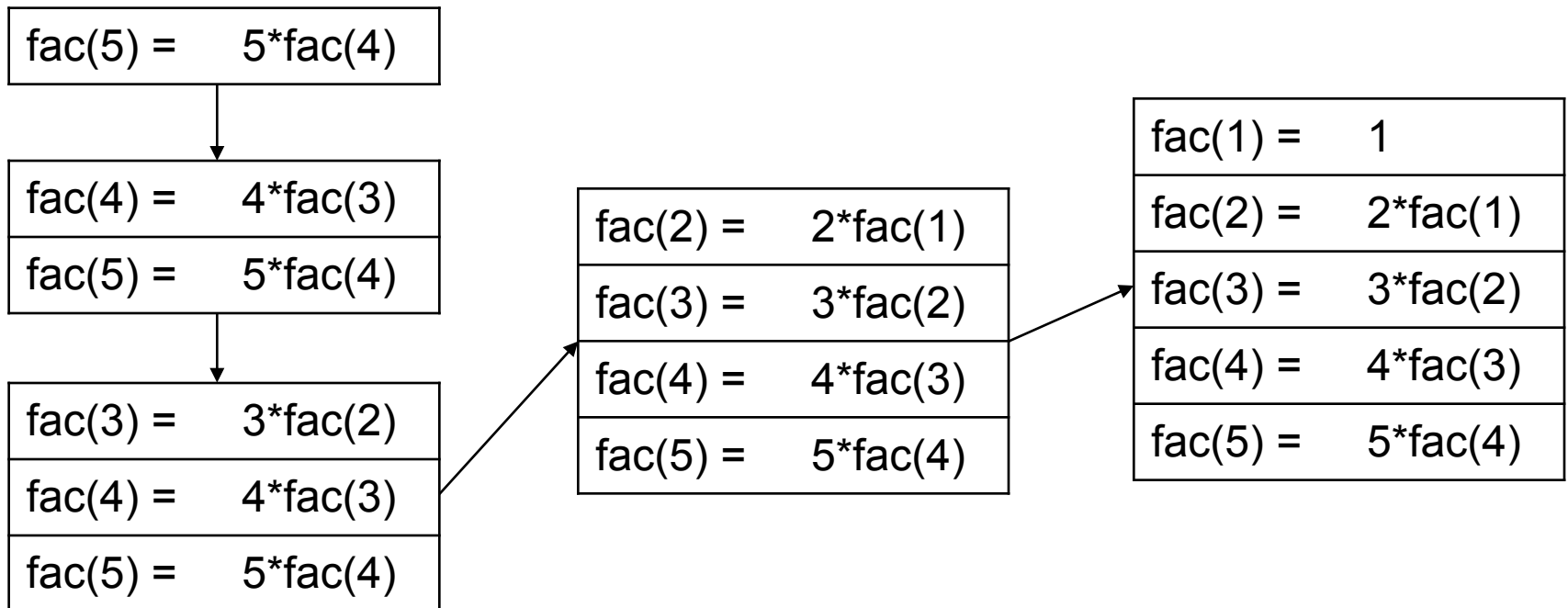
- a termination condition (determines when and how to stop the recursion).
- one (or more) recursive calls.

# Tracing the Execution



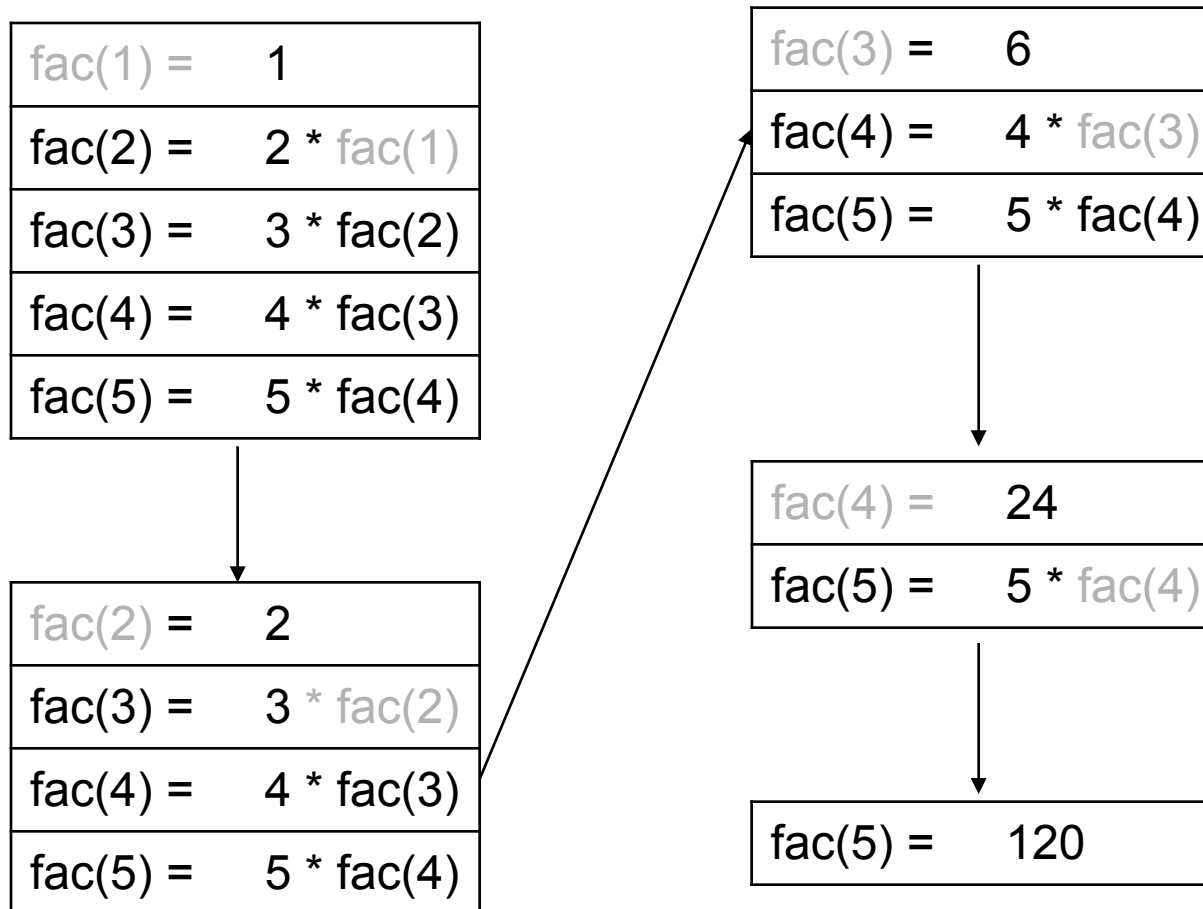
# Bookkeeping

The computer maintains an **activation stack** for active procedure calls ( $\rightarrow$  compiler construction).  
 Example for  $\text{fac}(5)$ . The stack is built up.



# Bookkeeping/2

Then the activation stack is reduced



# Variants of Factorial

**fac2**

*INPUT*: n - a natural number.

*OUTPUT*: n! (factorial of n)

**fac2(n)**

if n = 0 then return 1

return n \* fac2(n-1)

**fac3**

*INPUT*: n - a natural number.

*OUTPUT*: n! (factorial of n)

**fac3(n)**

if n = 0 then return 1

return n \* (n-1) \* fac3(n-2)

# Analysis of the Variants

`fac2` is correct

- The return statement in the if clause terminates the function and, thus, the entire recursion.

`fac3` is incorrect

- Infinite recursion.

The termination condition is never reached if  $n$  is odd:

```
fact(3)
= 3*2*fact(1)
= 3*2*1*0*fact(-1)
= ...
```

# Variants of Factorial/2

**fac4**

*INPUT*: n - a natural number.

*OUTPUT*: n! (factorial of n)

**fac4(n)**

if n <= 1 then return 1

return n\*(n-1)\*fac4(n-2)

**fac5**

*INPUT*: n - a natural number.

*OUTPUT*: n! (factorial of n)

**fac5(n)**

return n \* fac5(n-1)

if n <= 1 then return 1

# Analysis of the Variants/2

**fac4** is correct

- The return statement in the if clause terminates the function and, thus, the entire recursion.

**fac5** is incorrect

- Infinite recursion.  
The termination condition is never reached.

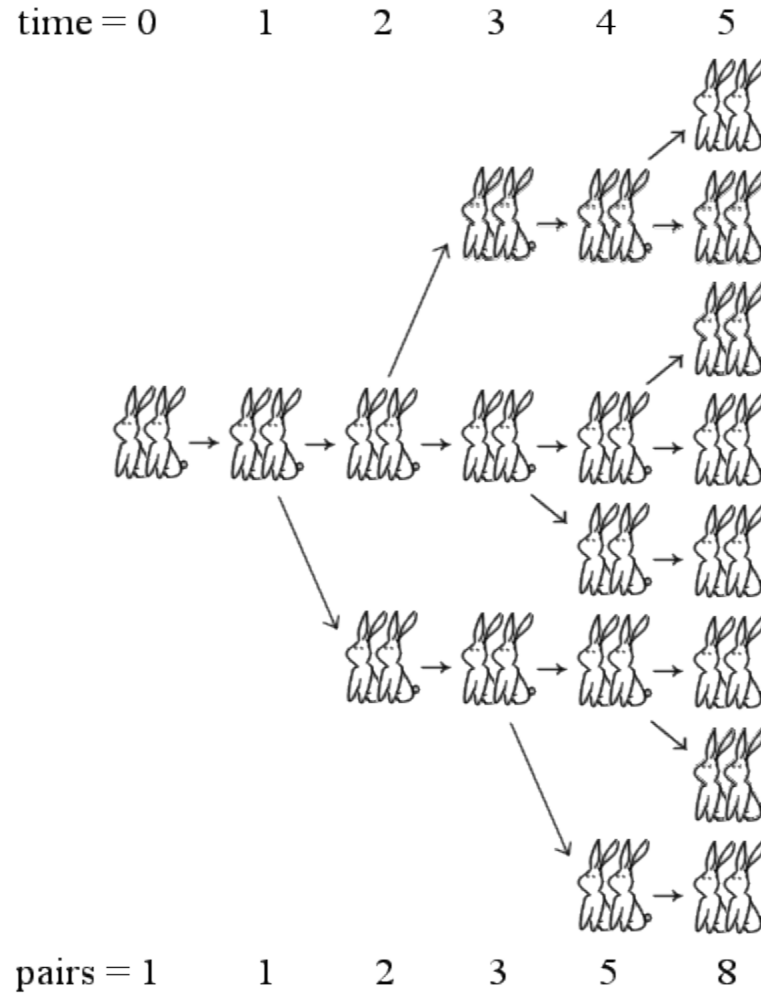


# Counting Rabbits

*Someone placed a pair of rabbits  
in a certain place,  
enclosed on all sides by a wall,  
so as to find out  
how many pairs of rabbits  
will be born there in the course of one year,  
it being assumed  
that every month  
a pair of rabbits produces another pair,  
and that rabbits begin to bear  
young two months after their own birth.*

*Leonardo di Pisa ("Fibonacci"),  
Liber abacci, 1202*

# Counting Rabbits/2



Source: <http://www.jimloy.com/algebra/fibo.htm>

# Fibonacci Numbers

## Definition

- $\text{fib}(0) = 1$
- $\text{fib}(1) = 1$
- $\text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2), n > 1$

## Numbers in the series:

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

# Fibonacci Procedure

**fib**

*INPUT*:  $n$  – a natural number greater or equal than 0.

*OUTPUT*:  $\text{fib}(n)$ , the  $n$ th Fibonacci number.

**fib( $n$ )**

**if**  $n \leq 1$  **then return** 1

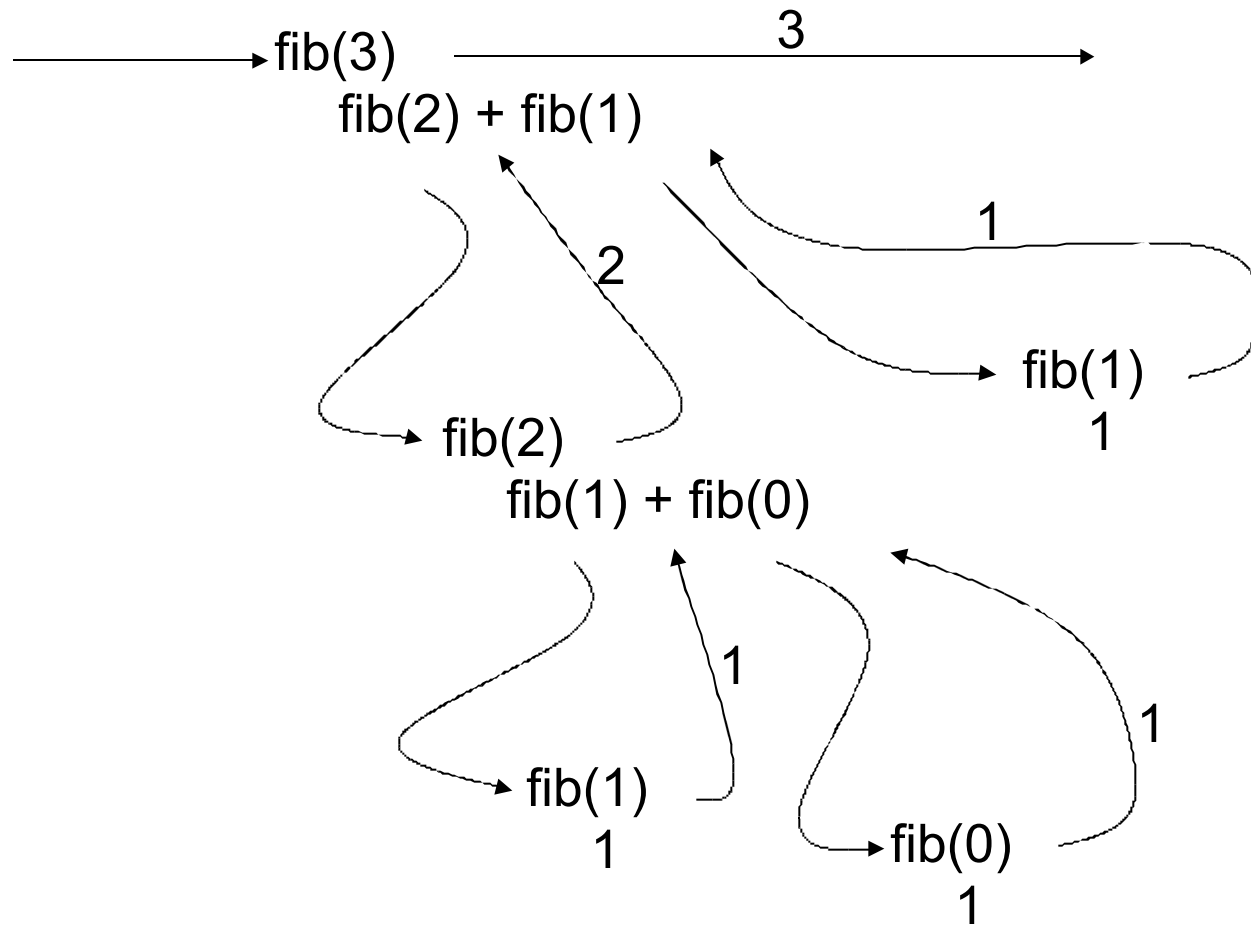
**else return**  $\text{fib}(n-1) + \text{fib}(n-2)$

A procedure with **multiple** recursive calls

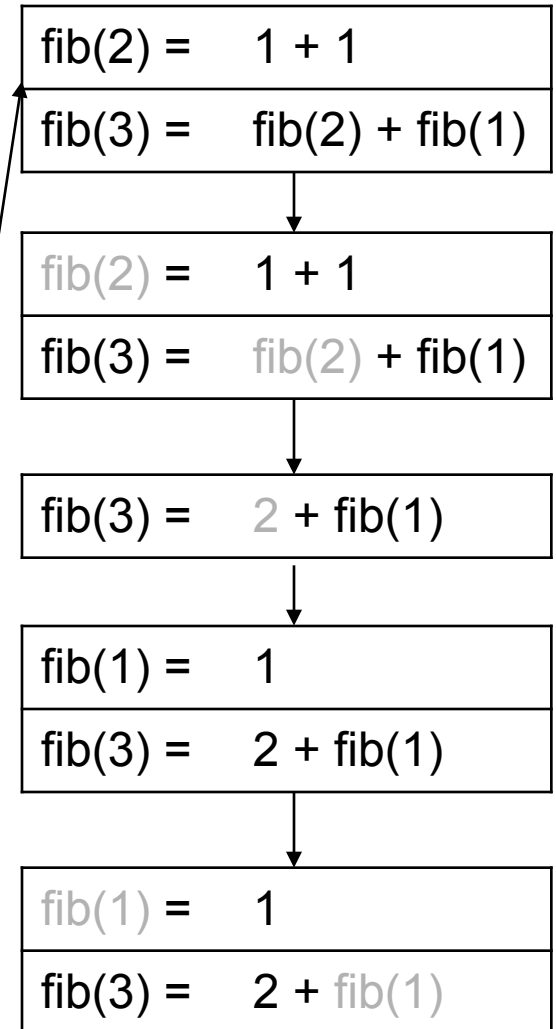
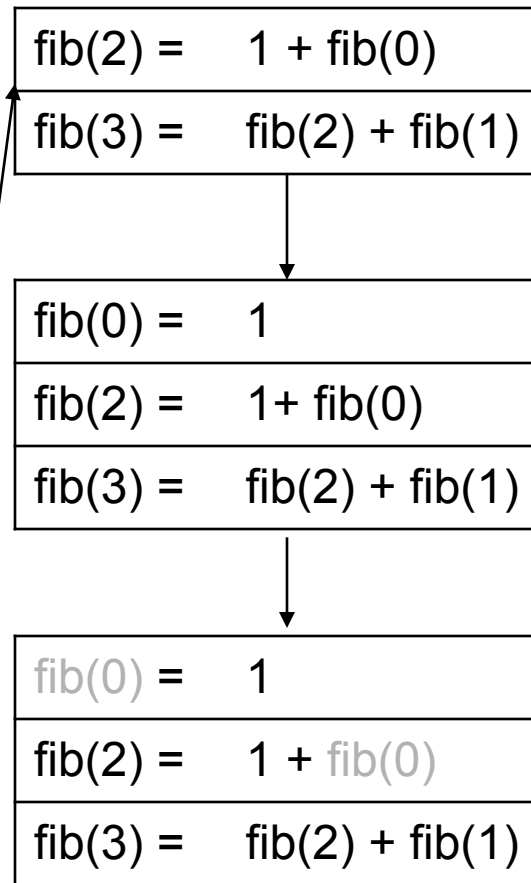
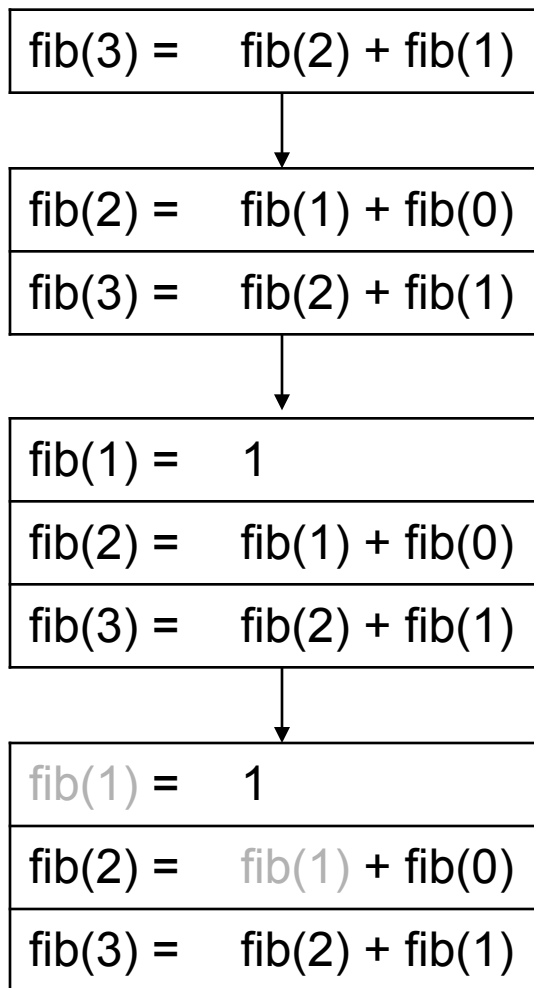
# Fibonacci Procedure/2

```
public class fibclassic {  
  
    static int fib(int n) {  
        if (n <= 1) {return 1;}  
        else {return fib(n - 1) + fib(n - 2);}  
    }  
  
    public static void main(String args[]) {  
        System.out.println("Fibonacci of 5 is "  
                            + fib(5));  
    }  
}
```

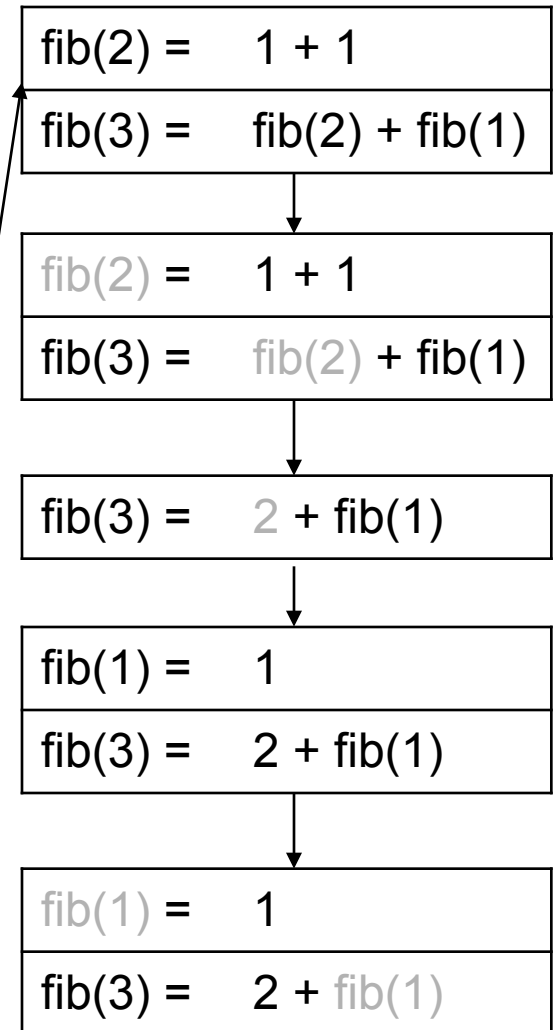
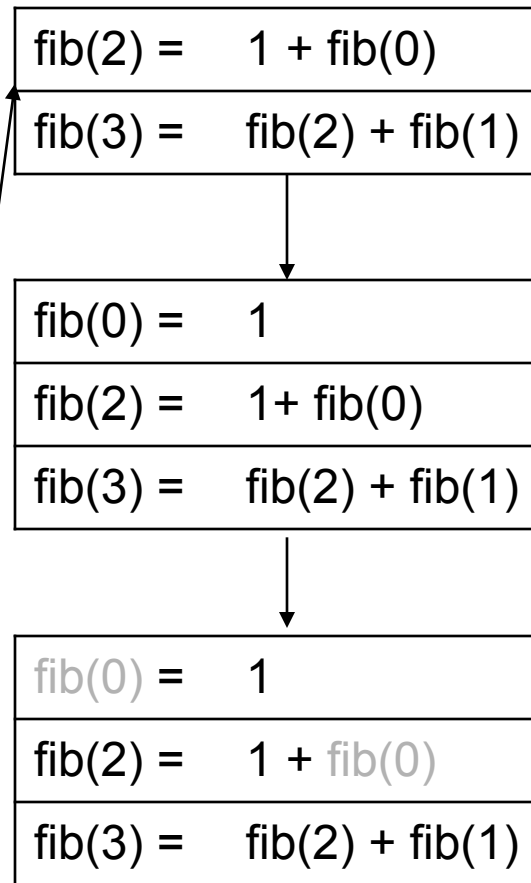
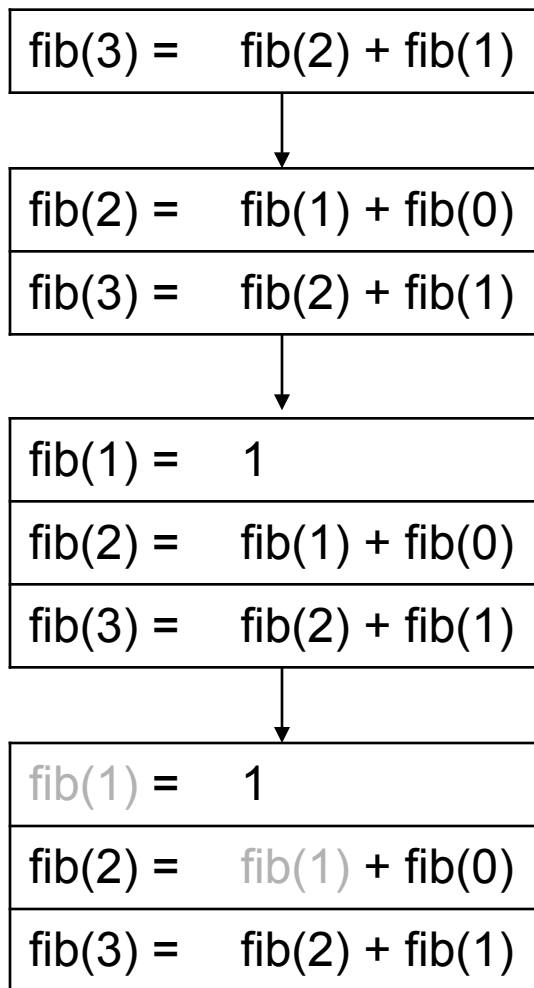
# Tracing fib(3)



# Bookkeeping



# Bookkeeping





# Questions

- What is the maximal height of the recursion stack during the computation of  $fib(n)$ ?
- How many recursive calls are made to compute  $fib(n)$ ?
- What does the tree of recursive calls (**recursion tree**) look like?
- Can we derive a lower bound for the number of calls from that?
- Can there be a procedure for  $fib$  with fewer operations?
- How is the size of the result  $fib(n)$  related to the size of the input  $n$ ?

# Saving Intermediate Values of fib(n)

**fib**

*INPUT*:  $n$  – a natural number greater or equal than 0.

*OUTPUT*: fib( $n$ ), the  $n$ th Fibonacci number.

**fib(n)**

```
if  $n \leq 1$  then return 1
```

```
int[] fibVal = new int[n+1]
```

```
// let's assume in this case that the array
```

```
// boundaries start with 0 :- (
```

```
fibVal[0] := 1; fibVal[1] := 1;
```

```
for j=2 to n do
```

```
    fibVal[j] := fibVal[j-1]+fibVal[j-2]
```

```
return fibVal[n];
```

*... but we only need the last two values of fib to compute the next one*

# Iterative Computation of fib(n)

**fib**

*INPUT*:  $n$  – a natural number greater or equal than 0.

*OUTPUT*: fib( $n$ ), the  $n$ th Fibonacci number.

**fib(n)**

```
if  $n \leq 1$  then return 1
```

```
int f, f1, f2;
```

```
f1 := 1; f2 := 1;
```

```
for j=2 to n do
```

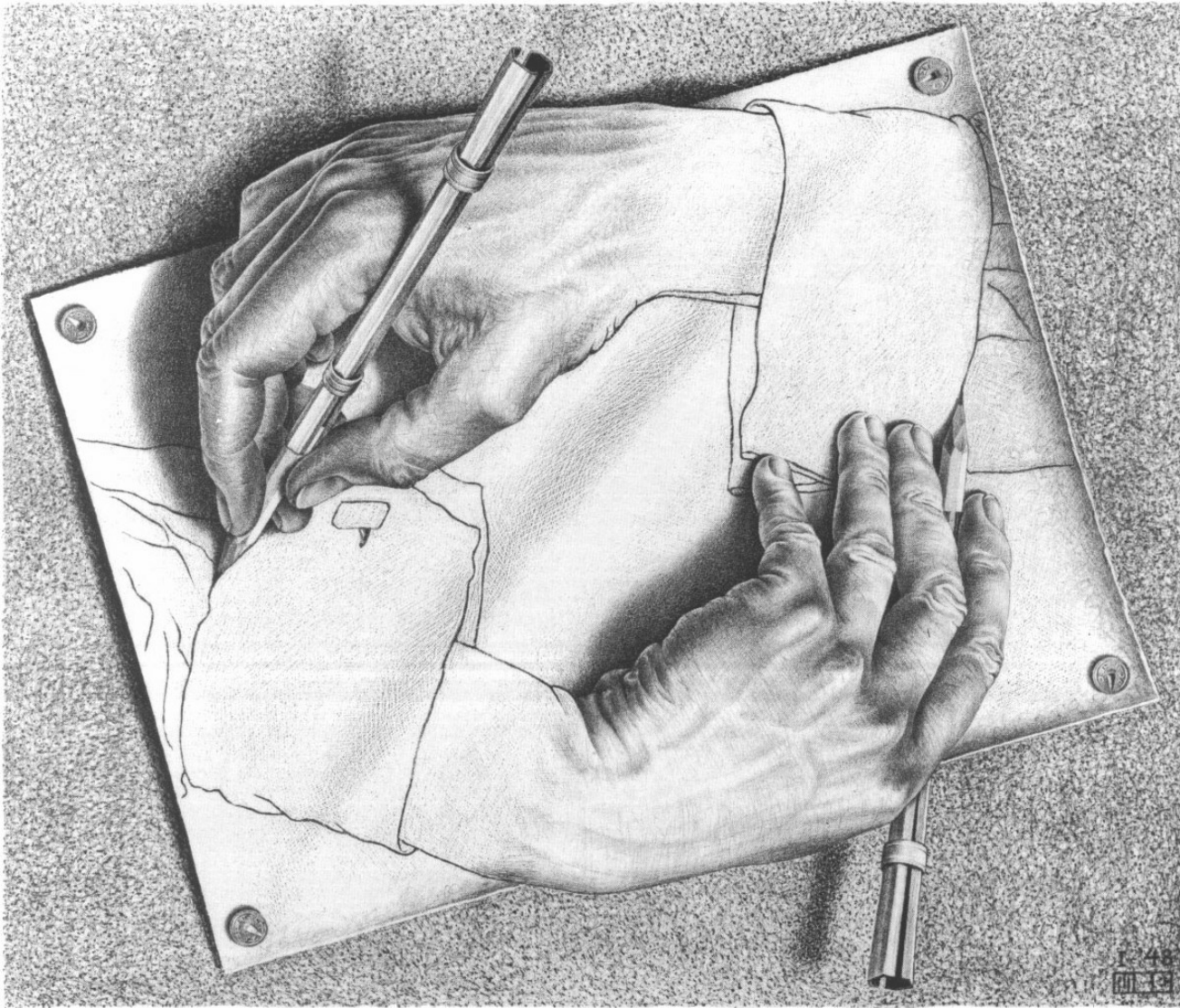
```
    f := f1+f2
```

```
    f2 := f1
```

```
    f1 := f
```

```
return f;
```

# Mutual Recursion



Source: [http://britton.disted.camosun.bc.ca/escher/drawing\\_hands.jpg](http://britton.disted.camosun.bc.ca/escher/drawing_hands.jpg)

# Mutual Recursion Example

- Problem: Determine whether a natural number is even
- Definition of even:
  - 0 is even
  - $n$  is even if  $n - 1$  is odd
  - $n$  is odd if  $n - 1$  is even

# Implementation of even

**even**

*INPUT*:  $n$  – a natural number.

*OUTPUT*: true if  $n$  is even; false otherwise

**even( $n$ )**

if  $n = 0$  then return **TRUE**

else return **odd( $n-1$ )**

**odd( $n$ )**

if  $n = 0$  then return **FALSE**

return **even( $n-1$ )**

How can we determine whether  $n$  is odd?

# Is Recursion Necessary?

- **Theory:** You can always resort to iteration and explicitly maintain a recursion stack.
- **Practice:** Recursion is elegant and in some cases the best solution by far.
- In the previous examples recursion was never appropriate since there exist simple iterative solutions.
- Recursion is more expensive than corresponding iterative solutions since bookkeeping is necessary.