Data Structures and Algorithms

Chapter 7

Hashing

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Acknowledgments

- The course follows the book “Introduction to Algorithms‘”, by **Cormen, Leiserson, Rivest and Stein**, MIT Press [CLRST]. Many examples displayed in these slides are taken from their book.
- These slides are based on those developed by Michael Böhlen for this course.

  (See http://www.inf.unibz.it/dis/teaching/DSA/)
- The slides also include a number of additions made by Roberto Sebastiani and Kurt Ranalter when they taught later editions of this course

  (See http://disi.unitn.it/~rseba/DIDATTICA/dsa2011_BZ//)

DSA, Chapter 7: Overview

1. Dictionaries
2. Hashing
3. Hash Functions
4. Collisions
5. Performance Analysis

DSA, Chapter 7: Overview

Dictionary

- A *dictionary* D is a dynamic data structure with operations:
  - search(D, k) – *returns a pointer x to an element such that x.key = k (null otherwise)*
  - insert(D, x) – *adds the element pointed to by x to D*
  - delete(D, x) – *removes the element pointed to by x from D*
- An element has a *key* and a *satellite data* part

Dictionaries

- Dictionaries store elements so that they can be located quickly using keys
- A dictionary may hold bank accounts.
  - Each account is an object that is identified by an account number.
  - Each account stores a lot of additional information.
  - An application wishing to operate on an account would have to provide the account number as a search key.

Dictionaries/2

- If order (methods like *min, max, successor, predecessor*) is not required, it is enough to check for equality.
- Operations that require ordering are still possible, but cannot use the dictionary access structure.
  - Usually all elements must be compared, which is slow.
  - Can be OK if it is rare enough

Dictionaries/3

- Dictionaires can be realized by different data structures
  - arrays
  - linked lists
  - binary trees
  - red/black trees
  - B-trees
  - hash tables
- In Java:
  - java.util.Map – interface defining Dictionary ADT

DSA, Chapter 7: Overview

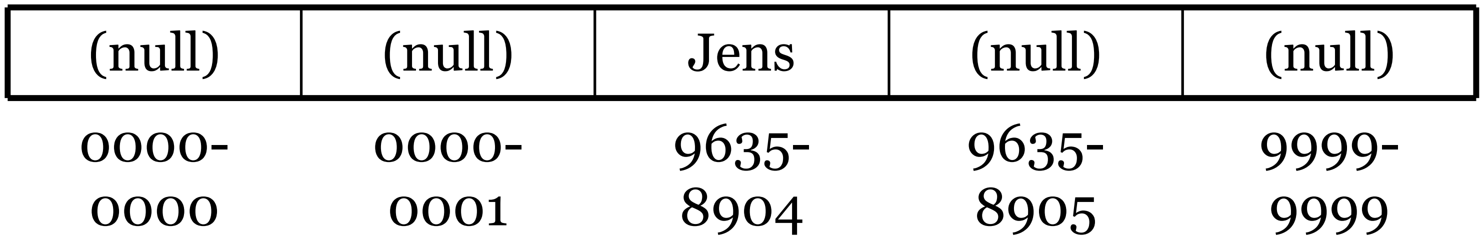
The Problem

XY Telecom, a large phone company,
wants to provide a caller ID capability:

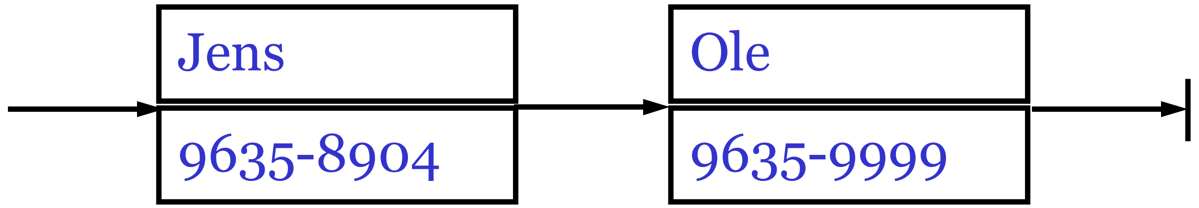
- given a phone number,
  return the caller's name
- phone numbers range from 0 to r = 10^8 -1
- do this as efficiently as possible

The Problem/2

- Two suboptimal ways to design this dictionary
  - direct addressing: an array indexed by key:
    - requires O(1) time,
    - requires O(*r*) space - huge amount of wasted space



  - a linked list: requires O(*n*) time, O(*n*) space



Another Solution: Hashing

- We can do better, with a hash table *of size m*
- Like an array, but with a **function** to map the large range into one which we can manage
    - e.g., take the original key, modulo the (relatively small) size of the table, and use that as an index
- Insert (9635-8904, Jens) into a hash table with, say, five slots (m = 5)
    - 96358904 mod 5 = 4

(null)	(null)	(null)	(null)	Jens
0	1	2	3	4

- O(1) expected time, O(n+m) space

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Hash Functions

- Need to choose a good hash function (HF)
  - quick to compute
  - distributes keys uniformly throughout the table
- How to deal with hashing non-integer keys:
  - find some way of turning the keys into integers
    - in our example, remove the hyphen in 9635-8904 to get 96358904
    - for a string, add up the ASCII values of the characters of your string (e.g., java.lang.String.hashCode())
  - then use a standard hash function on the integers

HF: Division Method

- Use the remainder: $h(k) = k \bmod m$
  - k is the key, m the size of the table
- Need to choose m
- $m = b^e$ **(bad)**
  - if m is a power of 2,
    $h(k)$ gives the e least significant bits of k
  - all keys with the same ending go to the same place
- m prime **(good)**
  - helps ensure uniform distribution
  - primes not too close to exact powers of 2 are best

HF: Division Method/2

- Example 1
  - hash table for n = 2000 character strings,
    ok to investigate an average of three attempts/search
  - m = 701
    - a prime near 2000/3
    - but not near any power of 2
- Further examples
  - m = 13
    - $h(3) = 3$
    - $h(12) = 12$
    - $h(13) = 0$

HF: Multiplication Method

- Use $h(k) = \lfloor m\,(k\,A \bmod 1) \rfloor$
  - k is the key
  - m the size of the table
  - A is a constant $1/2 < A < 1$
  - (k A mod 1): the fractional part of k A
- The steps involved
  - map $0 \ldots k_{max}$ into $0 \ldots k_{max}A$
  - take the fractional part (mod 1)
  - map it into $0 \ldots m-1$

HF: Multiplication Method/2

- Choice of *m* and *A*
  - Value of *m* is not critical:
    typically, for some p use $m = 2^p$
  - Optimal choice of *A* depends
    on the characteristics of the data
    - Knuth says use $A = \frac{\sqrt{5} - 1}{2}$ = 0.618033988

HF: Multiplication Method/3

- Assume 7-bit binary keys, $0 \leq k < 128$
- $m = 64 = 2^6$, $p = 6$
- $A = 89/128 = .1011001$, $k = 107 = 1101011$
- Computation of h(k):

```
          .1011001 A
           1101011 k
   1001010.0110011 kA
          .0110011 kA mod 1
    011001.1       m(kA mod 1)
```

- Thus, $h(k) = 25$

DSA, Chapter 7: Overview

Collisions

Assume a key is mapped
to an already occupied table location

– what to do?

Use a collision handling technique

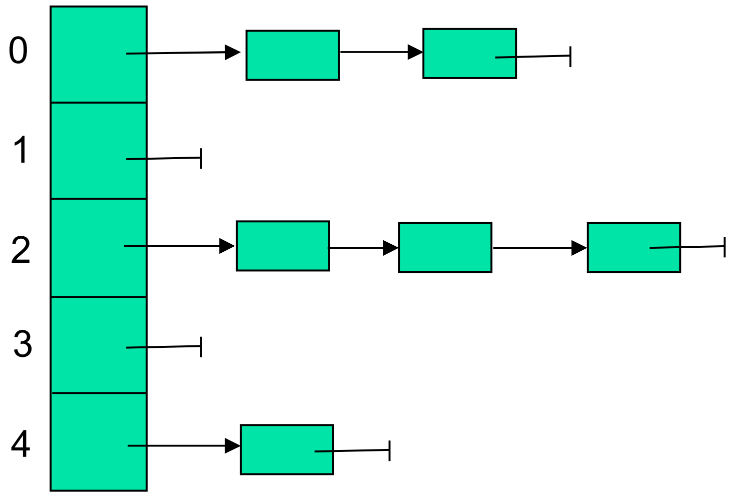
There are 3 techniques to deal with collisions:

– chaining
– open addressing/linear probing
– open addressing/double hashing

Chaining

Chaining maintains a table of links,

- indexed by the keys,
- to lists of items with the same key



Open Addressing

- All elements are stored in the hash table (can fill up), i.e., $n \leq m$
- Each table entry contains either an element or null
- When searching for an element, systematically probe table slots
- Modify hash function to take probe number i as second parameter

$$h: U \times \{ 0, 1, \ldots, m\text{-}1 \} \rightarrow \{ 0, 1, \ldots, m\text{-}1 \}$$

Open Addressing/2

- Hash function, h, determines the sequence of slots examined for a given key
- Probe sequence for a given key k is given by

  $$(h(k,0), h(k,1), \ldots, h(k,m-1)),$$

  which is a permutation of $(0, 1, \ldots, m-1)$

Linear Probing

```
LinearProbingInsert(k)
    if (table is full) then error
    probe := h(k)
    while (table[probe] occupied) do
        probe := (probe+1) mod m
    table[probe] = k
```

- If the current location is used, try the next table location:
  h(key,i) = (h1(key)+i) mod m
- Lookups walk along the table
  until the key or an empty slot is found
- Uses less memory than chaining
  - one does not have to store all those links
- Slower than chaining
  - one might have to probe the table for a long time

Linear Probing/2

- Problem "primary clustering":
  long lines of occupied slots
  - A slot preceded by i full slots has a high probability of getting filled: (i+1)/m
- Alternatives: (quadratic probing,) double hashing
- Example:
  - $h(k) = k \bmod 13$
  - insert keys: 18 41 22 44 59 32 31 73

Double Hashing

Use two hash functions:

$h(key,i) = (h1(key) + i*h2(key)) \bmod m, \; i = 0,1,...$

```
DoubleHashingInsert(k)
    if (table is full) then error
    probe := h1(k)
    offset := h2(k)
    while (table[probe] occupied) do
        probe := (probe + offset) mod m
    table[probe] := k
```

Distributes keys much more uniformly than linear probing.

Double Hashing/2

h2(k) must be relative prime to *m* to search the entire hash table

- Suppose *h2*(*k*) = *k*a* and *m* = *w*a*, *a* > *1*

Two ways to ensure this:

- *m* is power of 2, h2(k) is odd
- *m:* prime, *h2(k):* positive integer < *m*

Example

- *h*1(*k*) = *k* mod 13, *h2*(*k*) = 8 - (*k* mod 8)
- insert keys: 18 41 22 44 59 32 31 73

Open Addressing: Delete

Complex to delete from

- A slot may be reached from different points
  - We cannot simply store “NIL”: we'd loose the information necessary to retrieve other keys
- Possible solution: mark the deleted slot as “deleted”, insert also on “deleted”
  - Drawback: retrieval time no more depending on load factor: potentially lots of “jumps” on “deleted” slots

When deletion is admitted/frequent,
then chaining is preferred

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Analysis of Hashing

An element with key k is stored in slot $h(k)$
(instead of slot k without hashing)

The hash function h maps the universe U of keys into the slots of hash table $T[0...m-1]$

$$h: U \rightarrow \{ 0, 1, ..., m-1 \}$$

Assumption: Each key is equally likely to be hashed into any slot (bucket):
simple uniform hashing

Given hash table T with m slots holding n elements, the load factor is defined as $\alpha = n/m$

Analysis of Hashing/2

Assume time to compute $h(k)$ is $\Theta(1)$

To find an element

- using h, look up its position in table T
- search for the element in the linked list of the hashed slot
- *uniform* hashing yields an average list length of α = n/m
- expected number of elements to be examined is α
- search time is $O(1+\alpha)$

Analysis of Hashing/3

Assuming the number of hash table slots is proportional to the number of elements in the table

$$n = O(m)$$
$$\alpha = n/m = O(m)/m = O(1)$$

- searching takes constant time on average
- insertion takes $O(1)$ worst-case time
- deletion takes $O(1)$ worst-case time
  (pass the element not key, lists are doubly-linked)

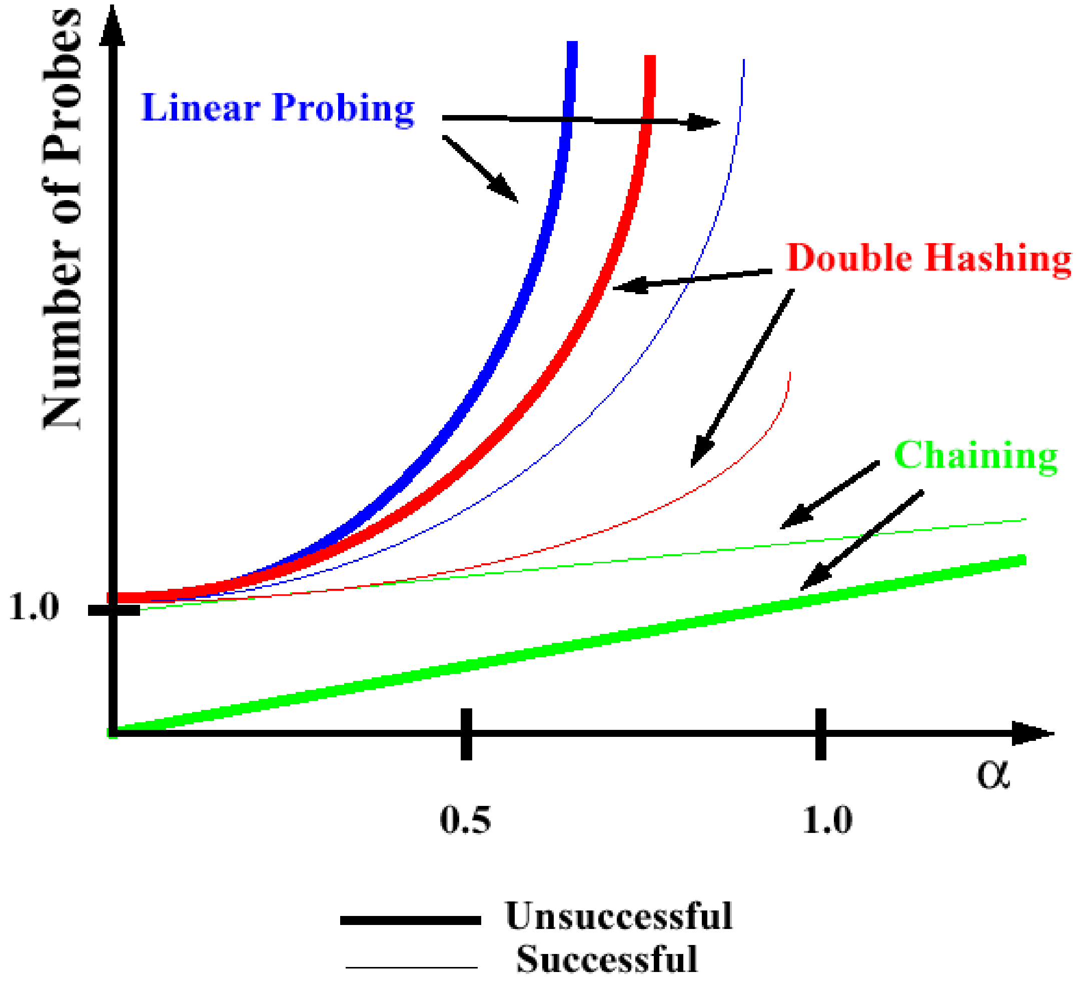
Expected Number of Probes

- Load factor $\alpha < 1$ for probing
- Analysis of probing uses *uniform hashing* assumption – any permutation is equally likely

	Unsuccessful	Successful
Chaining	$O(1+\alpha)$	$O(1+\alpha)$
Probing	$O(\frac{1}{1-\alpha})$	$O(\frac{1}{\alpha}\ln\frac{1}{1-\alpha})$

- Chaining: 1 (α=0%), 1.5 (α=50%), 2 (α=100%), n (α=n)
- Probing, unsucc: 1.25 (α=20%), 2 (α=50%), 5 (α=80%), 10 (α=90%)
- Probing, succ: 0.28 (α=20%), 1.39 (α=50%), 2.01 (α=80%), 2.56 (α=90%)

Expected Number of Probes/2



Summary

- Hashing is very efficient
  (not obvious, probability theory).
- Its functionality is limited (printing elements sorted according to key is not supported).
- The size of the hash table
  may not be easy to determine.
- A hash table is not really
  a dynamic data structure.

Suggested exercises

- Implement a Hash Table with the different techniques
- With paper & pencil, draw the evolution of a hash table when inserting, deleting and searching for new element, with the different techniques
- See also exercises of CLRS

Next Part

- Graphs:
  - Representation in memory
  - Breadth-first search
  - Depth-first search
  - Topological sort