# Data Structures and Algorithms

## Chapter 3

1. Divide and conquer
2. Merge sort, repeated substitutions
3. Tiling
4. Recurrences

# Recurrences

- Running times of algorithms with **recursive calls** can be described using recurrences.
- A **recurrence** is an equation or inequality that describes a function in terms of its value on smaller inputs.
- For divide and conquer algorithms:

$$T(n)=\begin{cases} \text{solving trivial problem} & \text{if } n=1 \\ \text{NumPieces} * T(n/\text{SubProbFactor}) + \text{divide} + \text{combine} & \text{if } n>1 \end{cases}$$

- Example: Merge Sort

$$T(n)=\begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2)+\Theta(n) & \text{if } n>1 \end{cases}$$

# Solving Recurrences

- Repeated (backward) substitution method
  - Expanding the recurrence by substitution and noticing a pattern (this is not a strictly formal proof).
- Substitution method
  - guessing the solutions
  - verifying the solution by the mathematical induction
- Recursion trees
- Master method
  - templates for different classes of recurrences

# Repeated Substitution

- Let's find the running time of merge sort (assume $n=2^b$).

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n > 1 \end{cases}$$

$$\begin{aligned}
T(n) &= 2T(n/2) + n \quad \text{substitute} \\
&= 2(2T(n/4) + n/2) + n \quad \text{expand} \\
&= 2^2T(n/4) + 2n \quad \text{substitute} \\
&= 2^2(2T(n/8) + n/4) + 2n \quad \text{expand} \\
&= 2^3T(n/8) + 3n \quad \text{observe pattern}
\end{aligned}$$

# Repeated Substitution/2

From $T(n) = 2^3 T(n/8) + 3n$

we get $T(n) = 2^i T(n/2^i) + i\, n$

An upper bound for $i$ is $log\ n$:

$$T(n) = 2^{log\ n} T(n/n) + n\ log\ n$$

$$T(n) = n + n\ log\ n$$

# Repeated Substitution Method



- The procedure is straightforward:
  - Substitute, Expand, Substitute, Expand, …
  - Observe a pattern and determine the expression after the *i*-th substitution.
  - Find out what the highest value of *i* (number of iterations, e.g., *log n*) should be to get to the base case of the recurrence (e.g., *T(1)*).
  - Insert the value of *T(1)* and the expression of *i* into your expression.

# Analysis of Sort Merge

- Let's find a more exact running time of merge sort (assume $n=2^b$).

$$T(n) = \begin{cases} 2 & \text{if } n = 1 \\ 2T(n/2) + 2n + 3 & \text{if } n > 1 \end{cases}$$

$$\begin{aligned}
T(n) &= 2T(n/2) + 2n + 3 && \text{substitute} \\
&= 2(2T(n/4) + n + 3) + 2n + 3 && \text{expand} \\
&= 2^2T(n/4) + 4n + 2*3 + 3 && \text{substitute} \\
&= 2^2(2T(n/8) + n/2 + 3) + 4n + 2*3 + 3 && \text{expand} \\
&= 2^3T(n/2^3) + 2*3n + (2^2+2^1+2^0)*3 && \text{observe pattern}
\end{aligned}$$

# Analysis of Sort Merge/2

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$$T(n) \quad = 2^i T(n/2^i) + 2in + 3 \sum_{j=0}^{i-1} 2^j$$

*An upper bound for i is log n*

$$= 2^{\log n} T(n/2^{\log n}) + 2n\log n + 3*(2^{\log n} - 1)$$

$$= 5n + 2n\log n - 3$$

$$= \Theta(n \log n)$$

# Substitution Method

- The substitution method to solve recurrences entails two steps:
  - Guess the solution.
  - Use induction to prove the solution.
- Example:
  - $T(n) = 4T(n/2) + n$

# Substitution Method/2

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1) *Guess $T(n) = O(n^3)$, i.e., $T(n)$ is of the form $cn^3$*

2) *Prove $T(n) \leq cn^3$ by induction*

$T(n) = 4T(n/2) + n$ recurrence

$\leq 4c(n/2)^3 + n$ induction hypothesis

$= 0.5cn^3 + n$ simplify

$= cn^3 - (0.5cn^3 - n)$ rearrange

$\leq cn^3$ if $c>=2$ and $n>=1$

Thus $T(n) = O(n^3)$

# Substitution Method/3

- Tighter bound for $T(n) = 4T(n/2) + n$:

  Try to show $T(n) = O(n^2)$

  Prove $T(n) \leq cn^2$ by induction

$$T(n) = 4T(n/2) + n$$
$$\leq 4c(n/2)^2 + n$$
$$= cn^2 + n$$
NOT $\leq cn^2$

=> contradiction

# Substitution Method/4

- What is the problem? Rewriting

  $T(n) = O(n^2) = cn^2 + (\text{something positive})$

  as $T(n) \leq cn^2$

  does not work with the inductive proof.
- Solution: Strengthen the hypothesis for the inductive proof:
  - $T(n) \leq$ (answer you want) - (something > 0)

# Substitution Method/5



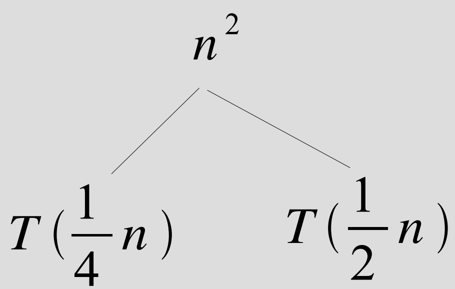
- Fixed proof: strengthen the inductive hypothesis by subtracting lower-order terms:

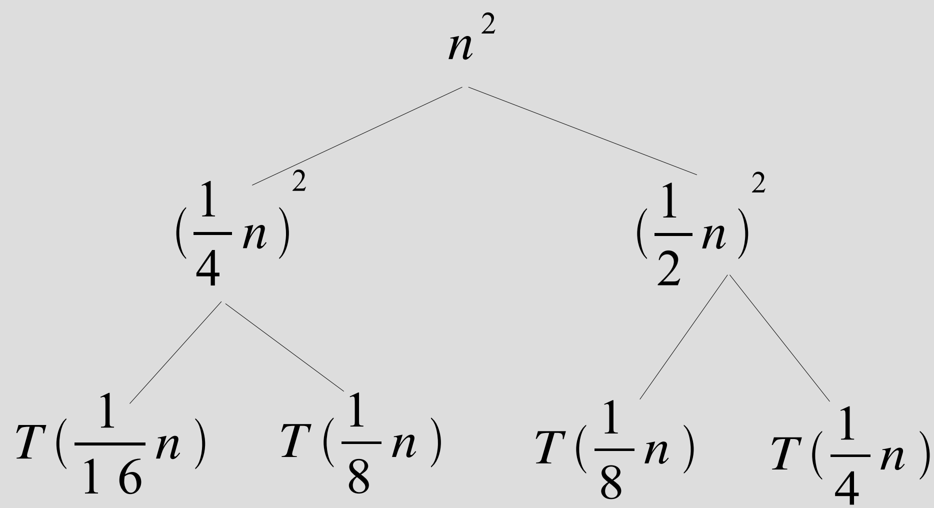
  Prove $T(n) \leq cn^2 - dn$ by induction

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\leq 4(c(n/2)^2 - d(n/2)) + n \\
&= cn^2 - 2dn + n \\
&= cn^2 - dn - (dn - n) \\
&\leq cn^2 - dn \text{ if } d \geq 1
\end{aligned}
$$

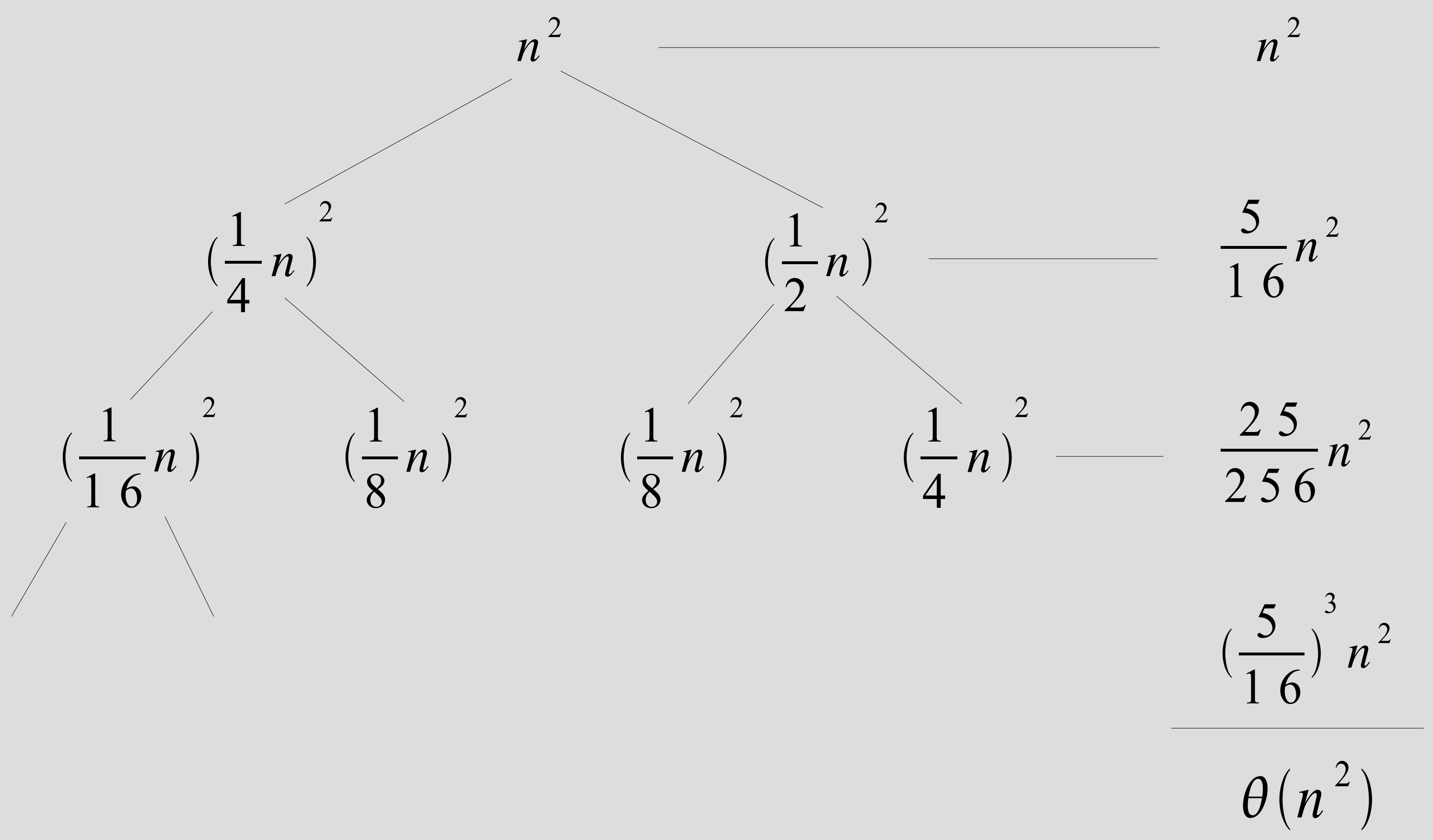
# Recursion Tree

- A recursion tree is a convenient way to visualize what happens when a recurrence is iterated.
  - Good for ”guessing” asymptotic solutions to recurrences



# Recursion Tree/2



# Master Method

- The idea is to solve a class of recurrences that have the form $T(n) = aT(n/b) + f(n)$
- *Assumptions:* $a \geq 1$ and $b > 1$, and $f(n)$ is asymptotically positive.
- Abstractly speaking, $T(n)$ is the runtime for an algorithm and we know that
  - $a$ subproblems of size $n/b$ are solved recursively, each in time $T(n/b)$.
  - $f(n)$ is the cost of dividing the problem and combining the results. In merge-sort $T(n) = 2T(n/2) + \Theta(n)$.

# Master Method/2

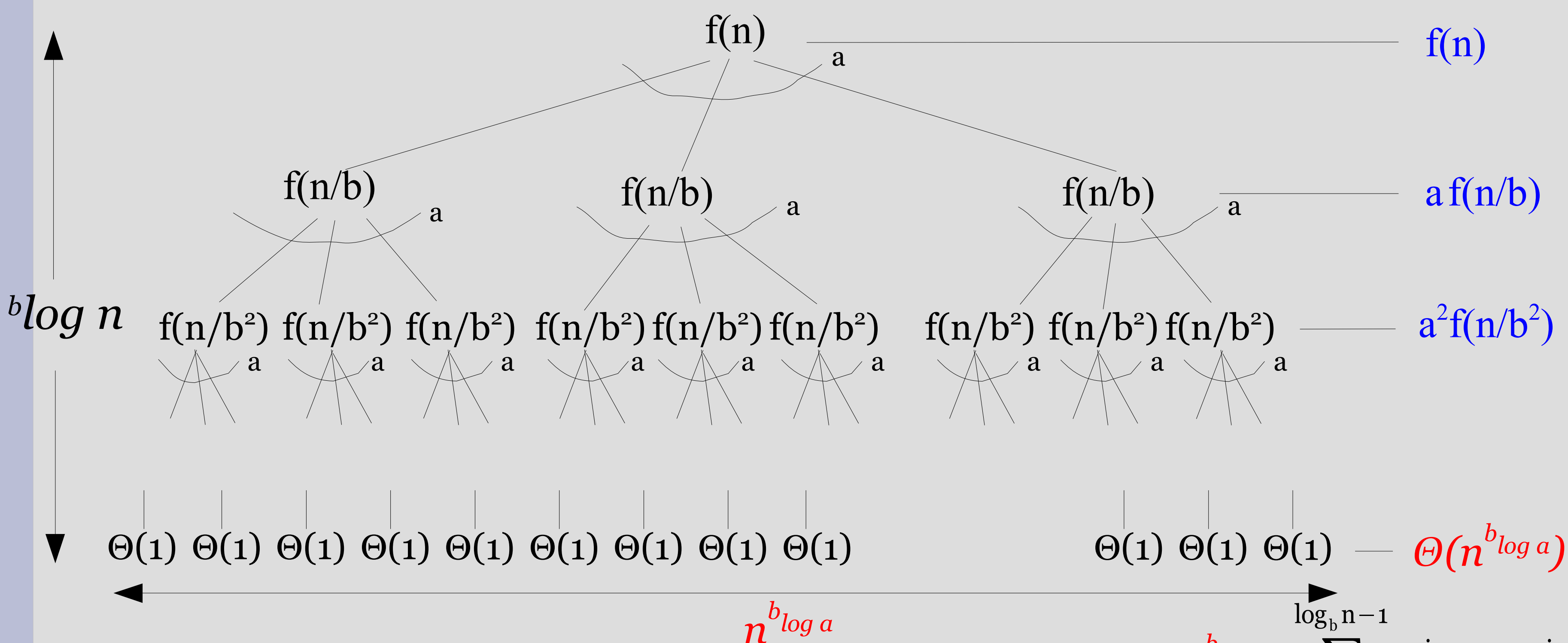
- Iterating the recurrence (expanding the tree) yields

$$T(n) = f(n) + aT(n/b)$$
$$= f(n) + af(n/b) + a^2T(n/b^2)$$
$$= f(n) + af(n/b) + a^2f(n/b^2) + \ldots$$
$$+ a^{{}^b\log n-1}f(n/a^{{}^b\log n-1}) + a^{{}^b\log n}T(1)$$

$$T(n) = \sum_{j=0}^{{}^b\log n-1} a^j f(n/b^j) + \Theta(n^{{}^b\log a})$$

- The first term is a division/recombination cost (totaled across all levels of the tree).
- The second term is the cost of doing all subproblems of size 1 (total of all work pushed to leaves).

# Master Method/3




$$\textit{Total: } \Theta(n^{{}^{b}\log a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)$$

Note: split into *a* parts, $^{b}\log n$ levels, $a^{{}^{b}\log n} = n^{{}^{b}\log a}$ leaves.

# Master Method, Intuition

- Three common cases:
  1. Running time dominated by cost at leaves.
  2. Running time evenly distributed throughout the tree.
  3. Running time dominated by cost at the root.
- To solve the recurrence, we need to identify the dominant term.
- In each case compare $f(n)$ with $O(n^{\log_b a})$.

# Master Method, Case 1

- $f(n) = O(n^{\log_b a-\varepsilon})$ for some constant $\varepsilon>0$
  - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by factor $n^{\varepsilon}$).
- **The work at the leaf level dominates**

$$T(n) = \Theta(n^{\log_b a})$$

Cost of all the leaves

# Master Method, Case 2

- $f(n) = \Theta(n^{\log_b a})$
  - $f(n)$ and $n^{\log_b a}$ are asymptotically the same

- **The work is distributed equally throughout the tree**

$$T(n) = \Theta(n^{\log_b a} \log n)$$

(level cost) $\times$ (number of levels)

# Master Method, Case 3

- $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for some constant $\varepsilon>0$
  - Inverse of Case 1
  - $f(n)$ grows polynomially faster than $n^{\log_b a}$
  - Also need a “regularity” condition

$$\exists c < 1 \text{ and } n_0 > 0 \text{ such that } af(n/b) \le cf(n)\ \forall n > n_0$$

- **The work at the root dominates**

$$T(n) = \Theta(f(n))$$

division/recombination cost

# Master Theorem Summarized

Given: recurrence of the form

$T(n) = aT(n/b) + f(n)$

1. $f(n) = O(n^{\log_b a - \varepsilon})$

   $\Rightarrow T(n) = \Theta(n^{\log_b a})$

2. $f(n) = \Theta(n^{\log_b a})$

   $\Rightarrow T(n) = \Theta(n^{\log_b a} \log n)$

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ and

   $a\, f(n/b) \leq \alpha\, f(n)$ for some $\alpha < 1$, $n > n_o$

   $\Rightarrow T(n) = \Theta(f(n))$

# Strategy

1. Extract *a*, *b*, and *f(n)* from a given recurrence
2. Determine $n^{{}^b\log a}$
3. Compare *f(n)* and $n^{{}^b\log a}$ asymptotically
4. Determine appropriate MT case and apply it

*Merge sort:* $T(n) = 2T(n/2) + \Theta(n)$

$a=2, b=2, f(n) = \Theta(n)$

$n^{{}^2\log 2} = n$

$\Theta(n) = \Theta(n)$

=> Case 2: $T(n) = \Theta(n^{{}^b\log a}\log n) = \Theta(n \log n)$

# Examples of Master Method

```
BinarySearch(A, l, r, q):
   m := (l+r)/2
   if A[m]=q then return m
   else if A[m]>q then
        BinarySearch(A, l, m-1, q)
   else BinarySearch(A, m+1, r, q)
```

$T(n) = T(n/2) + 1$

$a=1, b=2, f(n) = 1$

$n^{2log1} = 1$

$1 = \Theta(1)$

=> Case 2: $T(n) = \Theta(log\ n)$

# Examples of Master Method/2

$T(n) = 9T(n/3) + n$

$a=9, b=3, f(n) = n$

$n^{3\log 9} = n^2$

$n = O(n^{3\log 9 - \varepsilon})$ with $\varepsilon = 1$

=> Case 1: $T(n) = \Theta(n^2)$

# Examples of Master Method/3



$T(n) = 3T(n/4) + n \log n$

$a=3, b=4, f(n) = n \log n$

$n^{4\log 3} = n^{0.792}$

$n \log n = \Omega(n^{4\log 3 + \varepsilon})$ *with* $\varepsilon = 0.208$

=> Case 3:

regularity condition: $af(n/b) <= cf(n)$

$af(n/b) = 3(n/4)\log(n/4) <=$

$(3/4)n \log n = cf(n)$ *with* $c=3/4$

$T(n) = \Theta(n \log n)$

# BinarySearchRec1

- Find a number in a sorted array:
  - Trivial if the array contains one element.
  - Else **divide** into two equal halves and **solve** each half.
  - **Combine** the results.

```
INPUT:  A[1..n] - sorted array of integers, q - integer
OUTPUT: index j s.t. A[j] = q, NIL if ∀j(1≤j≤n): A[j] ≠ q
BinarySearchRec1(A, l, r, q):
   if l = r then
     if A[l] = q then return l else return NIL
   m := ⌊(l+r)/2⌋
   ret := BinarySearchRec1(A, l, m, q)
   if ret = NIL then return BinarySearchRec1(A, m+1, r, q)
   else return ret
```

# T(n) of BinarySearchRec1



- Example: BinarySearchRec1

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(1) & \text{if } n > 1 \end{cases}$$

- Solving the recurrence yields

  $T(n) = \Theta(n)$

# BinarySearchRec2

- T(n) = Θ(n) – not better than brute force!
- Better way to **conquer**:
  - Solve only one half!

```
INPUT:  A[1..n] – sorted array of integers, q – integer
OUTPUT: j s.t. A[j] = q, NIL if ∀j(1≤j≤n): A[j] ≠ q
BinarySearchRec2(A, l, r, q):
   if l = r then
      if A[l] = q then return l
      else return NIL
   m := ⌊(l+r)/2⌋
   if A[m] ≤ q then return BinarySearchRec2(A, l, m, q)
   else return BinarySearchRec2(A, m+1, r, q)
```

# T(n) of BinarySearchRec2



- $T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(n/2) + \Theta(1) & \text{if } n > 1 \end{cases}$

- Solving the recurrence yields

  $T(n) = \Theta(\log n)$

# Example: Finding Min and Max



- Given an unsorted array, find a minimum and a maximum element in the array.

```
INPUT: A[l..r] – an unsorted array of integers, l≤r.
OUTPUT: (min,max) s.t. ∀j(l≤j≤r): A[j]≥min and A[j]≤max

MinMax(A, l, r):
   if l = r then return (A[l], A[r]) // Trivial case
   m := ⌊(l+r)/2⌋                    // Divide
   (minl,maxl) := MinMax(A, l, m)    // Conquer
   (minr,maxr) := MinMax(A, m+1, r)  // Conquer
   if minl < minr then min = minl else min = minr // Combine
   if maxl > maxr then max = maxl else max = maxr // Combine
   return (min,max)
```

# Summary

- Divide and conquer
- Merge sort
- Tiling
- Recurrences
  - repeated substitutions
  - substitution
  - master method
- Example recurrences: Binary search

# Next Chapter

- Sorting
  - HeapSort
  - QuickSort