Data Structures and Algorithms

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Chapter 7

Academic Year 2012-2013

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Data Structures and Algorithms
Week 7

1. Dictionaries
2. Hashing
3. Hash Functions
4. Collisions
5. Performance Analysis

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Dictionary

- *Dictionary* – a dynamic data structure with methods:
  - **Search(S, k)** – *an access operation that returns a pointer x to an element where x.key = k*
  - **Insert(S, x)** – *a manipulation operation that adds the element pointed to by x to S*
  - **Delete(S, x)** – *a manipulation operation that removes the element pointed to by x from S*
- An element has a *key* part and a *satellite data* part.

Dictionaries

- Dictionaries store elements so that they can be located quickly using **keys.**
- A dictionary may hold bank accounts.
  - Each account is an object that is identified by an account number.
  - Each account stores a lot of additional information.
  - An application wishing to operate on an account would have to provide the account number as a search **key.**

Dictionaries/2

- If order (methods such as *min*, *max*, *successor*, *predecessor*) is not required, it is enough to check for **equality**.
- Operations that require ordering are still possible, but cannot use the dictionary access structure.
  - Usually all elements must be compared, which is slow.
  - Can be OK if it is rare enough

Dictionaries/3

- Dictionaires can be realized by different data structures
  - arrays
  - linked lists
  - **hash tables**
  - binary trees
  - red/black trees
  - B-trees
- In Java:
  - java.util.Map – interface defining Dictionary ADT

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The Problem

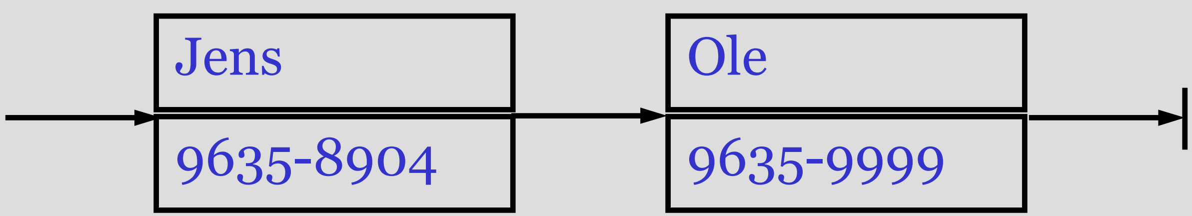
- XY Telecom, a large phone company, wants to provide a caller ID capability:
  - given a phone number,
    return the caller's name
  - phone numbers range from 0 to $r = 10^8 - 1$
  - do this as efficiently as possible

The Problem/2

- Two suboptimal ways to design this dictionary
  - direct addressing: an array indexed by key:
    - Requires O(1) time,
    - Requires O(*r*) space - huge amount of wasted space

(null)	(null)	Jens	(null)	(null)
0000-0000	0000-0001	9635-8904	9635-8905	9999-9999

  - a linked list: requires O(*n*) time, O(*n*) space



Another Solution: Hashing

- We can do better, with a **Hash table** *of size m*.
- Like an array, but with a **function** to map the large range into one which we can manage.
  - e.g., take the original key, modulo the (relatively small) size of the table, and use that as an index
- Insert (9635-8904, Jens) into a hash table with, say, five slots (m = 5)
  - 96358904 mod 5 = 4

(null)	(null)	(null)	(null)	Jens
0	1	2	3	4

- O(1) expected time, O(n+m) space

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Hash Functions

- Need to choose a good hash function (HF)
  - quick to compute
  - distributes keys uniformly throughout the table
- How to deal with hashing non-integer keys:
  - find some way of turning the keys into integers
    - in our example, remove the hyphen in 9635-8904 to get 96358904
    - for a string, add up the ASCII values of the characters of your string (e.g., java.lang.String.hashCode())
  - then use a standard hash function on the integers

HF: Division Method

- Use the remainder: $h(k) = k \bmod m$
  - k is the key, m the size of the table
- Need to choose m
- $m = b^e$ **(bad)**
  - if m is a power of 2, $h(k)$ gives the ***e*** least significant bits of k
  - all keys with the same ending go to the same place
- m prime **(good)**
  - helps ensure uniform distribution
  - primes not too close to exact powers of 2 are best

HF: Division Method/2

- Example 1
  - hash table for n = 2000 character strings, ok to investigate an average of three attempts/search
  - m = 701
    - a prime near 2000/3
    - but not near any power of 2
- Further examples
  - m = 13
    - $h(3) = 3$
    - $h(12) = 12$
    - $h(13) = 0$

HF: Multiplication Method

- Use $h(k) = \lfloor m\ (k\ A \bmod 1)\ \rfloor$
  - k is the key
  - m the size of the table
  - A is a constant $0 < A < 1$
  - (k A mod 1): the fractional part of k A
- The steps involved
  - map $0 \ldots k_{max}$ into $0 \ldots k_{max} A$
  - take the fractional part (mod 1)
  - map it into $0 \ldots m$-1

HF: Multiplication Method/2

- Choice of m and A
  - Value of m is not critical: typically, for some p use $m = 2^p$
  - Optimal choice of A depends on the characteristics of the data
    - Knuth says use $A = \frac{\sqrt{5} - 1}{2} = 0.618033988$

HF: Multiplication Method/3

- Assume 7-bit binary keys, $0 \le k < 128$
- $m = 64 = 2^6$, $p = 6$
- $A = 89/128 = .1011001$, $k = 107 = 1101011$
- Computation of h(k):

```
        .1011001 A
         1101011 k
 1001010.0110011 kA
        .0110011 kA mod 1
  011001.1       m(kA mod 1)
```

- Thus, $h(k) = 25$

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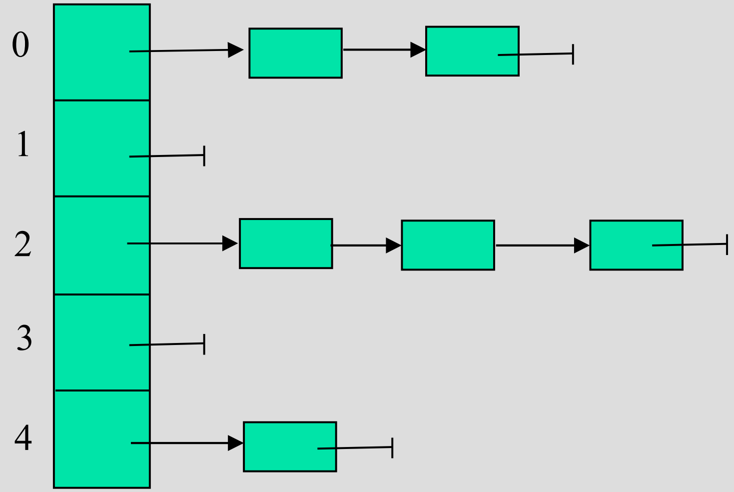
Collisions

- Assume a key is mapped to an already occupied table location
  - what to do?
- Use a **collision handling** technique
- 3 techniques to deal with collisions:
  - chaining
  - open addressing/linear probing
  - open addressing/double hashing

Chaining

- **Chaining** maintains a table of links, indexed by the keys, to **lists** of items with the same key



Open Addressing

- All elements are stored in the hash table (can fill up), i.e., $n \leq m$
- Each table entry contains either an element or null
- When searching for an element, systematically probe table slots
- Modify hash function to take probe number i as second parameter

$$h: U \times \{ 0, 1, \ldots, m-1 \} \rightarrow \{ 0, 1, \ldots, m-1 \}$$

Open Addressing/2

- Hash function, *h*, determines the sequence of slots examined for a given key
- Probe sequence for a given key *k* given by

  $$(h(k,0), h(k,1), \ldots, h(k,m-1))$$

  a permutation of $(0, 1, \ldots, m-1)$

Linear Probing

```
LinearProbingInsert(k)
    if (table is full) error
    probe = h(k)
    while (table[probe] occupied)
        probe = (probe+1) mod m
    table[probe] = k
```

- If the current location is used, try the next table location: $h(key,i) = (h1(key)+i) \bmod m$
- Lookups walk along the table until the key or an empty slot is found
- Uses less memory than chaining
  - one does not have to store all those links
- Slower than chaining
  - one might have to probe the table for a long time

Linear Probing/2

- Problem “primary clustering”:
  long lines of occupied slots
  - A slot preceded by i full slots has a high probability of getting filled: (i+1)/m
- Alternatives: (quadratic probing,) double hashing
- Example:
  - $h(k) = k \bmod 13$
  - insert keys: 18 41 22 44 59 32 31 73

Double Hashing

- Use two hash functions:
  *h(key,i) = (h1(key) + i*h2(key)) mod m*, i=0,1,...

```
DoubleHashingInsert(k)
if (table is full) error
probe = h1(k)
offset = h2(k)
while (table[probe] occupied)
   probe = (probe + offset) mod m
table[probe] = k
```

- Distributes keys much more uniformly than linear probing.

Double Hashing/2

- *h2(k)* must be relative prime to *m* to search the entire hash table
  - Suppose $h2(k) = k*a$ and $m = w*a$, $a > 1$
- Two ways to ensure this:
  - *m* is power of 2, h2(k) is odd
  - *m:* prime, *h2(k):* positive integer < *m*
- Example
  - $h1(k) = k \bmod 13$, $h2(k) = 8 - (k \bmod 8)$
  - insert keys: 18 41 22 44 59 32 31 73

Open addressing: delete

- Complex to delete from
  - A slot may be reached from different points
    - We cannot simply store “NIL”: we'd loose the information necessary to retrieve other keys
  - Possible solution: mark the deleted slot as “deleted”, insert also on “deleted”
    - Drawback: retrieval time no more depending on load factor: potentially lots of “jumps” on “deleted” slots
- When deletion admitted/frequent, chaining preferred

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Analysis of Hashing:

- An element with key k is stored in slot $h(k)$ (instead of slot k without hashing)
- The hash function h maps the universe U of keys into the slots of hash table $T[0...m-1]$

  $h: U \rightarrow \{ 0, 1, ..., m-1 \}$

- Assumption: Each key is equally likely to be hashed into any slot (bucket):

  simple uniform hashing

- Given hash table T with m slots holding n elements, the **load factor** is defined as $\alpha = n/m$

Analysis of Hashing/2

- Assume time to compute $h(k)$ is $\Theta(1)$
- To find an element
  - using h, look up its position in table T
  - search for the element in the linked list of the hashed slot
  - *uniform* hashing yields an average list length $\alpha = n/m$
  - expected number of elements to be examined α
  - search time $O(1+\alpha)$

Analysis of Hashing/3

- Assuming the number of hash table slots is proportional to the number of elements in the table

  $n = O(m)$

  $\alpha = n/m = O(m)/m = O(1)$

  – searching takes constant time on average
  – insertion takes $O(1)$ worst-case time
  – deletion takes $O(1)$ worst-case time (pass the element not key, lists are doubly-linked)

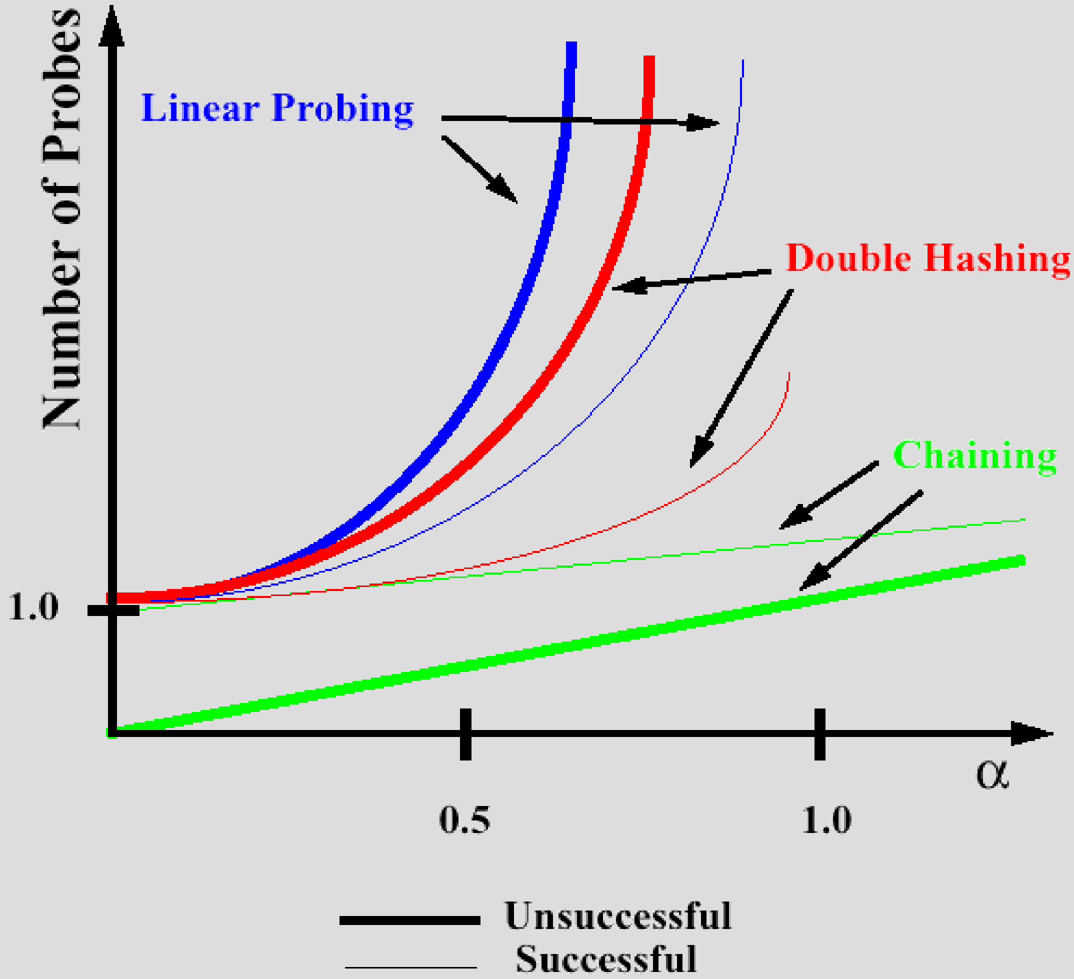
Expected Number of Probes

- Load factor α < 1 for probing
- Analysis of probing uses *uniform hashing* assumption – any permutation is equally likely

	Unsuccessful	**Successful**
Chaining	$O(1+\alpha)$	$O(1+\alpha)$
Probing	$O(\frac{1}{1-\alpha})$	$O(\frac{1}{\alpha}\ln\frac{1}{1-\alpha})$

- Chaining: 1 (α=0%), 1.5 (α=50%), 2 (α=100%), n (α=n)
- Probing, unsucc: 1.25 (α=20%), 2 (α=50%), 5 (α=80%), 10 (α=90%)
- Probing, succ: 0.28 (α=20%), 1.39 (α=50%), 2.01 (α=80%), 2.56 (α=90%)

Expected Number of Probes/2



Summary

- Hashing is very efficient
  (not obvious, probability theory).
- Its functionality is limited (printing elements sorted according to key is not supported).
- The size of the hash table
  may not be easy to determine.
- A hash table is not really
  a dynamic data structure.

Suggested exercises

- Implement a Hash Table with the different techniques
- With paper & pencil, draw the evolution of a hash table when inserting, deleting and searching for new element, with the different techniques
- See also exercises of CLRS

Next Part

- Graphs:
  - Representation in memory
  - Breadth-first search
  - Depth-first search
  - Topological sort