Data Structures and Algorithms

Part 2

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Acknowledgments

- The course follows the book “Introduction to Algorithms‘”, by **Cormen, Leiserson, Rivest and Stein**, MIT Press [CLRST]. Many examples displayed in these slides are taken from their book.
- These slides are based on those developed by Michael Böhlen for this course.

(See http://www.inf.unibz.it/dis/teaching/DSA/)

- The slides also include a number of additions made by Roberto Sebastiani and Kurt Ranalter when they taught later editions of this course

(See http://disi.unitn.it/~rseba/DIDATTICA/dsa2011_BZ//)

DSA, Part 2: Overview

- Complexity of algorithms
- Asymptotic analysis
- Correctness of algorithms
- Special case analysis

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Analysis of Algorithms

- Efficiency:
  - Running time
  - Space used
- Efficiency is defined as a function of the input size:
  - Number of data elements (numbers, points)
  - The number of bits of an input number

The RAM Model

It is important to choose the level of detail.

The RAM (Random Access Machine) model:

- Instructions (each taking constant time) – we usually choose one type of instruction as a **characteristic** operation that is counted:
  - Arithmetic (add, subtract, multiply, etc.)
  - Data movement (assign)
  - Control flow (branch, subroutine call, return)
  - Comparison
- Data types – integers, characters, and floats

Analysis of Insertion Sort

- **Running time** as a function of the **input size** (exact analysis).

	cost	times
for j := 2 **to** *n* **do**	c1	n
key := A[j]	c2	n-1
// Insert A[j] into A[1..j-1]	0	n-1
i := j-1	c3	n-1
while i>0 and A[i]>key **do**	c4	$\sum_{j=2}^{n} t_j$
A[i+1] := A[i]	c5	$\sum_{j=2}^{n} (t_j - 1)$
i--	c6	$\sum_{j=2}^{n} (t_j - 1)$
A[i+1]:= key	c7	n-1

t_j is the number of times the while loop is executed, i.e.,

$(T_j - 1)$ is number of elements in the initial segment greater than A[j]

Analysis of Insertion Sort/2

- The running time of an algorithm for a given input is the sum of the running times of each statement.
- A statement
  - with cost c
  - that is executed n times

  contributes c*n to the running time.
- The total running time T(n) of insertion sort is

$$T(n) = c1*n + c2*(n-1) + c3*(n-1) + c4 * \sum_{j=2}^{n} t_j$$

$$+ c5 \sum_{j=2}^{n} (t_j - 1) + c6 \sum_{j=2}^{n} (t_j - 1) + c7*(n - 1)$$

Analysis of Insertion Sort/3

- The running time is not necessarily equal for every input of size *n*
- The performance on the details of the input (not only length *n*)
- This is modeled by t_j.
- In the case of insertion sort the time t_j depends on the original sorting of the input array

Performance Analysis

- Often it is sufficient to count the number of iterations of the core (innermost) part
  - No distinction between comparisons, assignments, etc (that means roughly the same cost for all of them)
  - Gives precise enough results
- In some cases the cost of selected operations dominates all other costs.
  - Disk I/O versus RAM operations
  - Database systems

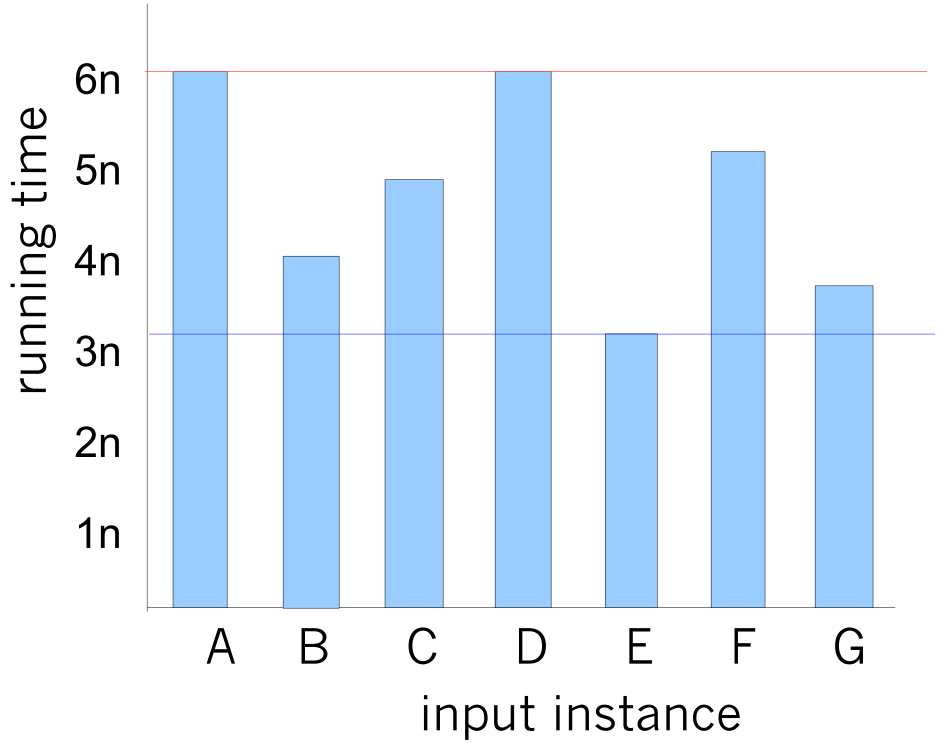
Worst/Average/Best Case

- Analyzing insertion sort's
  - Worst case: elements sorted in inverse order, $t_j=j$, total running time is *quadratic* (time = an^2+bn+c)
  - Average case: $t_j=j/2$, total running time is *quadratic* (time = an^2+bn+c)
  - Best case: elements already sorted, $t_j=1$, innermost loop is zero, total running time is *linear* (time = $an+b$)
- How can we define these concepts formally?

  … and how much sense does "Best case" make?

Worst/Average/Best Case/2

For a specific size of input size *n*, investigate running times for different input instances:



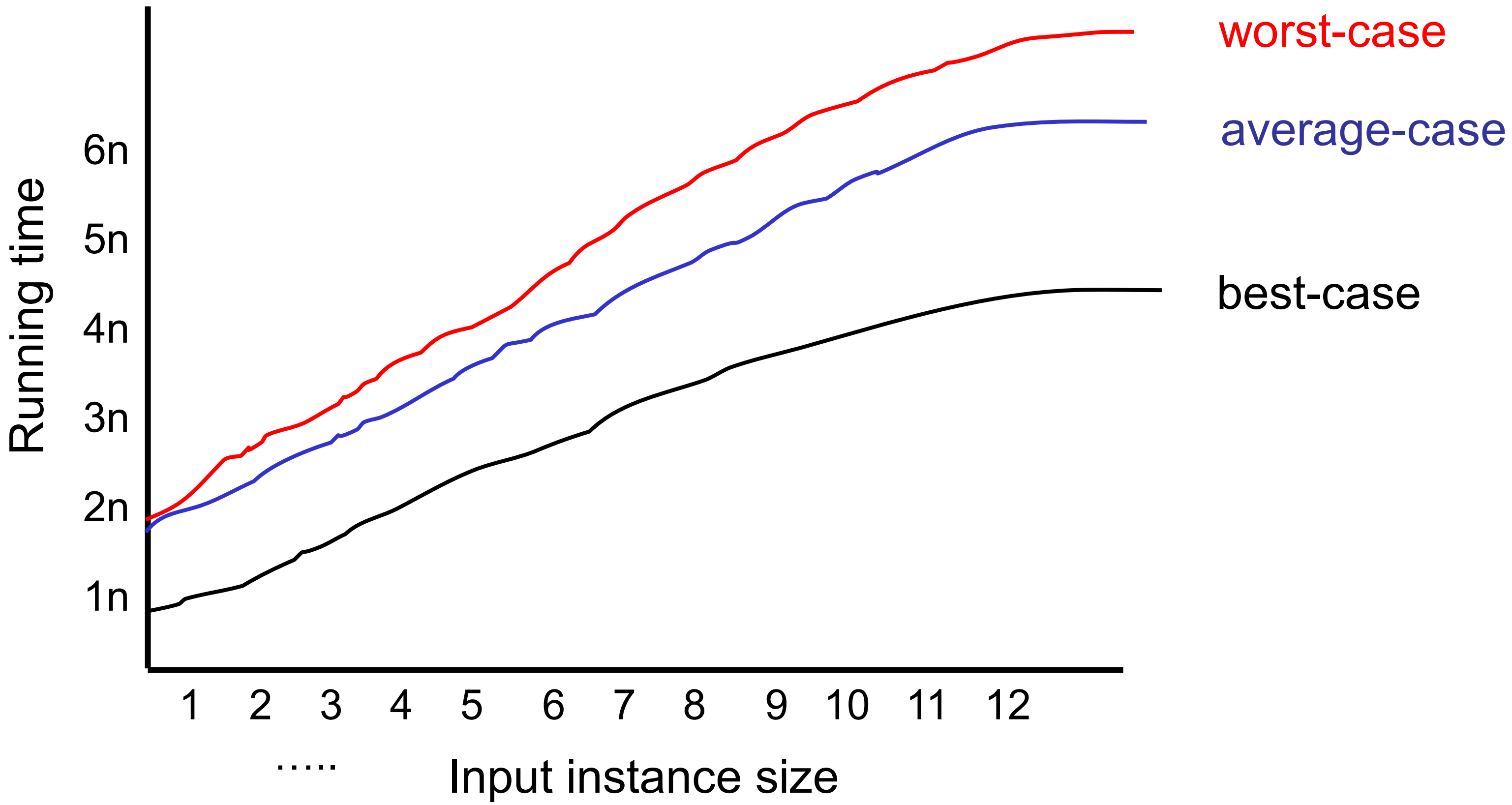
worst case = 4 3 2 1

average case ???

best case = 1 2 3 4

Worst/Average/Best Case/3

For inputs of all sizes:



Best/Worst/Average Case/4

Worst case is most often used:

- It is an upper-bound
- In certain application domains (e.g., air traffic control, surgery) knowing the worst-case time complexity is of crucial importance.
- For some algorithms worst case occurs fairly often
- The average case is often as bad as the worst case

The average case depends on assumptions

- What are the possible input cases?
- What is the probability of each input?

Analysis of Linear Search

```
INPUT: A[1..n] – a sorted array of integers,
         q – an integer.
OUTPUT: j s.t. A[j]=q. NIL if ∀j(1≤j≤n): A[j]≠q

j := 1
while j <= n and A[j] != q do j++
if j <= n then return j
else return NIL
```

- Worst case running time: *n*
- Average case running time: *n/2* (if *q* is present)

… under which assumption?

Binary Search

- Idea: Left and right bounds l, r.
  Elements to the right of r are bigger than search element, ...
- In each step, the range of the search space is cut in half

***INPUT*: A[1..n] – sorted (increasing) array of integers, *q* – integer.**
OUTPUT: an index j such that $A[j] = q$. *NIL*, if $\forall j\ (1 \le j \le n)$: $A[j] \neq q$

```
l := 1; r := n
do
 m := ⌊(l+r)/2⌋
      if A[m] = q then return m
      else if A[m] > q then r := m-1
      else l := m+1
while l <= r
return NIL
```

Analysis of Binary Search

How many times is the loop executed?

- With each execution
  the difference between `l` and `r` is cut in half
  - Initially the difference is *n*
  - *The loop stops when the difference becomes 0 (less than 1)*
- How many times do you have to cut *n* in half to get 0?
- *log n* – better than the brute-force approach of linear search (*n*).

Linear vs Binary Search

- Costs of linear search: n
- Costs of binary search: log(n)
- Should we care?
- Phone book with *n* entries:
  - n = 200’000,   log n = log 200’000 = ??
  - n = 2M,   log 2M = ??
  - n = 20M,   log 20M = ??

Suggested Exercises

- Implement binary search in 3 versions:
  - as in previous slides
  - without return statements inside the loop
  - Recursive
- As before, returning nil if q<a[l] or q>a[r] (trace the different executions)
- Implement a function printSubArray printing only the subarray from l to r, leaving blanks for the others
  - use it to trace the behaviour of binary search

DSA, Part 2: Overview

- Complexity of algorithms
- Asymptotic analysis
- Correctness of algorithms
- Special case analysis

Asymptotic Analysis

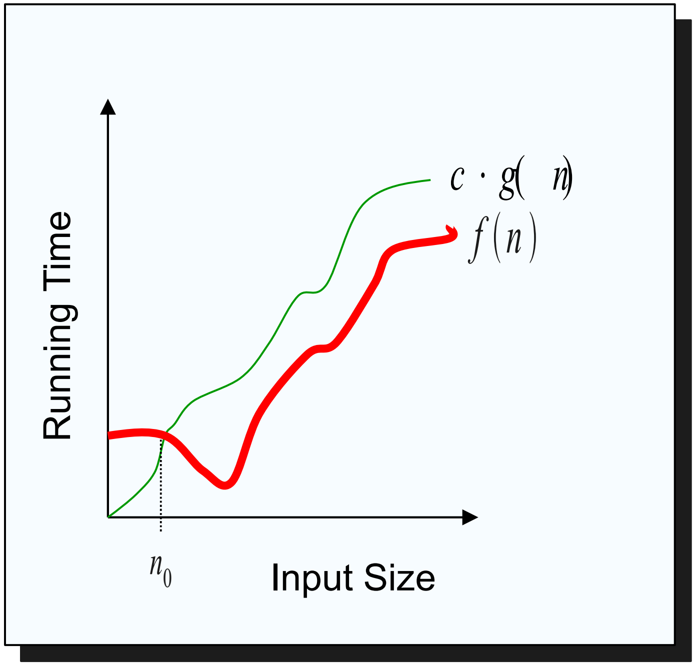
- Goal: simplify the analysis of the running time by getting rid of details, which are affected by specific implementation and hardware
  - "rounding" of numbers: 1,000,001 $\approx$ 1,000,000
  - "rounding" of functions: $3n^2 \approx n^2$
- Capturing the essence: how the running time of an algorithm increases with the size of the input *in the limit*
  - Asymptotically more efficient algorithms are best for all but small inputs

Asymptotic Notation

The “big-Oh” *O*-Notation

- talks about asymptotic upper bounds
- $f(n) = O(g(n))$ iff there exist $c > 0$ and $n_0 > 0$,

  s.t. $f(n) \leq c\, g(n)$ for $n \geq n_0$
- $f(n)$ and $g(n)$ are functions over non-negative integers



Used for *worst-case* analysis

Asymptotic Notation

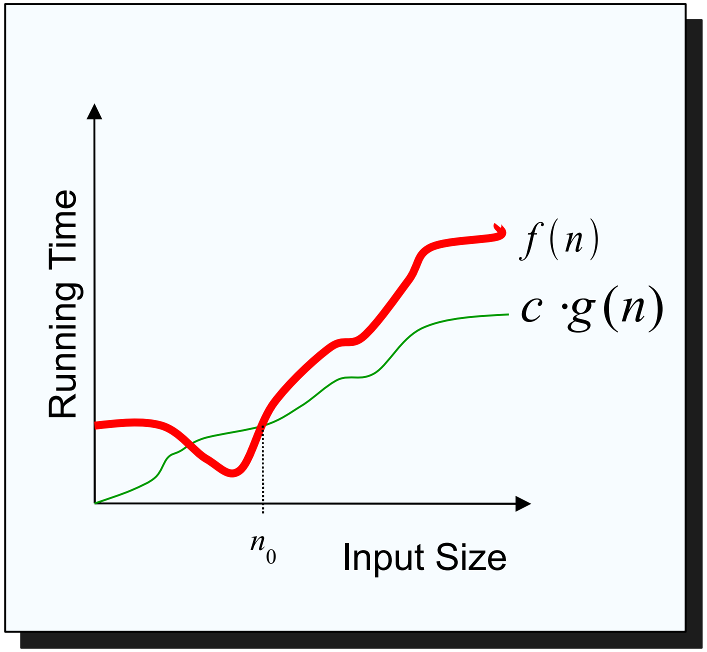
- Simple Rule: We can always drop lower order terms and constant factors, without changing big Oh:
  - $50\, n \log n$ is $O(n \log n)$
  - $7n - 3$ is $O(n)$
  - $8n^2 \log n + 5n^2 + n$ is $O(n^2 \log n)$

- Note:
  - $50\, n \log n$ is $O(n^2)$
  - $50\, n \log n$ is $O(n^{100})$

  but this is less informative than saying
  - $50\, n \log n$ is $O(n \log n)$

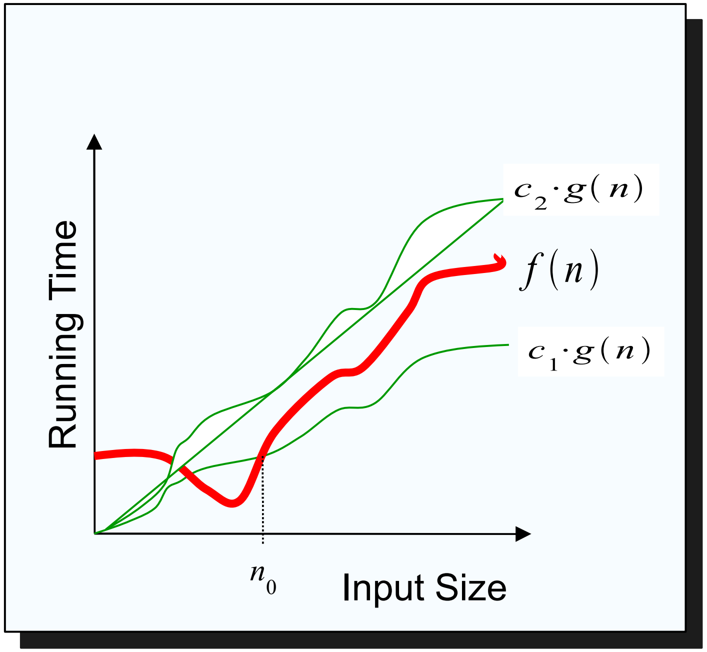
Asymptotic Notation/3

- The "big-Omega" Ω-Notation
  - asymptotic lower bound
  - $f(n) = \Omega(g(n))$ iff
    there exist $c > 0$ and $n_0 > 0$,
    s.t. $c\ g(n) \leq f(n)$, for $n \geq n_0$
- Used to describe lower bounds of algorithmic problems
  - E.g., searching in an unsorted array with search2 is $\Omega(n)$, with search1 it is $\Omega(\log n)$



Asymptotic Notation/4

- The "big-Theta" Θ-Notation
  - asymptoticly tight bound
  - $f(n) = \Theta(g(n))$ if there exists $c_1>0$, $c_2>0$, and $n_0>0$, *s.t.* for $n \geq n_0$
    $c_1\, g(n) \leq f(n) \leq c_2\, g(n)$



- $f(n) = \Theta(g(n))$ iff
  $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
- *Note:* $O(f(n))$ is often used
  when $\Theta(f(n))$ *is meant*

Two More Asymptotic Notations

$f(n)$ is *much* smaller than $g(n)$

- "Little-Oh" notation $f(n)=o(g(n))$
  non-tight analogue of Big-Oh
  - For every $c > 0$, there exists $n_0>0$, s.t.

    $$f(n) < c\, g(n)$$

    for $n \geq n_0$
  - If $f(n) = o(g(n))$, it is said that *$g(n)$ dominates $f(n)$*

$f(n)$ is *much* bigger than $g(n)$

- "Little-omega" notation $f(n)=\omega(g(n))$
  non-tight analogue of Big-Omega

Asymptotic Notation/6

- Analogy with real numbers
  - $f(n) = O(g(n)) \quad \cong f \leq g$
  - $f(n) = \Omega(g(n)) \quad \cong f \geq g$
  - $f(n) = \Theta(g(n)) \quad \cong f = g$
  - $f(n) = o(g(n)) \quad \cong f < g$
  - $f(n) = \omega(g(n)) \quad \cong f > g$

- Abuse of notation:

  $f(n) = O(g(n))$ actually means

  $$f(n) \in O(g(n))$$

Comparison of Running Times

Determining the maximal problem size

Running Time $T(n)$ in μs	1 second	1 minute	1 hour
$400n$	2500	150’000	9’000’000
$20n \log n$	4096	166’666	7’826’087
$2n^2$	707	5477	42’426
n^4	31	88	244
2^n	19	25	31

DSA, Part 2: Overview

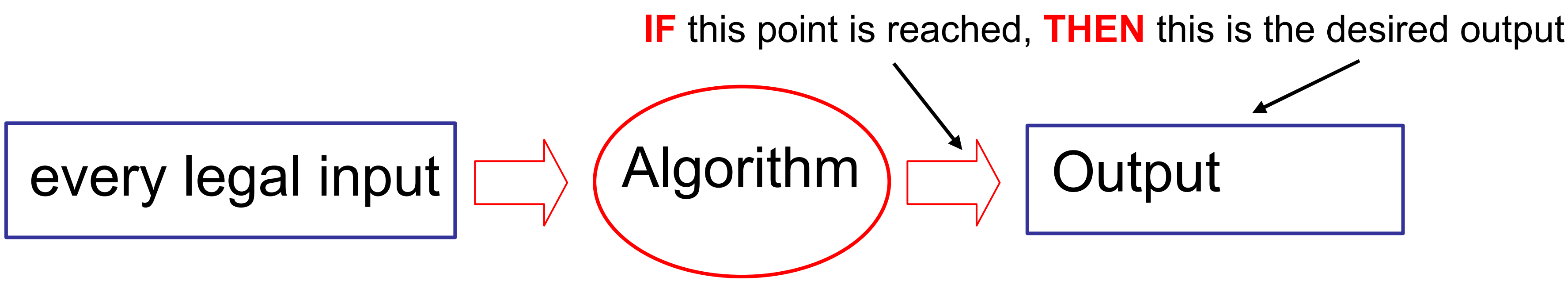
- Complexity of algorithms
- Asymptotic analysis
- Correctness of algorithms
- Special case analysis

Correctness of Algorithms

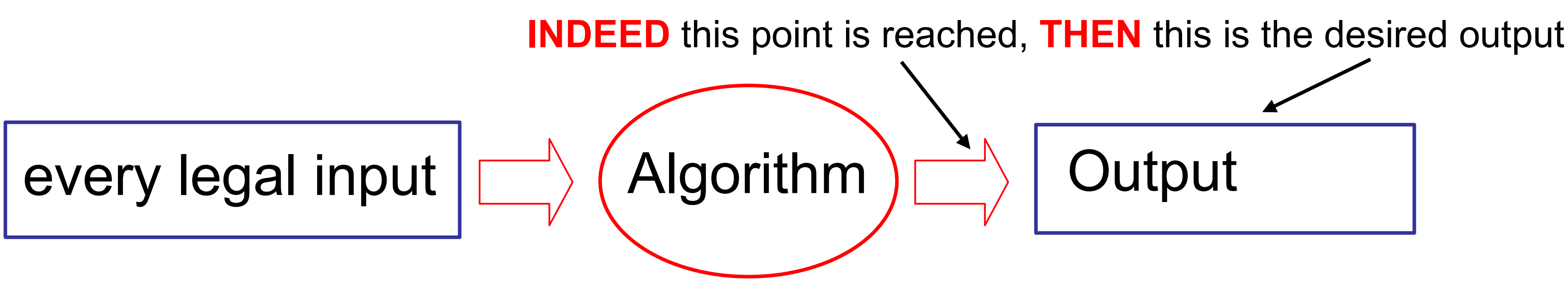
- An algorithm is *correct* if for every legal input, it terminates and produces the desired output.
- Automatic proof of correctness is not possible.
- There are practical techniques and rigorous formalisms that help to reason about the correctness of (parts of) algorithms.

Partial and Total Correctness

- Partial correctness



- Total correctness



Assertions

- To prove partial correctness we associate a number of **assertions** (statements about the state of the execution) with specific checkpoints in the algorithm.
  - E.g., *A*[1], …, *A*[j] form an increasing sequence
- **Preconditions** – assertions that must be valid *before* the execution of an algorithm or a subroutine (INPUT)
- **Postconditions** – assertions that must be valid *after* the execution of an algorithm or a subroutine (OUTPUT)

Loop Invariants

- **Invariants:** assertions that are valid every time they are reached (many times during the execution of an algorithm, e.g., in loops)
- We must show three things about loop invariants:
  - **Initialization:** it is true prior to the first iteration.
  - **Maintenance:** *if* it is true before an iteration, *then* it is true after the iteration.
  - **Termination:** when a loop terminates the invariant gives a useful property to show the correctness of the algorithm

Example: Binary Search/1

```
l := 1; r := n;
do
   m := ⌊(l+r)/2⌋
   if A[m]=q then return m
   else if A[m]>q then r := m-1
   else l := m+1
while l <= r
return NIL
```

- We want to show that q is not in A if NIL is returned.
- **Invariant:**
  $\forall i \in [1..l-1]: A[i]<q$ (ia)
  $\forall i \in [r+1..n]: A[i]>q$ (ib)
- **Initialization**: $l = 1, r = n$
  the invariant holds because
  there are no elements to the left of *l* or to the right of *r*.

  l = 1 yields $\forall i \in [1..0]: A[i]<q$
  this holds because [1..0] is empty

  r = n yields $\forall i \in [n+1..n]: A[i]>q$
  this holds because [n+1..n] is empty

Example: Binary Search/2

- **Invariant:**
  $\forall i \in [1..l-1]: A[i] < q$ (ia)
  $\forall i \in [r+1..n]: A[i] > q$ (ib)

```
l := 1; r := n;
do
  m := ⌊(l+r)/2⌋
  if A[m]=q then return m
  else if A[m]>q then r := m-1
  else l := m+1
while l <= r
return NIL
```

- **Maintenance**: $1 \leq l$, $r \leq n$, $m = \lfloor (l+r)/2 \rfloor$
  - $A[m]$!= q & $q < A[m]$ implies $r = m-1$
    A sorted implies $\forall k \in [r+1..n]: A[k] > q$ (ib)
  - $A[m]$!= q & $A[m] < q$ implies $l = m+1$
    A sorted implies $\forall k \in [1..l-1]: A[k] < q$ (ia)

Example: Binary Search/3

- **Invariant:**
  $\forall i \in [1..l-1]: A[i] < q$ (ia)
  $\forall i \in [r+1..n]: A[i] > q$ (ib)

```
l := 1; r := n;
do
  m := ⌊(l+r)/2⌋
  if A[m]=q then return m
  else if A[m]>q then r := m-1
  else l := m+1
while l <= r
return NIL
```

- **Termination**: $1 \le l,\ r \le n,\ l \le r$

  Two cases:

  $l := m+1$ implies $l_{new} = \lfloor (l+r)/2 \rfloor + 1 > l_{old}$

  $r := m-1$ implies $r_{new} = \lfloor (l+r)/2 \rfloor - 1 < r_{old}$

- The range gets smaller during each iteration and the loop will terminate when $l \le r$ no longer holds

Example: Insertion Sort/1

Loop invariants:

External “for” loop

A[1..j-1] is sorted

A[1..j-1] $\in$ A^{orig}[1..j-1]

```
for j := 2 to n do
  key := A[j]
  i := j-1
  while i>0 and A[i]>key do
    A[i+1] := A[i]
    i--
  A[i+1] := key
```

Internal “while” loop

- A[1...i], key, A[i+1…j-1]
- A[1..i] is sorted
- A[i+1..j-1] is sorted
- A[1..i] is sorted
- A[k] > key   for all k in {i+1,…,j-1}

Example: Insertion Sort/2

External for loop:

(i) A[1...j-1] is sorted

(ii) A[1...j-1] $\in$ A^{orig}[1..j-1]

Internal while loop:

- A[1...i], key, A[i+1…j-1]
- A[1..i] is sorted
- A[i+1..j-1] is sorted
- A[1..i] is sorted
- A[k] > key for all k in {i+1,...,j-1}

```
for j := 2 to n do
  key := A[j]
  i := j-1
  while i>0 and A[i]>key do
    A[i+1] := A[i]
    i--
  A[i+1] := key
```

Initialization:

(i), (ii) j = 2: A[1..1] $\in$ A^{orig}[1..1] and is trivially sorted

i=j-1: A[1...j-1], key, A[j...j-1] where key=A[j]

A[j...j-1] is empty (and thus trivially sorted)

A[1...j-1] is sorted (invariant of outer loop)

Example: Insertion Sort/3

External for loop:

(i) A[1...j-1] is sorted

(ii) A[1...j-1] $\in$ A^{orig}[1..j-1]

Internal while loop:

- A[1...i], key, A[i+1…j-1]
- A[1..i] is sorted
- A[i+1..j-1] is sorted
- A[1..i] is sorted
- A[k] > key   for all k in {i+1,...,j-1}

```
for j := 2 to n do
  key := A[j]
  i := j-1
  while i>0 and A[i]>key do
    A[i+1] := A[i]
    i--
  A[i+1] := key
```

Maintenance: A to A'

- (A[1...j-1] sorted) and (insert A[j]) implies (A[1...j] sorted)
- A[1...i-1], key, A[i,i+1...j-1] satisfies conditions because of condition A[i] > key and A[1...j-1] being sorted.

Example: Insertion Sort/4

External for loop:

(i) A[1...j-1] is sorted

(ii) A[1...j-1] $\in$ A^{orig}[1..j-1]

Internal while loop:

- A[1...i], key, A[i+1…j-1]
- A[1..i] is sorted
- A[i+1..j-1] is sorted
- A[1..i] is sorted
- A[k] > key for all k in {i+1,...,j-1}

```
for j := 2 to n do
  key := A[j]
  i := j-1
  while i>0 and A[i]>key do
    A[i+1] := A[i]
    i--
  A[i+1] := key
```

Termination:

- main loop, j=n+1: A[1...n] sorted.
- A[i]$\leq$key: (A[1...i], key, A[i+1...j-1]) = A[1...j-1] is sorted
- i=0: (key, A[1...j-1]) = A[1...j] is sorted.

Exercise

- Apply the same approach to prove the correctness of bubble sort.

Special Case Analysis

- Consider extreme cases and make sure your solution works in all cases.
- The problem: identify special cases.
- This is related to INPUT and OUTPUT specifications.

Special Cases

- empty data structure (array, file, list, …)
- single element data structure
- completely filled data structure
- entering a function
- termination of a function
- zero, empty string
- negative number
- border of domain
- start of loop
- end of loop
- first iteration of loop

Sortedness

The following algorithm checks whether an array is sorted.

```
INPUT: A[1..n] – an array of integers.
OUTPUT: TRUE if A is sorted; FALSE otherwise

for i := 1 to n
  if A[i] ≥ A[i+1] then return FALSE
return TRUE
```

Analyze the algorithm by considering special cases.

Sortedness/2

```
INPUT: A[1..n] – an array of integers.
OUTPUT: TRUE if A is sorted; FALSE otherwise

for i := 1 to n
  if A[i] ≥ A[i+1] then return FALSE
return TRUE
```

- Start of loop, i=1 ➔ OK
- End of loop, i=n ➔ ERROR (tries to access A[n+1])

Sortedness/3

```
INPUT: A[1..n] – an array of integers.
OUTPUT: TRUE if A is sorted; FALSE otherwise

for i := 1 to n-1
  if A[i] ≥ A[i+1] then return FALSE
return TRUE
```

- Start of loop, i=1 ➡ OK
- End of loop, i=n-1 ➡ OK
- A=[1,1,1] ➡ First iteration, from i=1 to i=2 ➡ OK
- A=[1,1,1] ➡ ERROR (if A[i]=A[i+1] for some i)

Sortedness/3

```
INPUT: A[1..n] – an array of integers.
OUTPUT: TRUE if A is sorted; FALSE otherwise

for i := 1 to n
  if A[i] ≥ A[i+1] then return FALSE
return TRUE
```

- Start of loop, i=1 ➔ OK
- End of loop, i=n-1 ➔ OK
- First iteration, from i=1 to i=2 ➔ OK
- A=[1,1,1] ➔ ERROR (if A[i]=A[i+1] for some i)

Sortedness/4

```
INPUT: A[1..n] – an array of integers.
OUTPUT: TRUE if A is sorted; FALSE otherwise

for i := 1 to n
  if A[i] > A[i+1] then return FALSE
return TRUE
```

- Start of loop, i=1 ➔ OK
- End of loop, i=n-1 ➔ OK
- First iteration, from i=1 to i=2 ➔ OK
- A=[1,1,1] ➔ OK
- Empty data structure, n=0 ➔ ? (for loop)
- A=[-1,0,1,-3] ➔ OK

Binary Search, Variant 1

Analyze the following algorithm
by considering special cases.

```
l := 1; r := n
do
  m := ⌊(l+r)/2⌋
  if A[m] = q then return m
  else if A[m] > q then r := m-1
  else l := m+1
while l < r
return NIL
```

Binary Search, Variant 1

```
l := 1; r := n
do
  m := ⌊(l+r)/2⌋
  if A[m] = q then return m
  else if A[m] > q then r := m-1
  else l := m+1
while l < r
return NIL
```

- Start of loop ➔ OK
- End of loop, l=r ➔ Error! Example: search 3 in [3 5 7]

Binary Search, Variant 1

```
l := 1; r := n
do
  m := ⌊(l+r)/2⌋
  if A[m] = q then return m
  else if A[m] > q then r := m-1
  else l := m+1
while l <= r
return NIL
```

- Start of loop ➔ OK
- End of loop, l=r ➡ OK
- First iteration ➡ OK
- A=[1,1,1] ➡ OK
- Empty data structure, n=0 ➡ Error! Tries to access A[0]
- One-element data structure, n=1 ➡ OK

Binary Search, Variant 1

```
l := 1; r := n
If r < 1 then return NIL;
do
  m := ⌊(l+r)/2⌋
  if A[m] = q then return m
  else if A[m] > q then r := m-1
  else l := m+1
while l <= r
return NIL
```

- Start of loop ➔ OK
- End of loop, l=r ➡ OK
- First iteration ➡ OK
- A=[1,1,1] ➡ OK
- Empty data structure, n=0 ➡ OK
- One-element data structure, n=1 ➡ OK

Binary Search, Variant 2

Analyze the following algorithm
by considering special cases

```
l := 1; r := n
while l < r do
  m := ⌊(l+r)/2⌋
  if A[m] <= q
    then l := m+1 else r := m
if A[l-1] = q
  then return q else return NIL
```

Binary Search, Variant 3

Analyze the following algorithm by considering special cases

```
l := 1; r := n
while l <= r do
  m := ⌊(l+r)/2⌋
  if A[m] <= q
    then l := m+1 else r := m
if A[l-1] = q
  then return q else return NIL
```

Insertion Sort, Slight Variant

- Analyze the following algorithm by considering special cases
- Hint: beware of lazy evaluations

```
INPUT:  A[1..n] – an array of integers
OUTPUT: permutation of A s.t.
        A[1]≤A[2] ≤… ≤A[n]

for j := 2 to n do
  key := A[j]; i := j-1;
  while A[i] > key and i > 0 do
    A[i+1] := A[i];  i--;
  A[i+1] := key
```

Merge

Analyze the following algorithm
by considering special cases.

```
INPUT: A[1..n1], B[1..n2] sorted arrays of
integers
OUTPUT: permutation C of A.B s.t.
  C[1] ≤ C[2] ≤...≤ C[n1+n2]

i:=1;j:=1;
for k:=1 to n1 + n2 do
      if A[i] <= B[j]
      then C[k] := A[i]; i++;
      else C[k] := B[j]; j++;
return C;
```

Merge/2

```
INPUT: A[1..n1], B[1..n2] sorted arrays of
integers
OUTPUT: permutation C of A.B s.t.
  C[1] ≤ C[2] ≤...≤ C[n1+n2]

i:=1;j:=1;
for k:= 1 to n1 + n2 do
      if j > n2 or (i <= n1 and A[i]<=B[j])
      then C[k] := A[i]; i++;
      else C[k] := B[j]; j++;
return C;
```

Math Refresher

- Arithmetic progression

$$\sum_{i=0}^{n} i = 0 + 1 + \ldots + n = n(n+1)/2$$

- Geometric progression (for a number a ≠ 1)

$$\sum_{i=0}^{n} a^{i} = 1 + a^{2} + \ldots + a^{n} = (1 - a^{n+1})/(1 - a)$$

Induction Principle

We want to show that property P is true for all integers $n \geq n0$.

Basis: prove that P is true for $n0$.

Inductive step: prove that if P is true for all k such that $n0 \leq k \leq n - 1$ then P is also true for n.

Exercise: Prove that every Fibonacci number of the form fib(3n) is even

Summary

- Algorithmic complexity
- Asymptotic analysis
  - Big O and Theta notation
  - Growth of functions and asymptotic notation
- Correctness of algorithms
  - Pre/Post conditions
  - Invariants
- Special case analysis