Data Structures and Algorithms

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Part 10

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Data Structures and Algorithms
Week 10

1. Weighted Graphs
2. Minimum Spanning Trees
   - Greedy Choice Theorem
   - Kruskal’s algorithm
   - Prim’s algorithm
3. Shortest Paths
   - Dijkstra’s algorithm
   - Bellman-Ford's algorithm

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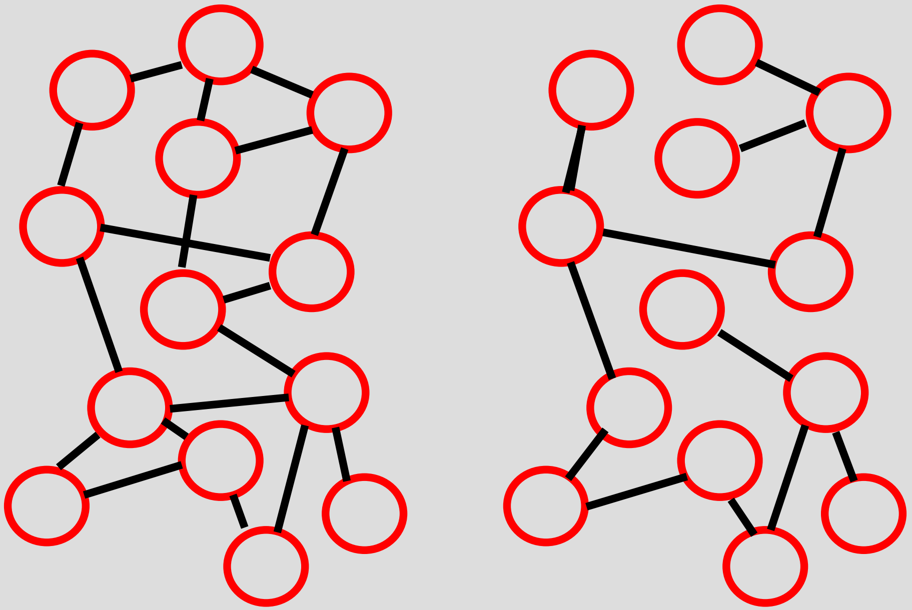
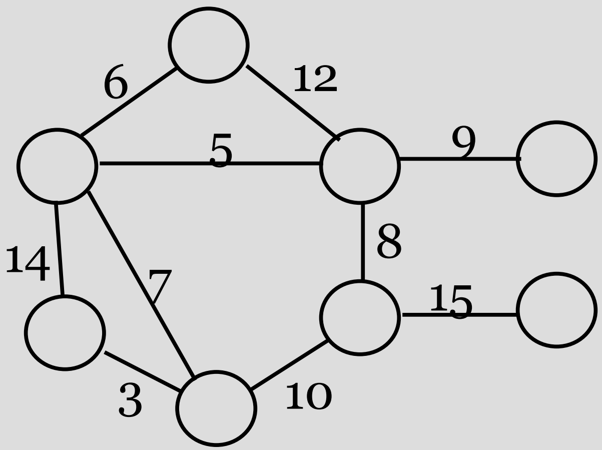
Spanning Tree

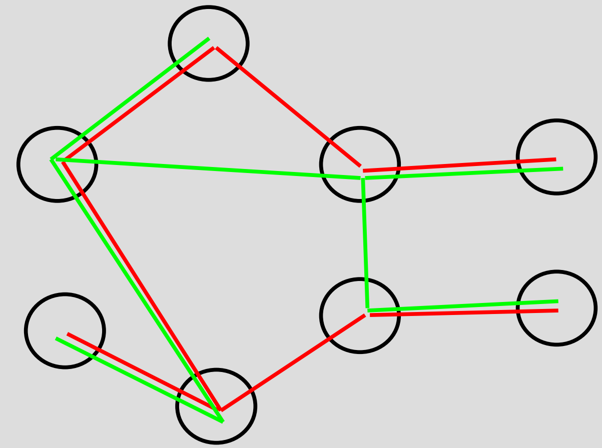
- A **spanning tree** of **G** is a subgraph which
  - contains all vertices of **G**
  - is a tree
- How many edges are there in a spanning tree, if there are *V* vertices?



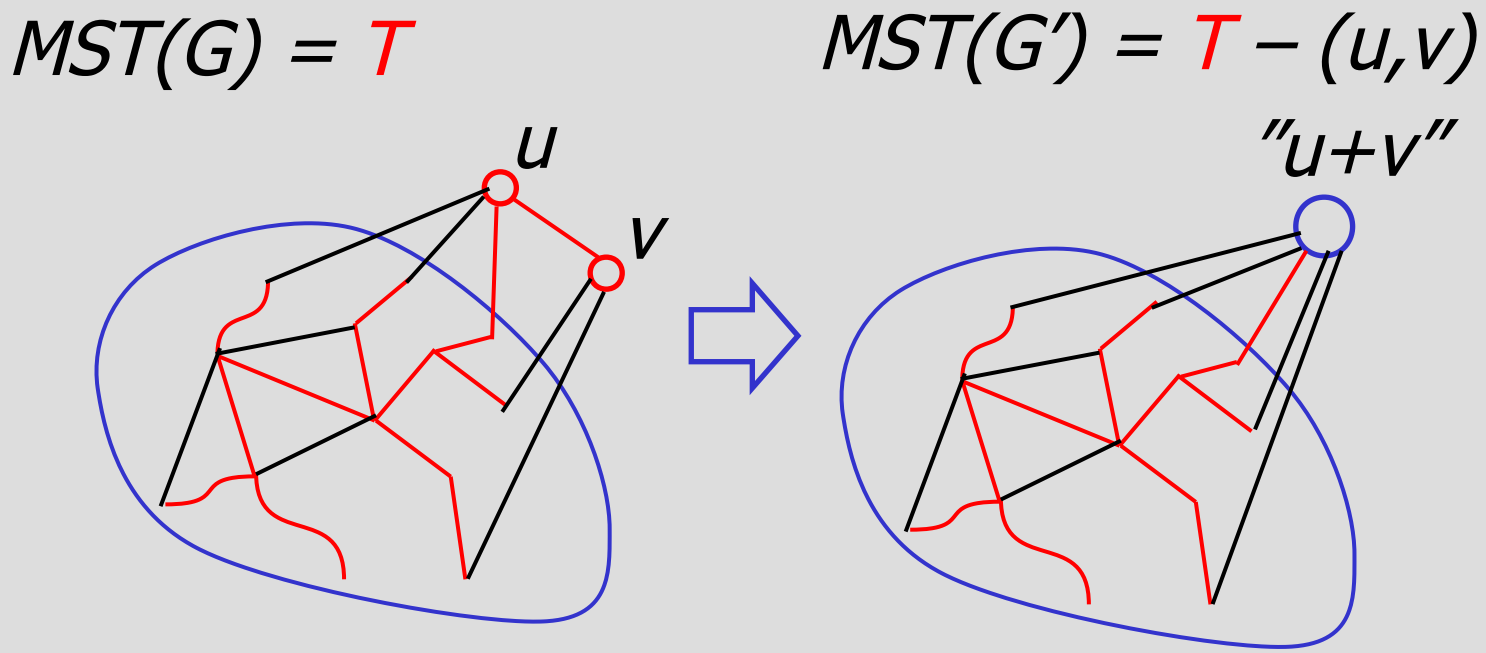
Minimum Spanning Trees

- Undirected, connected graph $G = (V,E)$
- **Weight** function $W: E \rightarrow R$ (assigning cost or length or other values to edges)
- Spanning tree: tree that connects all vertexes
- **Minimum spanning tree** (MST): spanning tree T that minimizes $w(T) = \sum_{(u,v) \in T} w(u,v)$





Optimal Substructure



- Rationale:
  - If *G'* had a cheaper ST *T'*, then we would get a cheaper ST of *G*: *T'* + (*u*,*v*)

Idea for an Algorithm

- We have to make $V-1$ choices (edges of the MST) to arrive at the optimization goal
- After each choice we have a sub-problem that is one vertex smaller than the original problem.
  - A dynamic programming algorithm would consider all possible choices (edges) at each vertex.
  - Goal: at each vertex cheaply determine an edge that definitely belongs to an MST

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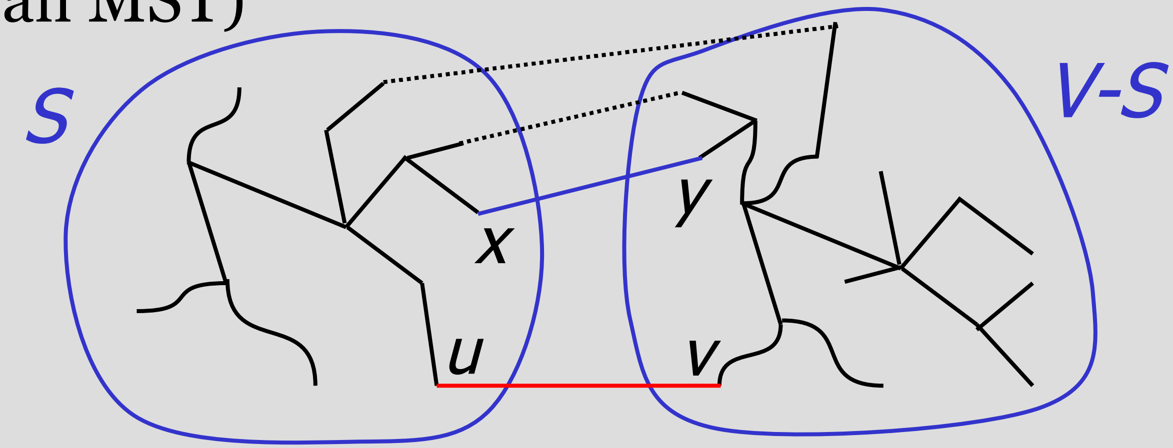
Greedy Choice

- Greedy choice property: locally optimal (greedy) choice yields a globally optimal solution.
- Theorem
  - Let $G=(V, E)$ and $S \subseteq V$
  - S is a **cut** of G (it splits G into parts S and V-S)
  - (u,v) is a **light** edge if it is a *min*-weight edge of G that connects S and $V-S$
  - Then (u,v) belongs to *a* MST T of G

Greedy Choice/2

- Proof
  - Suppose *(u,v)* is light but (*u*,*v*) ∉ any MST
  - look at path from *u* to *v* in some MST *T*
  - Let (*x*, *y*) be the first edge on a path from *u* to *v* in *T* that crosses from *S* to *V*–*S*. Swap (*x*, *y*) with (*u*,*v*) in *T*.
  - this improves cost of *T* ➔ contradiction (*T* is supposed to be an MST)



Generic MST Algorithm

```
Generic-MST(G, w)
1 A := ∅  // Contains edges that belong to a MST
2 while (A does not form a spanning tree) do
3    Find an edge (u,v) that is safe for A
4    A := A ∪{(u,v)}
5 return A
```

A safe edge is an edge that does not destroy *A*'s property.

```
MoreSpecific-MST(G, w)
1    A := ∅  // Contains edges that belong to a MST
2    while A does not form a spanning tree do
3.1     Make a cut (S, V-S) of G that respects A
3.2     Take the min-weight edge (u,v) connecting S to V-S
4       A := A ∪{(u,v)}
5 return A
```

Prim-Jarnik Algorithm

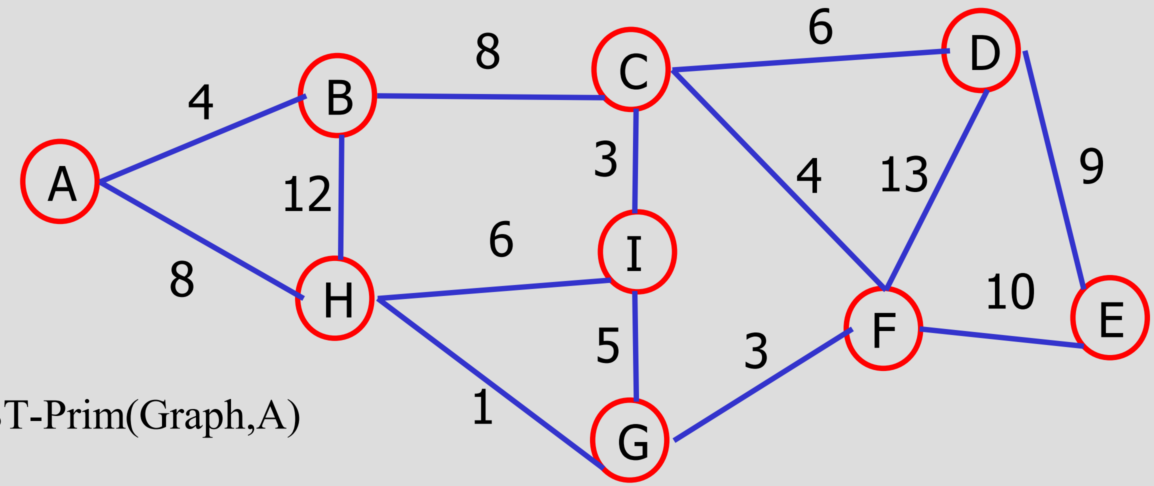
- Vertex-based algorithm
- Grows a single MST T one vertex at a time
- The set A covers the portion of T that was already computed
- Annotate all vertices $\boldsymbol{v}$ outside of the set A with $\boldsymbol{v}$.**key**, the current minimum weight of an edge that connects v to a vertex in A ($\boldsymbol{v}$.**key** $= \infty$ if no such edge exists)

Prim-Jarnik Algorithm/2

```
MST-Prim(G,s)
for each vertex u ∈ G.V
  u.key := ∞
  u.pred := NIL
s.key := 0
init(Q, G.V) // Q is a priority queue
while not isEmpty(Q)
  u := extractMin(Q) // add u to T
  for each v ∈ u.adj do
    if v ∈ Q and w(u,v) < v.key then
      v.key := w(u,v)
      modifyKey(Q,v)
      v.pred := u
```

updating keys

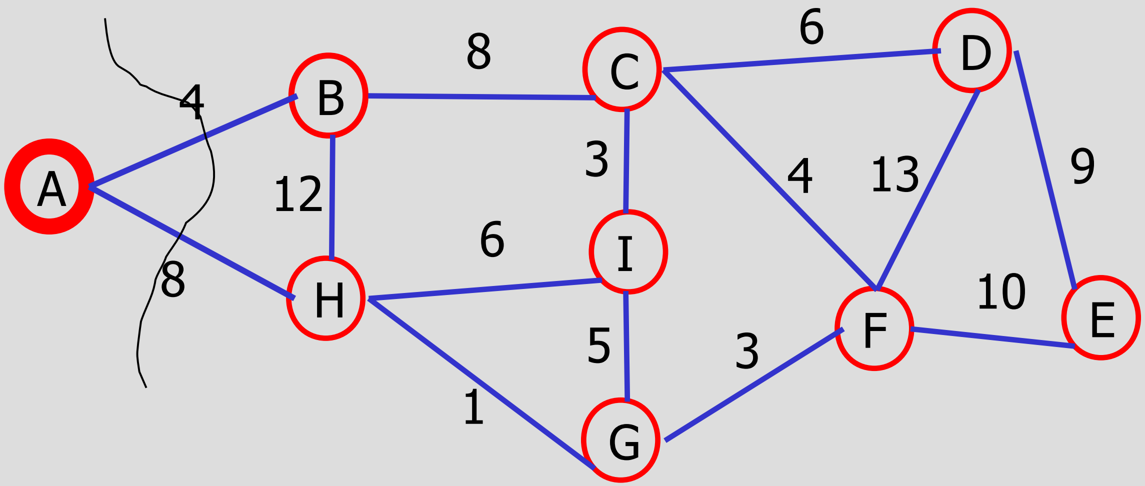
Prim-Jarnik Example



MST-Prim(Graph,A)

A = {}
Q = A-NIL/0, B-NIL/∞, C-NIL/∞, D-NIL/∞, E-NIL/∞, F-NIL/∞, G-NIL/∞, H-NIL/∞, I-NIL/∞

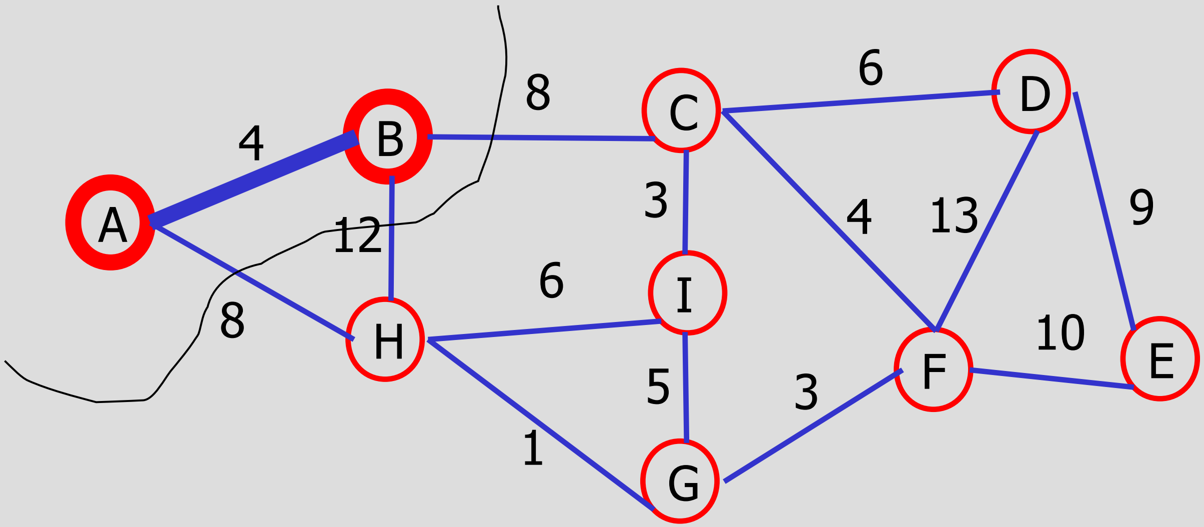
Prim-Jarnik Example/2



A = A-NIL/0

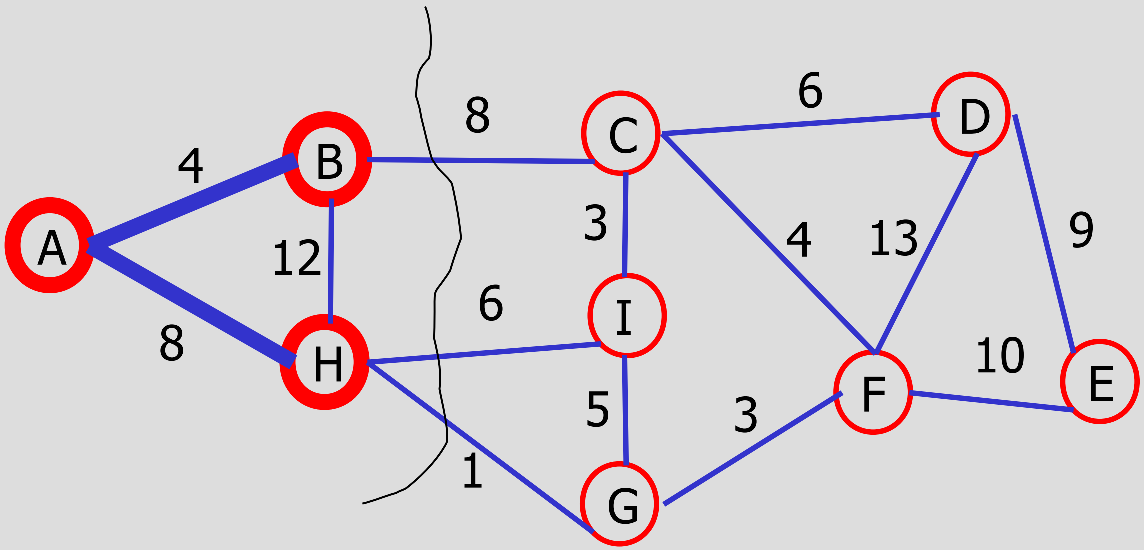
Q = B-A/4, H-A/8, C-NIL/∞, D-NIL/∞, E-NIL/∞, F-NIL/∞, G-NIL/∞, I-NIL/∞

Prim-Jarnik Example/3



A = A-NIL/0, B-A/4
Q = H-A/8, C-B/8, D-NIL/∞, E-NIL/∞,
F-NIL/∞, G-NIL/∞, I-NIL/∞

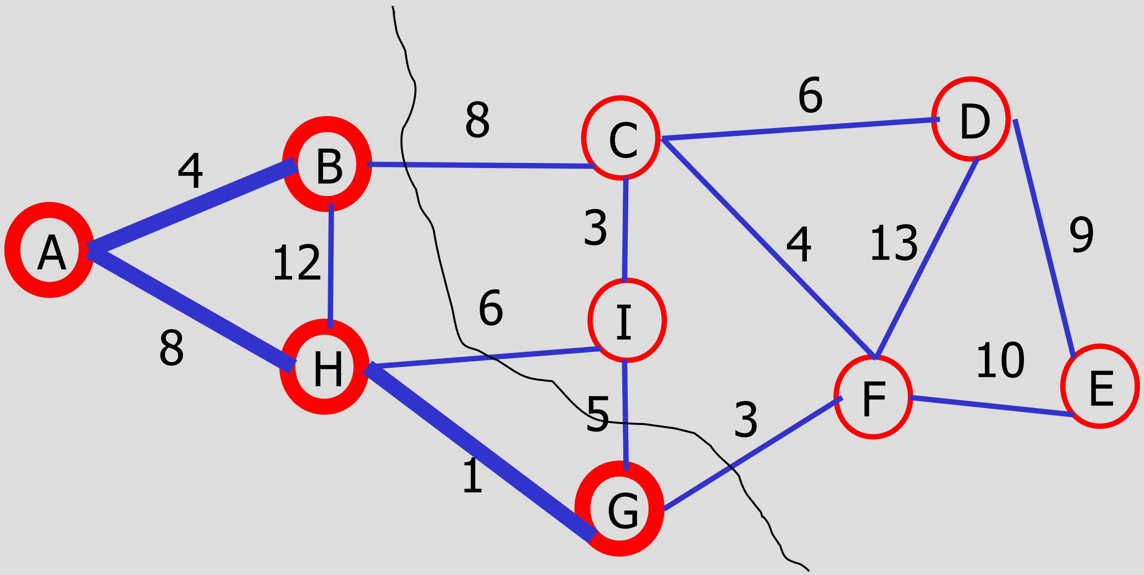
Prim-Jarnik Example/4



A = A-NIL/0, B-A/4, H-A/8

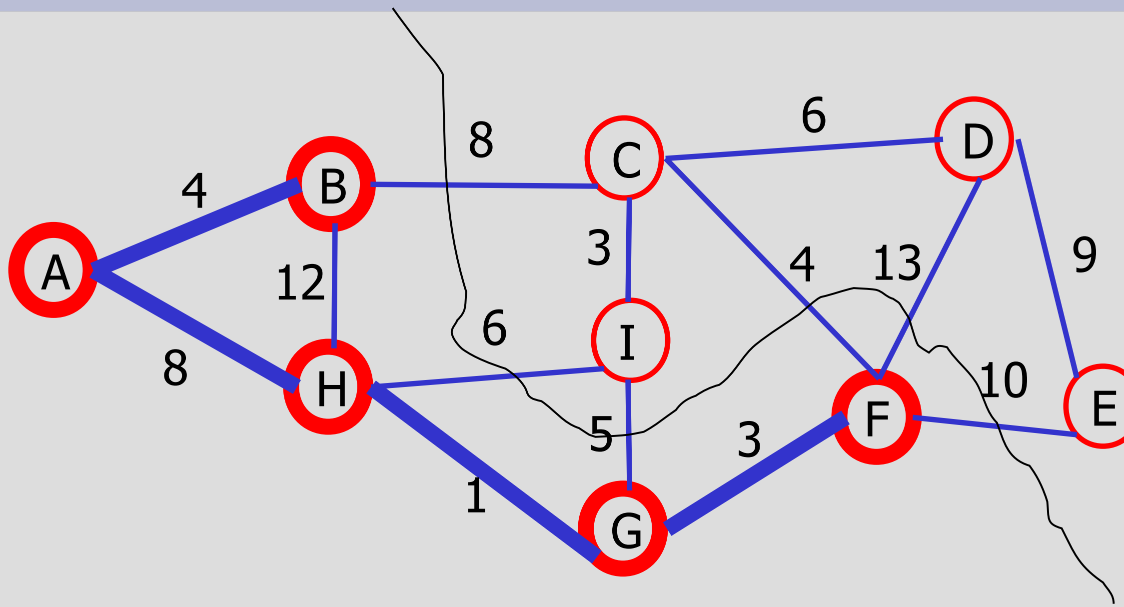
Q = G-H/1, I-H/6, C-B/8, D-NIL/∞, E-NIL/∞, F-NIL/∞

Prim-Jarnik Example/5



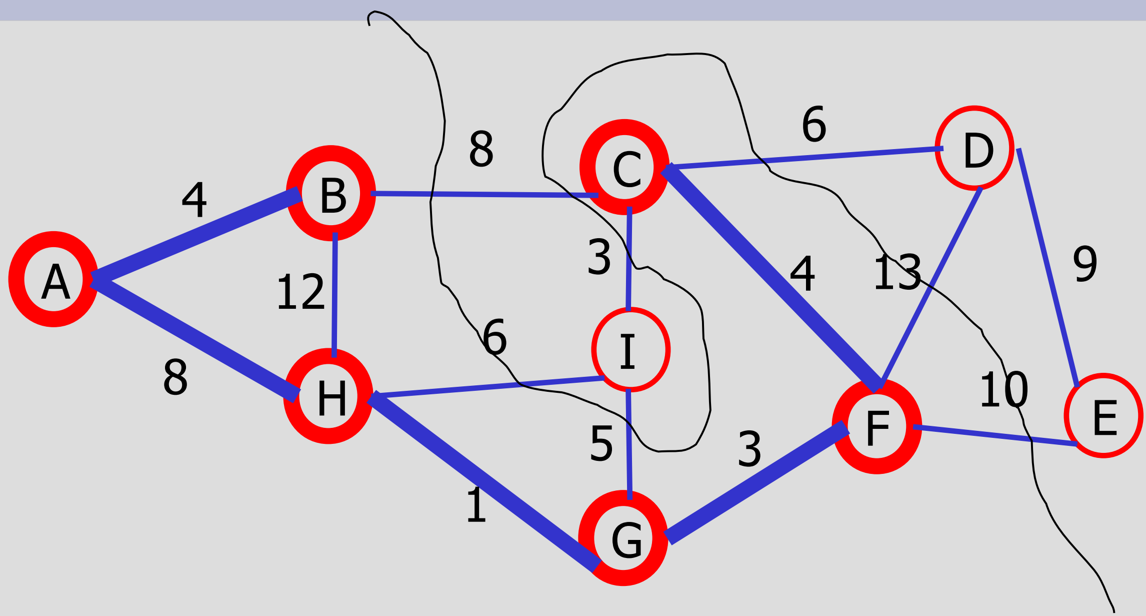
A = A-NIL/0, B-A/4, H-A/8, G-H/1
Q = F-G/3, I-G/5, C-B/8, D-NIL/∞, E-NIL/∞

Prim-Jarnik Example/6



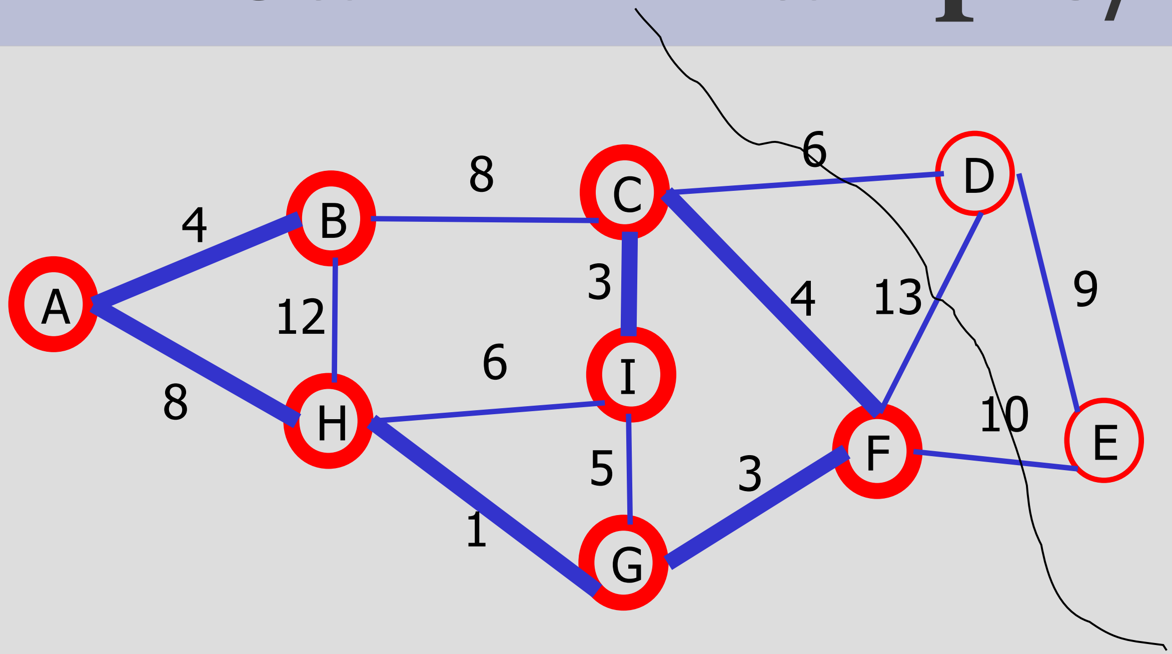
A = A-NIL/0, B-A/4, H-A/8, G-H/1, F-G/3
Q = C-F/4, I-G/5, E-F/10, D-F/13

Prim-Jarnik Example/7



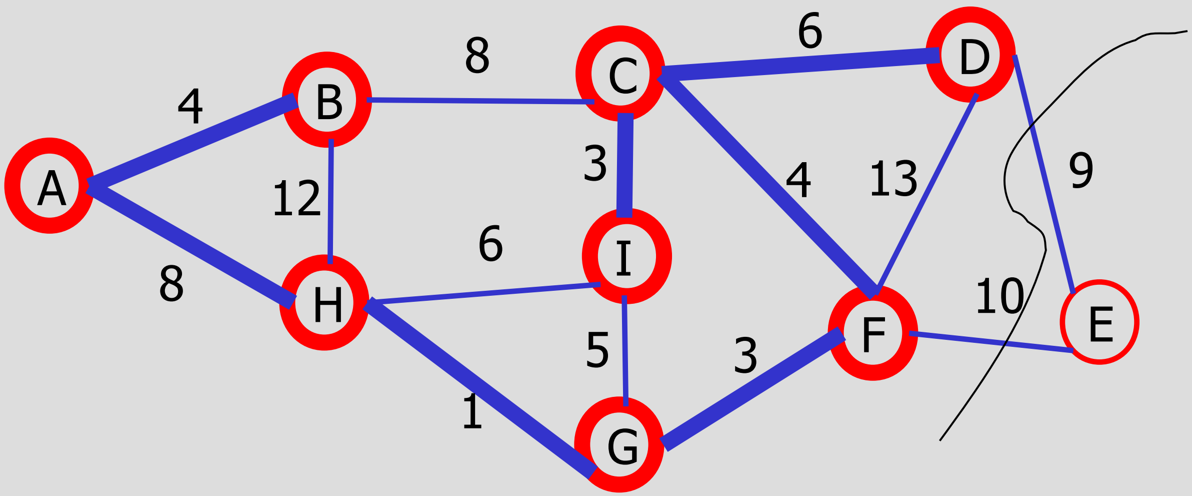
A = A-NIL/0, B-A/4, H-A/8, G-H/1, F-G/3, C-F/4
Q = I-C/3, D-C/6, E-F/10

Prim-Jarnik Example/8



A = A-NIL/0, B-A/4, H-A/8, G-H/1, F-G/3, C-F/4, I-C/3
Q = D-C/6, E-F/10

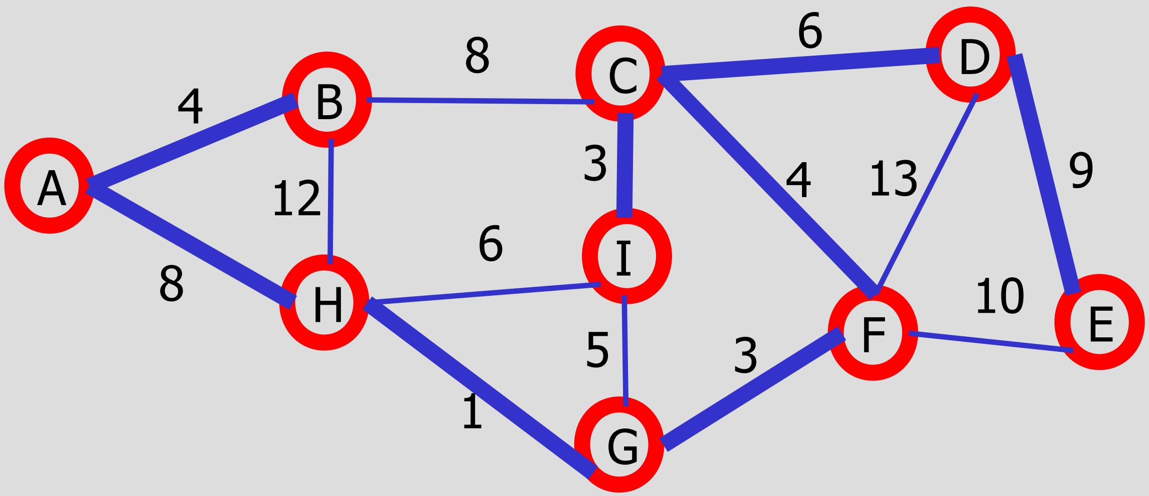
Prim-Jarnik Example/9



A = A-NIL/0, B-A/4, H-A/8, G-H/1, F-G/3, C-F/4, I-C/3, D-C/6

Q = E-D/9

Prim-Jarnik Example/10



A = A-NIL/0, B-A/4, H-A/8, G-H/1, F-G/3, C-F/4,
I-C/3, D-C/6, E-D/9
Q = {}

Implementation Issues

```
MST-Prim(G,r)

for u ∈ G.V do u.key := ∞; u.pred := NIL
r.key := 0
init(Q, G.V) // Q is a min-priority queue
while not isEmpty(Q) do
  u := extractMin(Q) // add u to T
  for v ∈ u.adj do
    if v ∈ Q and w(u,v) < v.key then
      v.key := w(u,v)
      modifyKey(Q,v)
      v.pred := u
```

Priority Queues

- A priority queue maintains a set S of elements, each with an associated key value.
- We need PQ to support the following operations
  - **init**(Q:PriorityQueue, *S*:*VertexSet*)
  - **extractMin**(Q:PriorityQueue): *Vertex*
  - **modifyKey**(Q:PriorityQueue, *v*:*Vertex*)
- To choose how to implement a PQ, we need to count how many times the operations are performed:
  - **init** is performed just once and runs in $O(V)$

Prim-Jarnik Running Time

- Time = $|V| * T(\text{extractMin}) + O(E) * T(\text{modifyKey})$

Q	T(extractMin)	T(modifyKey)	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\log V)$	$O(\log V)$	$O(E \log V)$

- $E \geq V-1$, $E < V^2$, $E = O(V^2)$
- Binary heap implementation:
  - Time = $O(V \log V + E \log V) = O(V^2 \log V) = O(E \log V)$

About Greedy Algorithms

- Greedy algorithms make a locally optimal choice (cheapest path, etc).
- In general, a locally optimal choice does not give a globally optimal solution.
- **Greedy** algorithms can be used to solve optimization problems, if:
  - There is an *optimal substructure*
  - We can prove that a *greedy choice* at each iteration leads to an optimal solution.

Kruskal's Algorithm

- Edge based algorithm
- Add edges one at a time in increasing weight order.
- The algorithm maintains *A*: a **forest of trees**. An edge is accepted it if connects vertices of distinct trees (the cut respects A).

Disjoint Sets

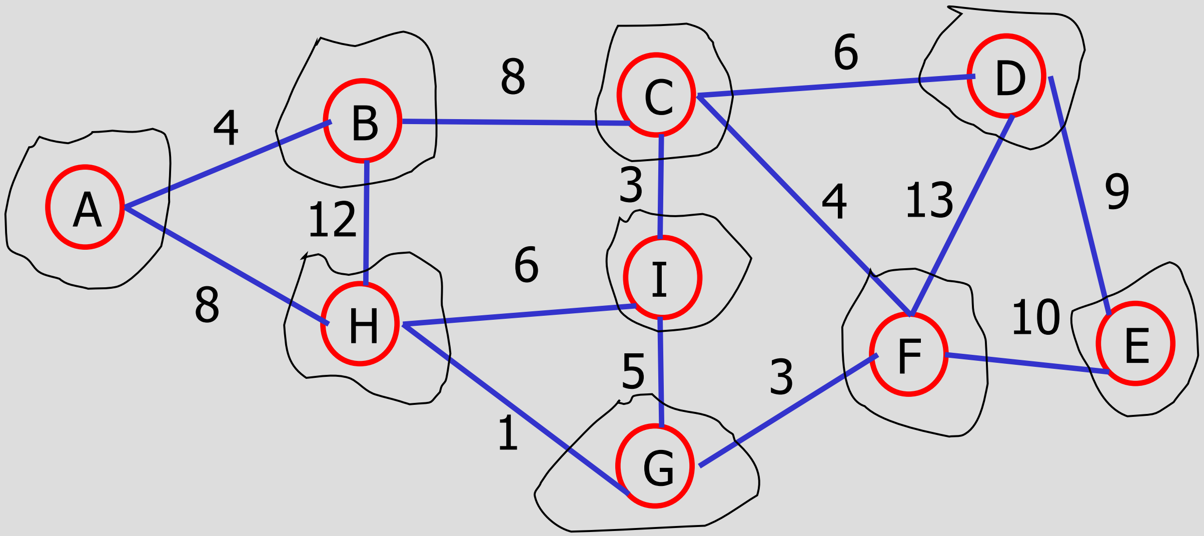
- We need to maintain a disjoint partitioning of a set, i.e., a collection S of disjoint sets.
  Operations:
  - **addSingletonSet**(S:Set, *x*:*Vertex*)
    - $S := S \cup \{\{x\}\}$
  - **findSet**(S:Set, *x*:*Vertex*): *Set*
    - returns $X \in S$ *such that* $x \in \mathrm{X}$
  - **unionSets**(S:Set, *x*:*Vertex*, *y*:*Vertex*)
    - $X :=$ **findSet**(S:Set, *x*)
    - $Y :=$ **findSet**(S:Set, y)
    - $S := (S - \{X, Y\}) \cup \{X \cup Y\}$

Kruskal's Algorithm/2

- The algorithm keeps adding the cheapest edge that connects two trees of the forest

```
MST-Kruskal(G)
A := ∅
init(S) // Init disjoint-set
for v ∈ G.V do addSingletonSet(S,v)
sort edges of G.E by non-decreasing w(u,v)
for (u,v) ∈ G.E in sorted order do
  if findSet(S,u) ≠ findSet(S,v) then
     A := A ∪ {(u,v)}
     unionSets(S,u,v)
return A
```

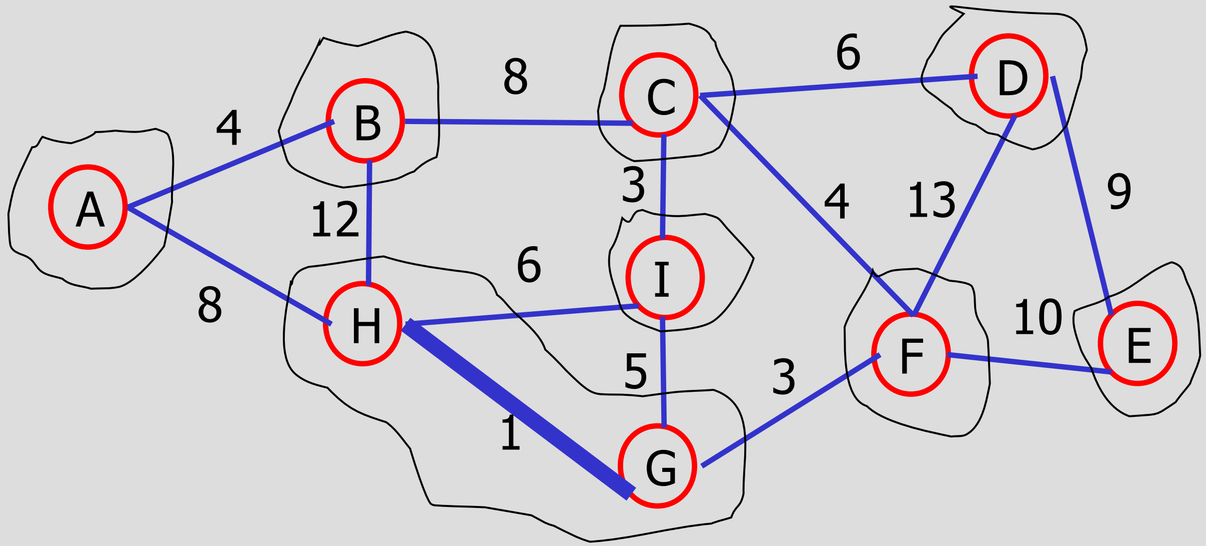
Kruskal Example/1



S = { A B C D E F G H I }

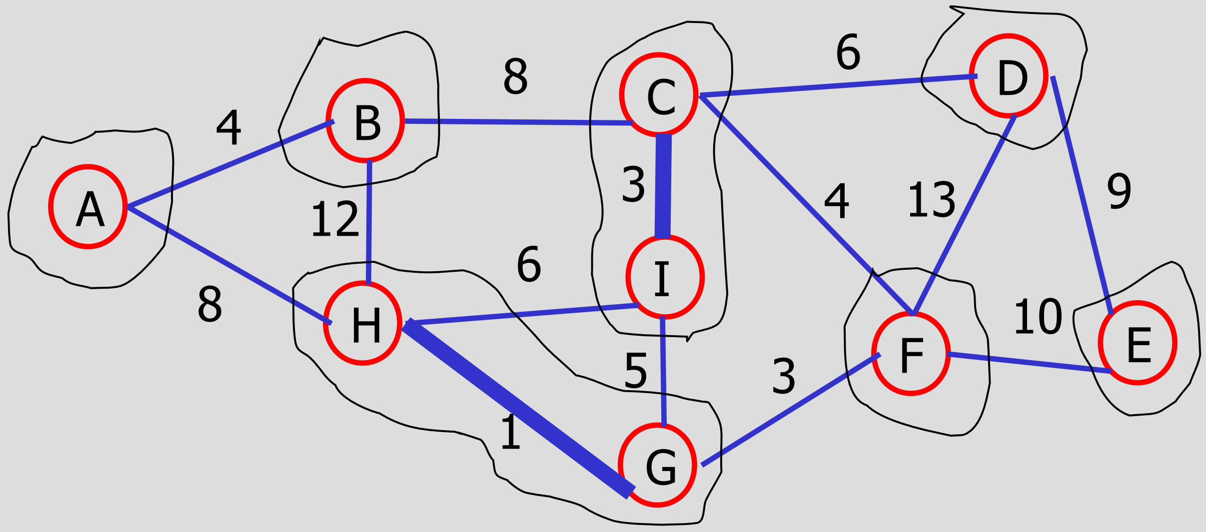
E' = { HG CI GF CF AB HI CD BC AH DE EF BH DF }

Kruskal Example/2



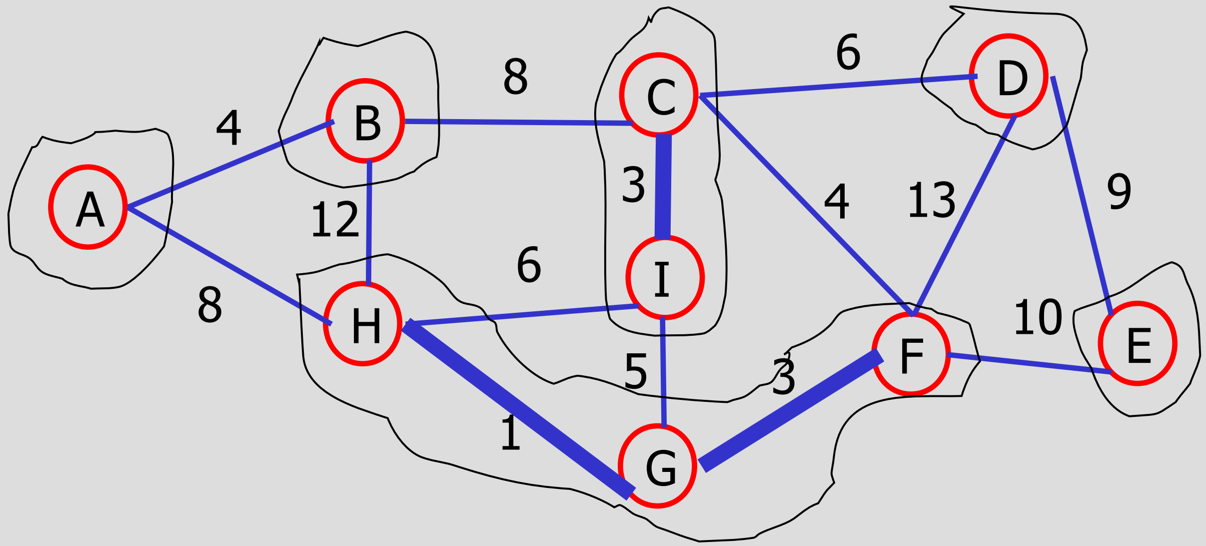
S = { A B C D E F GH I }
E' = { CI GF CF AB HI CD BC AH DE EF BH DF }

Kruskal Example/3



S = { A B CI D E F GH }
E' = { GF CF AB HI CD BC AH DE EF BH DF }

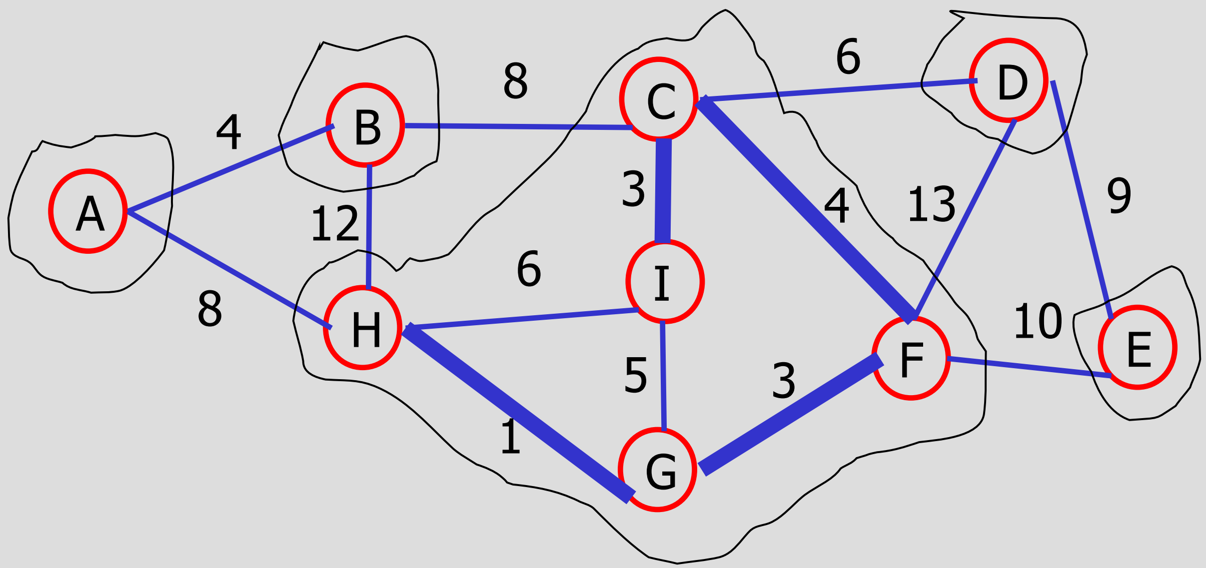
Kruskal Example/4



S = { A B CI D E FGH }

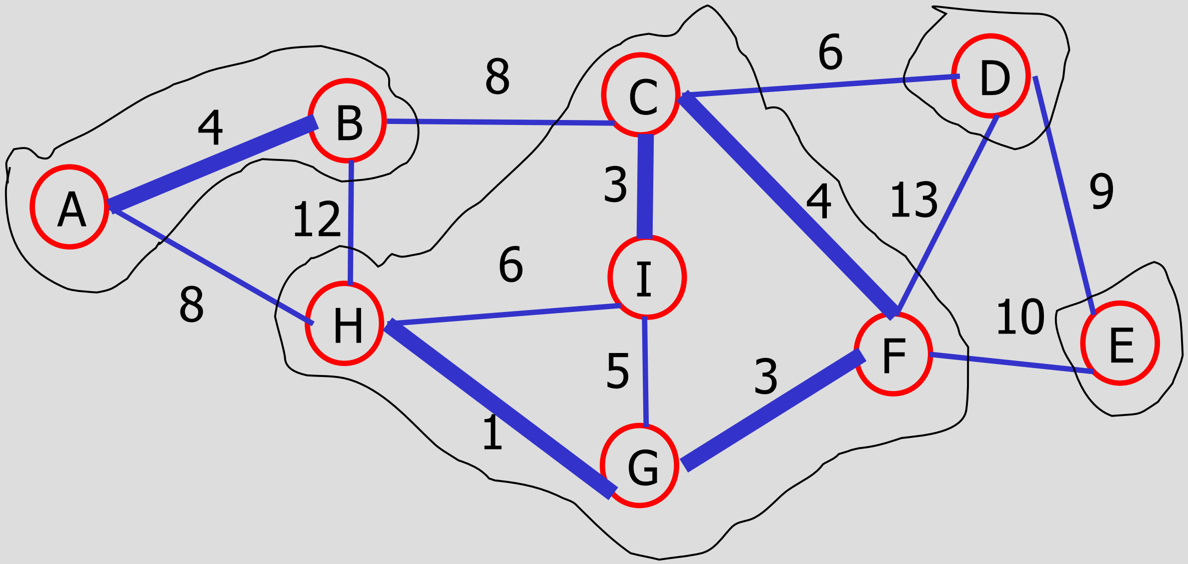
E' = { CF AB HI CD BC AH DE EF BH DF }

Kruskal Example/5



S = { A B CFGHI D E }
E' = { AB HI CD BC AH DE EF BH DF }

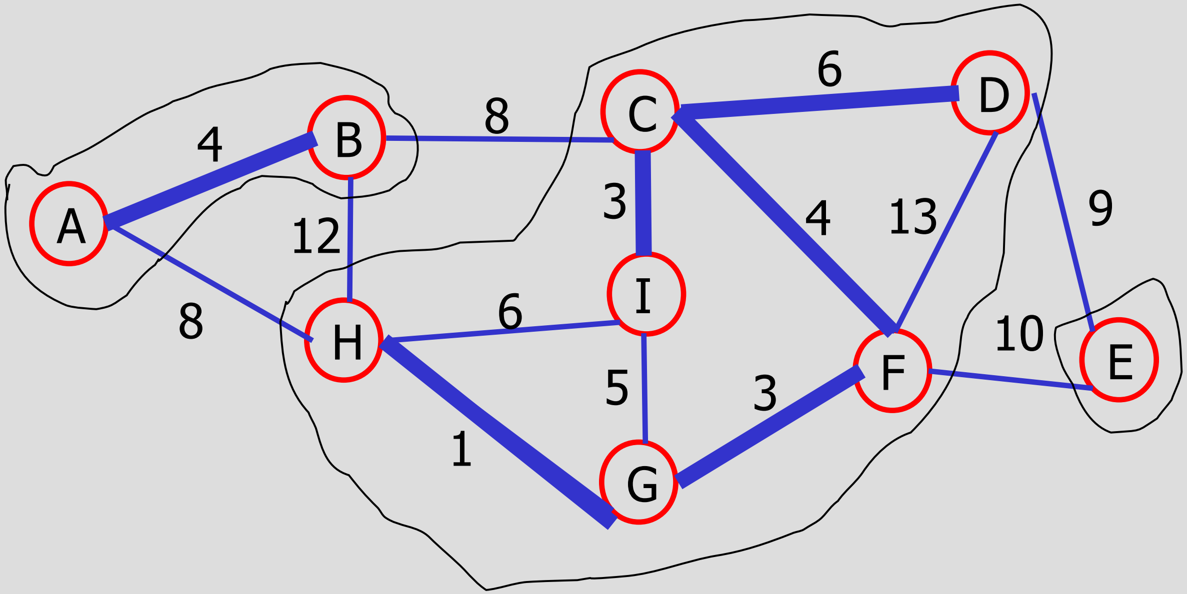
Kruskal Example/6



S = { AB CFGHI D E }

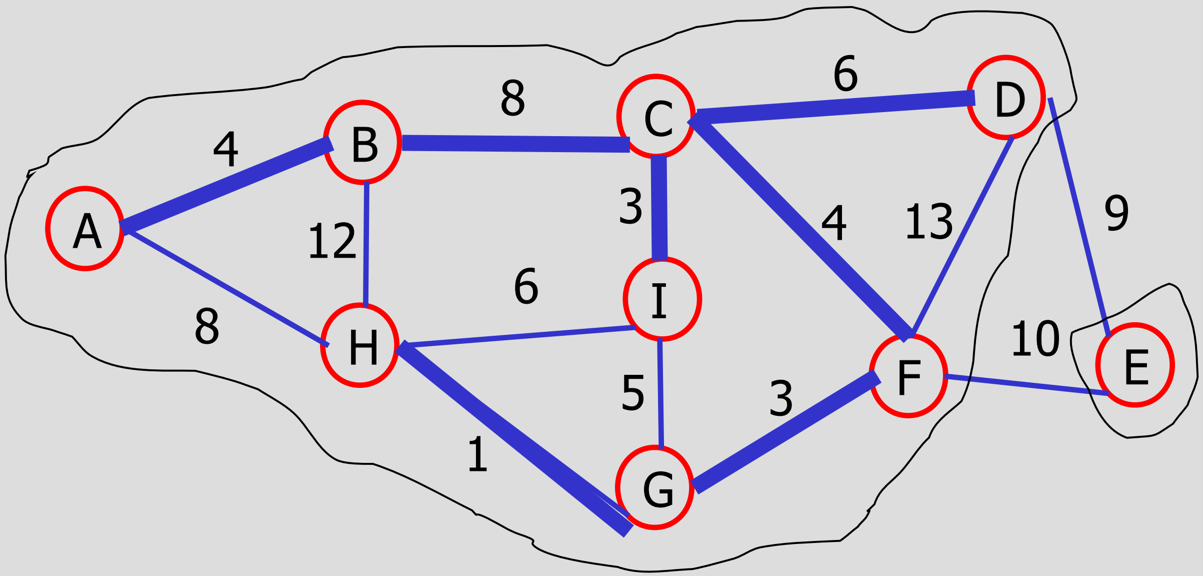
E' = { HI CD BC AH DE EF BH DF }

Kruskal Example/7



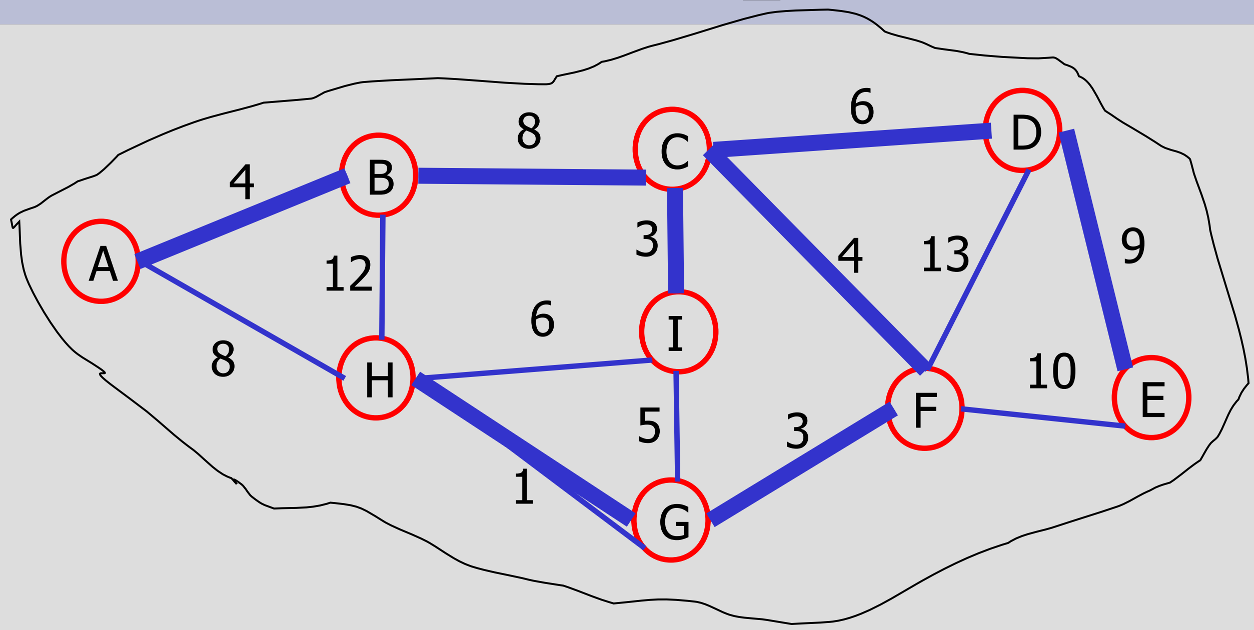
S = { AB CDFGHI E }
E' = { BC AH DE EF BH DF }

Kruskal Example/8



S = { ABCDFGHI E }
E' = { AH DE EF BH DF }

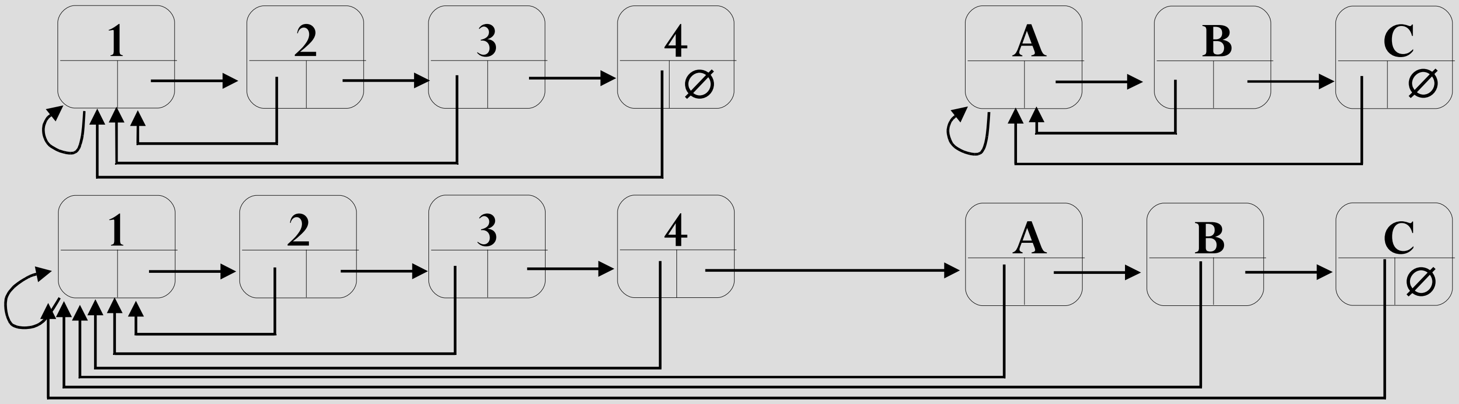
Kruskal Example/9



S = { ABCDEFGHI }
E' = { EF BH DF }

Disjoint Sets as Lists

- Each set is a list of elements identified by the first element, all elements in the list point to the first element
- UnionSets: add a shorter list to a longer one, $O(\min\{|C(u)|, |C(v)|\})$
- AddSingletonSet/FindSet: $O(1)$



Kruskal Running Time

- Initialization $O(V)$ time
- Sorting the edges $\Theta(E \log E) = \Theta(E \log V)$
- $O(E)$ calls to FindSet
- Union costs
  - Let $t(v)$ be the number of times v is moved to a new cluster
  - Each time a vertex is moved to a new cluster the size of the cluster containing the vertex at least doubles: $t(v) \leq \log V$
  - Total time spent doing Union $\sum_{v \in V} t(v) \leq |V| \log |V|$
- Total time: $O(E \log V)$

Suggested exercises

- Implement both Prim's (tough) and Kruskal's algorithms (very tough)
- Using paper & pencil, simulate the behaviour of both Prim's and Kruskal's algorithms on some examples

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Shortest Path

- Generalize distance to weighted setting
- Digraph $G=(V,E)$ with weight function $W: E \rightarrow R$ (assigning real values to edges)
- Weight of path $p = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1})$$

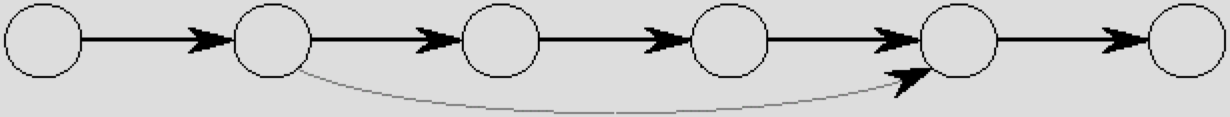
- Shortest path = a path of minimum weight (cost)
- Applications
  - static/dynamic network routing
  - robot motion planning
  - map/route generation in traffic

Shortest-Path Problems

- Shortest-Path problems
  - **Single-source (single-destination).** Find a shortest path from a given source (vertex *s*) to each of the vertices.
  - **Single-pair.** Given two vertices, find a shortest path between them. Solution to single-source problem solves this problem efficiently, too.
  - **All-pairs.** Find shortest-paths for every pair of vertices. Dynamic programming algorithm.
  - Unweighted shortest-paths – BFS.

Optimal Substructure

- *Theorem*: subpaths of shortest paths are shortest paths
- Proof:
  - if some subpath were not the shortest path, one could substitute the shorter subpath and create a shorter total path

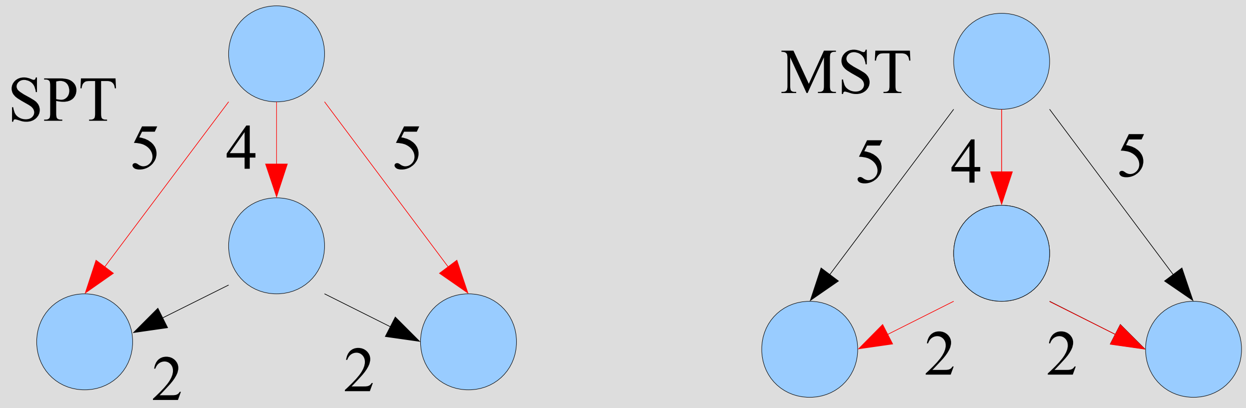


Negative Weights and Cycles

- Negative edges are OK, as long as there are no *negative weight cycles* (otherwise paths with arbitrary small "lengths" would be possible).
- Shortest-paths can have no cycles (otherwise we could improve them by removing cycles).
  - Any shortest-path in graph G can be no longer than $n - 1$ edges, where n is the number of vertices.

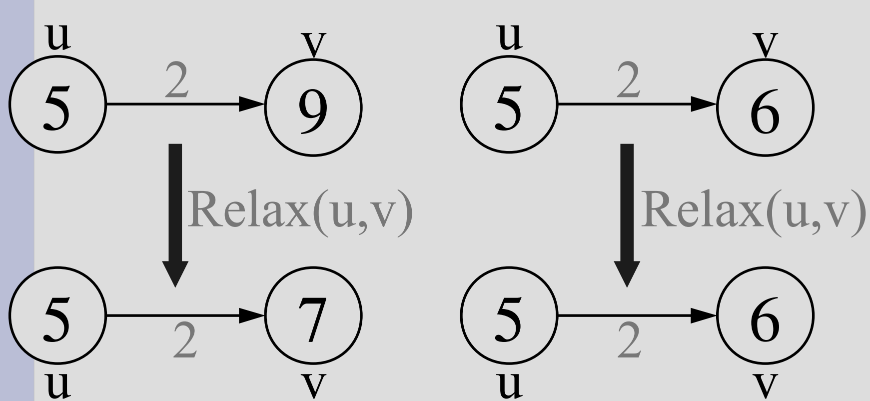
Shortest Path Tree

- The result of the algorithms is a *shortest path tree (SPT)*. For each vertex v, it
  - records a shortest path from the start vertex s to v.
  - v.**pred** is the predecessor of v on this shortest path
  - v.**dist** is the shortest path length from s to v
- *Note: SPT is different from minimum spanning tree (MST)!*



Relaxation

- For each vertex *v* in the graph, we maintain *v*.**dist**, the estimate of the shortest path from *s*. *It is* initialized to ∞ at the start.
- Relaxing an edge (*u*,*v*) means testing whether we can improve the shortest path to *v* found so far by going through *u*.




```
Relax (u,v,G)
if v.dist > u.dist + w(u,v) then
   v.dist := u.dist + w(u,v)
   v.pred := u
```

Dijkstra's Algorithm

- Assumption: non-negative edge weights
- Greedy, similar to Prim's algorithm for MST
- Like breadth-first search (if all weights = 1, one can simply use BFS)
- Use Q, a priority queue with keys v.**dist** (BFS used FIFO queue, here we use a PQ, which is re-organized whenever some **dist** decreases)
- Basic idea
  - maintain a set S of solved vertices
  - at each step select "closest" vertex u, add it to S, and relax all edges from u

Dijkstra's Pseudo Code

Input: Graph *G*, start vertex *s*

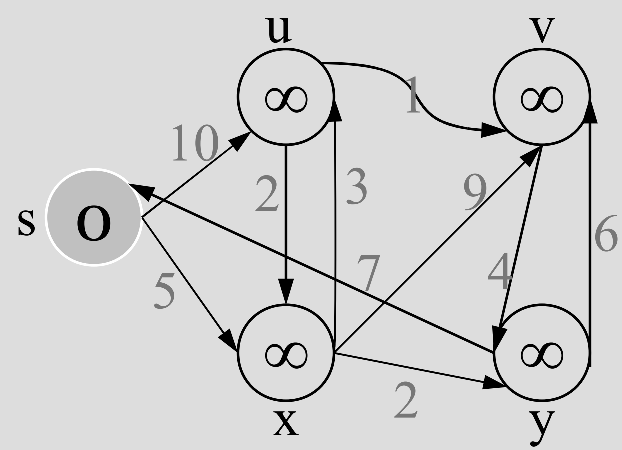
```
Dijkstra(G,s)
for u ∈ G.V
   u.dist := ∞
   u.pred := NIL
s.dist := 0
init(Q, G.V) // initialize priority queue Q
while not isEmpty(Q) do
   u := extractMin(Q)
   for v ∈ u.adj do
      Relax(u, v, G)
      modifyKey(Q, v)
```

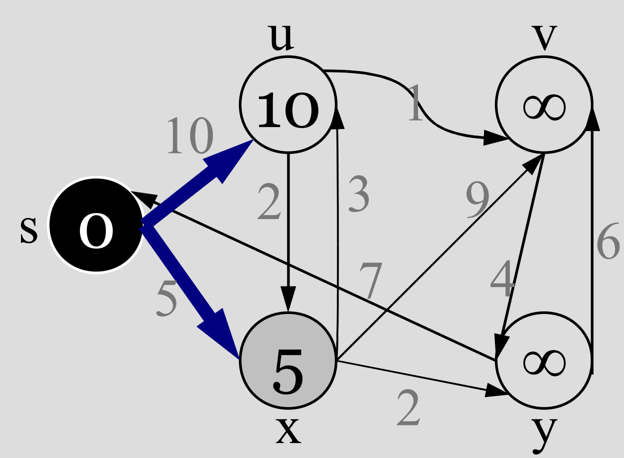
initialize graph (lines for u ∈ G.V … s.dist := 0)

relaxing edges (lines for v ∈ u.adj do … modifyKey(Q, v))

Dijkstra's Example/1

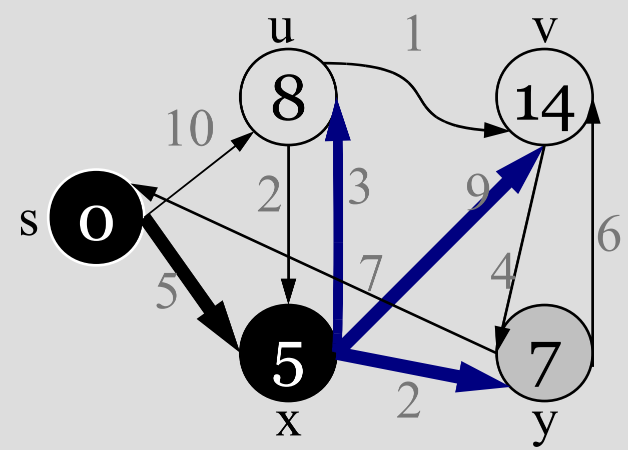
```
Dijkstra(G,s)
for u ∈ G.V
    u.dist := ∞
    u.pred := NIL
s.dist := 0
init(Q, G.V)
while not isEmpty(Q) do
    u := extractMin(Q)
    for v ∈ u.adj do
        Relax(u, v, G)
        modifyKey(Q, v)
```

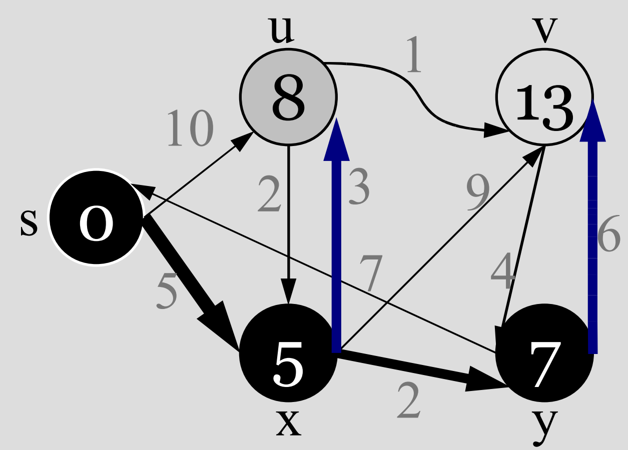




Dijkstra's Example/2

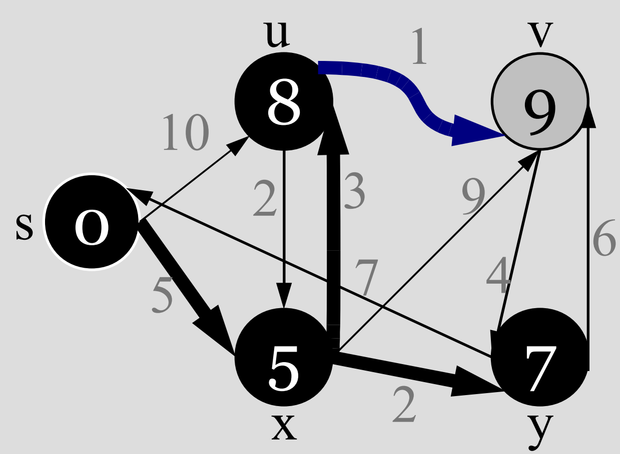
```
Dijkstra(G,s)
for u ∈ G.V
    u.dist := ∞
    u.pred := NIL
s.dist := 0
init(Q, G.V)
while not isEmpty(Q) do
    u := extractMin(Q)
    for v ∈ u.adj do
        Relax(u, v, G)
        modifyKey(Q, v)
```

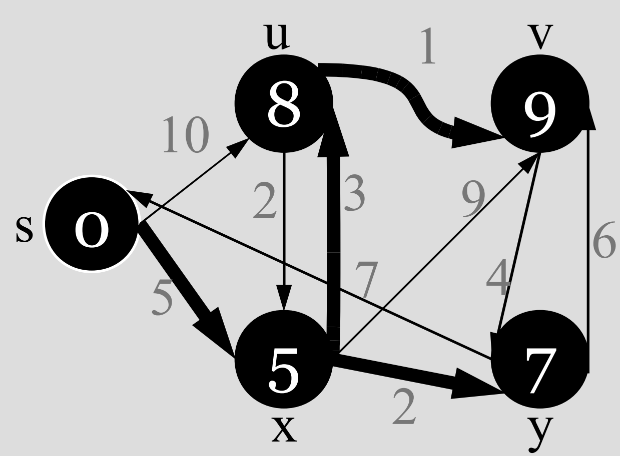




Dijkstra's Example/3

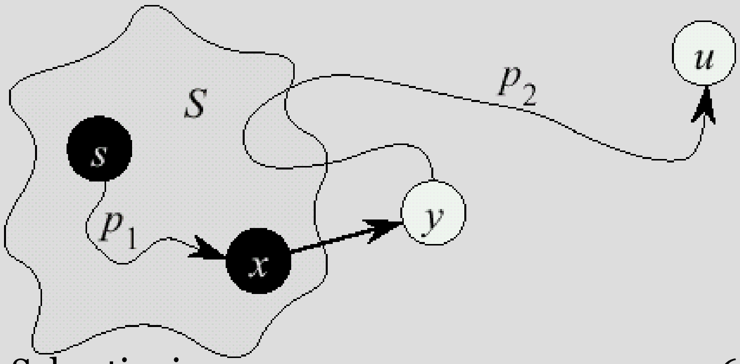
```
Dijkstra(G,s)
for u ∈ G.V
    u.dist := ∞
    u.pred := NIL
s.dist := 0
init(Q, G.V)
while not isEmpty(Q) do
    u := extractMin(Q)
    for v ∈ u.adj do
        Relax(u, v, G)
        modifyKey(Q, v)
```





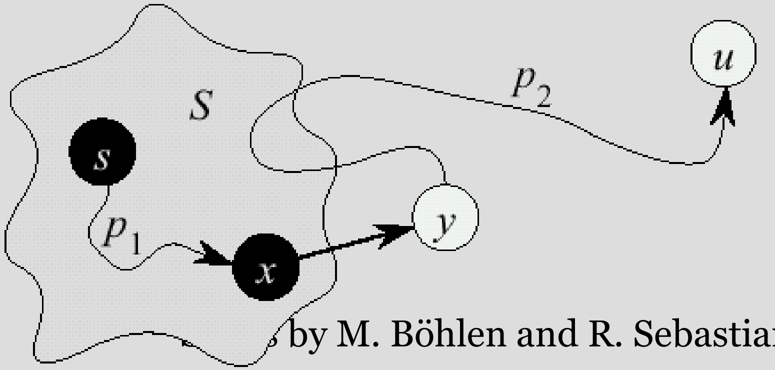
Dijkstra's Correctness/1

- We prove that **whenever *u* is added to *S***, *u*.**dist** = $\delta(s,u)$, i.e., *dist* is minimum.
- Proof (by contradiction)
  - Initially $\forall v$: *v*.**dist** $\geq \delta(s,v)$
  - Let *u* be the **first** vertex such that there is a shorter path than *u*.**dist**, i.e., *u*.**dist** $> \delta(s,u)$
  - We will show that this assumption leads to a contradiction



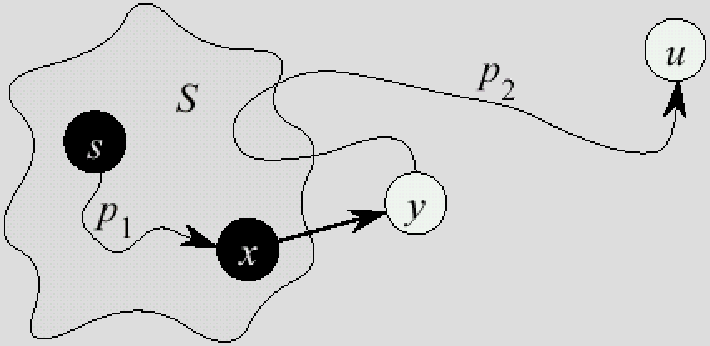
Dijkstra Correctness/2

- Let y be the first vertex $\in V-S$ on the actual shortest path from s to u, then it must be that y.**dist** $= \delta(s,y)$ because
  - x.**dist** is set correctly for y's predecessor $x \in S$ on the shortest path (by choice of u as the first vertex for which **dist** is set incorrectly)
  - when the algorithm inserted x into S, it relaxed the edge (x,y), setting y.**dist** to the correct value



Dijkstra Correctness/3

u.**dist** > (*s*,*u*)	initial assumption
= (*s*,*y*) + (*y*,*u*)	optimal substructure
= y.dist + (*y*,*u*)	*correctness of y.dist*
≥ *y.dist*	*no negative weights*



- But *u*.**dist** > *y*.**dist** ⇒ algorithm would have chosen *y* (from the PQ) to process next, not *u* ⇒ contradiction
- Thus, *u*.**dist** = δ(*s*,*u*) at time of insertion of *u* into *S*, and Dijkstra's algorithm is correct

Dijkstra's Running Time

- Extract-Min executed $|V|$ time
- Modify-Key executed $|E|$ time
- Time = $|V|\ T_{\text{Extract-Min}} + |E|\ T_{\text{Modify-Key}}$
- T depends on different Q implementations

Q	T(Extract-Min)	T(Modify-Key)	Total
array	$O(V)$	$O(1)$	$O(V^2)$
heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$

Bellman-Ford Algorithm/1

- Dijkstra's doesn't work when there are negative edges:
  - Intuition – we cannot be greedy anymore on the assumption that the lengths of paths will only increase in the future
- Bellman-Ford algorithm detects negative cycles (returns *false*) or returns the shortest path-tree

Bellman-Ford Algorithm/2

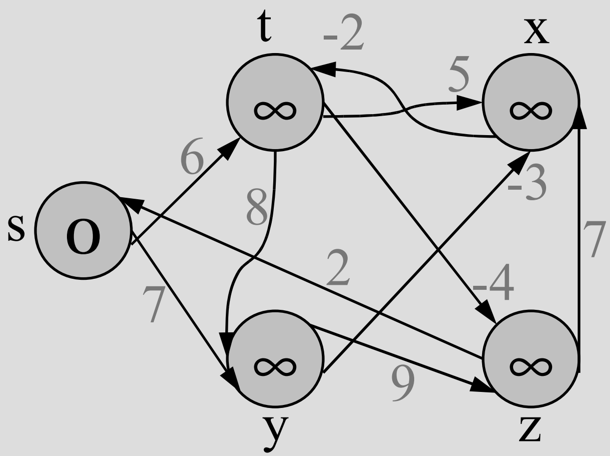
```
Bellman-Ford(G,s)
for each vertex u ∈ G.V
  u.dist := ∞
  u.pred := NIL
s.dist := 0
for i := 1 to |G.V|-1 do
  for each edge (u,v) ∈ G.E do
    Relax (u,v,G)
for each edge (u,v) ∈ G.E do
  if v.dist > u.dist + w(u,v) then
    return false
return true
```

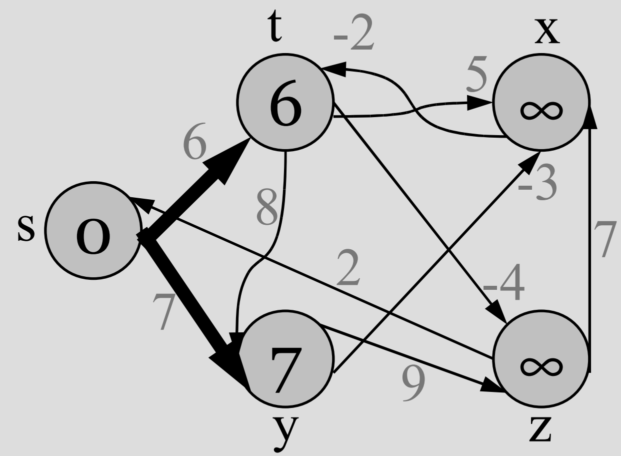
initialization

compute distances

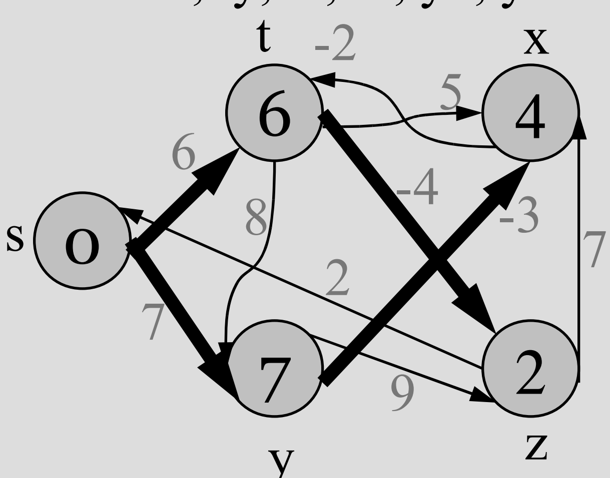
check for negative cycles

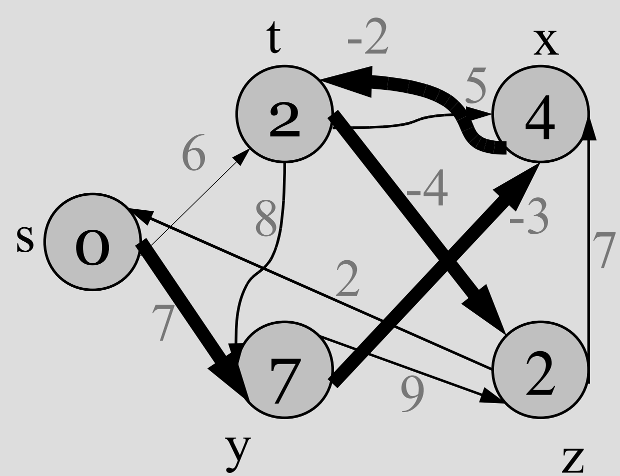
Bellman-Ford Example



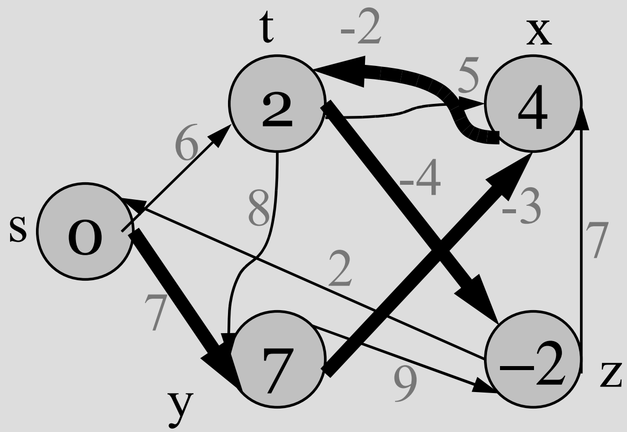


Order: tx, ty, tz, xt, yx, yz zx, zs, st, sy





Bellman-Ford Example/2



Order: tx, ty, tz, xt, yx, yz zx, zs, st, sy

Bellman-Ford Running time

- Bellman-Ford running time:
  - $(|V|-1)|E| + |E| = \Theta(VE)$

Correctness of Bellman-Ford/1

- Let $\delta_i(s,u)$ denote the length of path from *s* to *u*, that is shortest among all paths, that contain at most *i* edges
- Prove by induction that *u*.**dist** = $\delta_i(s,u)$ after the i^{th} iteration of Bellman-Ford
  - Base case (i=0) trivial
  - Inductive step (say *u*.**dist** = $\delta_{i-1}(s,u)$):
    - Either $\delta_i(s,u) = \delta_{i-1}(s,u)$
    - Or $\delta_i(s,u) = \delta_{i-1}(s,z) + w(z,u)$
    - In an iteration we try to relax each edge (*(z,u)* also), so we handle both cases, thus *u*.**dist** = $\delta_i(s,u)$

Correctness of Bellman-Ford/2

- After *n-1* iterations, *u*.**dist** = $\delta_{n-1}(s,u)$, for each vertex *u*.
- If there is some edge to relax in the graph, then there is a vertex *u*, such that $\delta_{n}(s,u) < \delta_{n-1}(s,u)$. But there are only *n* vertices in *G* – we have a cycle, and it is negative.
- Otherwise, *u*.**dist** = $\delta_{n-1}(s,u) = \delta(s,u)$, for all *u*, since any shortest path will have at most *n-1* edges

Shortest-Path in DAG's

- Finding shortest paths in DAG's is much easier, because it is easy to find an order in which to do relaxations – Topological sorting!

```
DAG-Shortest-Paths(G,w,s)
for each vertex u ∈ G.V
  u.dist := ∞
  u.pred := NIL
s.dist := 0
topologically sort G
for each vertex u in topological order do
  for each v ∈ u.adj do
    Relax(u, v, G)
```

Shortest-Paths in DAG’s/2

- Running time:
  - $\Theta(V+E)$ – only one relaxation for each edge, V times faster than Bellman-Ford

Suggested exercises

- Implement both Dijkstra's and Bellman-Ford's algorithms
- Implement the algorithm based on topological sorting for DAGs
- Using paper & pencil
  - simulate the behaviour of both Dijkstra's and Bellman-Ford's algorithms on some examples
  - Simulate the behaviour of the T.S. algorithm on some example DAGs

Summary and Outlook

- Greedy algorithms
- MST: Kruskal
- MST: Prim
- Shortest path: Dijkstra
- Shortest path: Bellman-Ford

Next Week

- Dynamic Programming