Data Structures and Algorithms

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Part 6

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Data Structures and Algorithms
Week 6

- Binary Search Trees
  - Tree traversals
  - Searching
  - Insertion
  - Deletion
- Red-Black Trees
  - Properties
  - Rotations
  - Insertion
  - Deletion

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Dictionaries

- A *dictionary D* is a dynamic data structure with operations:
  - **Search(D, k)** – *returns a pointer x to an element such that x.key = k (null otherwise)*
  - **Insert(D, x)** – *adds the element pointed to by x to D*
  - **Delete(D, x)** – *removes the element pointed to by x from D*
- An element has a *key* and *data* part.

Ordered Dictionaries

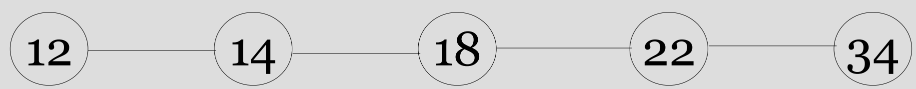
- In addition to dictionary functionality, we may want to support operations:
  - **Min(D)**
  - **Max(D)**
- and
  - **Predecessor(D, k)**
  - **Successor(D, k)**
- These operations require keys that are *comparable (ordered domain)*.

A List-Based Implementation

- Unordered list

  - search, min, max, predecessor, successor: *O(n)*
  - insertion, deletion: *O(1)*

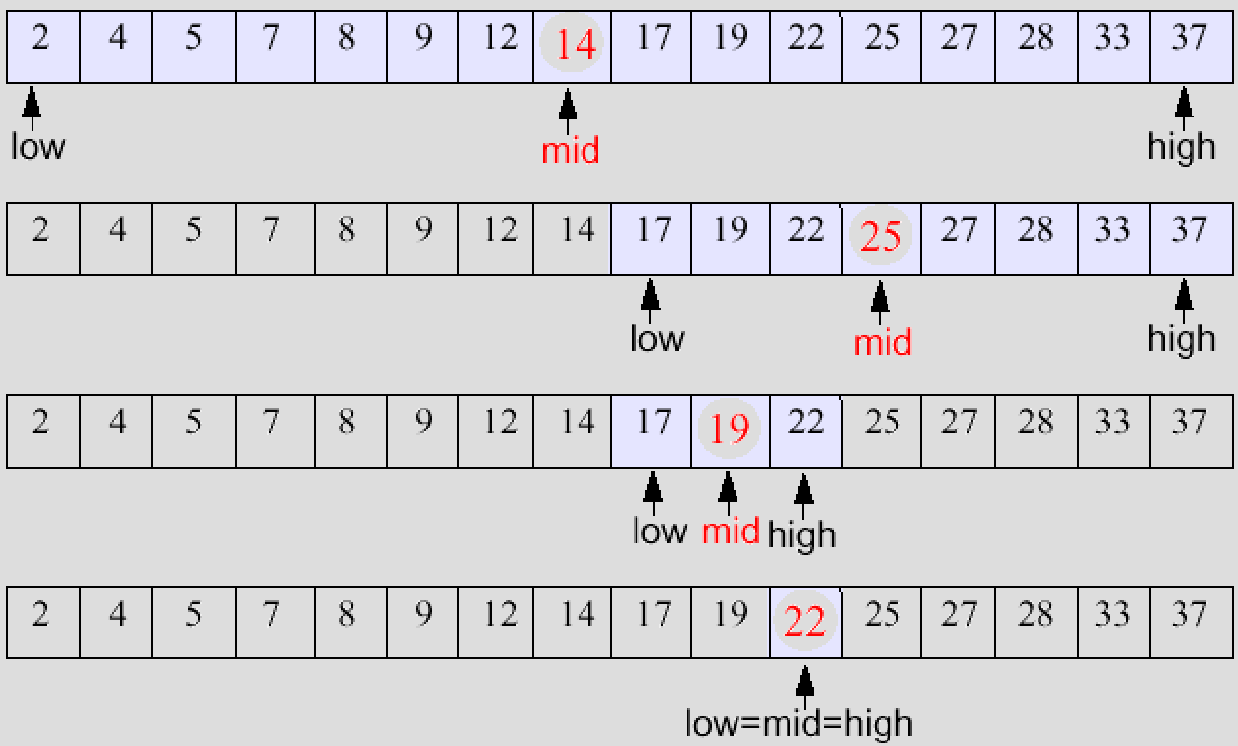
- Ordered list



  - search, insertion: *O(n)*
  - min, max, predecessor, successor, deletion: *O(1)*

Refresher: Binary Search

- Narrow down the search range in stages
  - findElement(22)

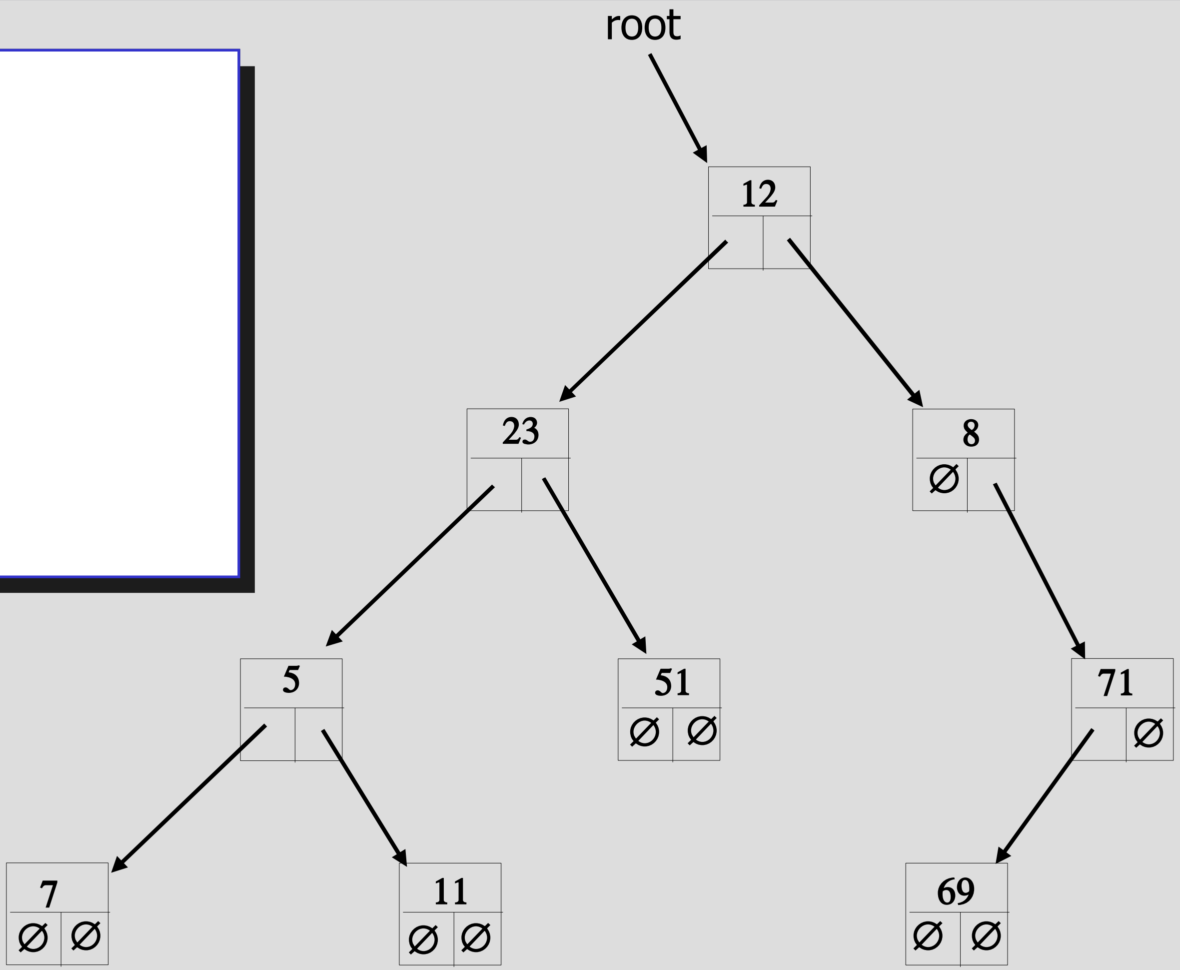


Run Time of Binary Search

- The range of candidate items to be searched is halved after comparing the key with the middle element.
- Binary search runs in *O(log n)* time.
- What about insertion and deletion?
  - search: *O(log n)*
  - insert, delete: *O(n)*
  - min, max, predecessor, successor: *O(1)*
- The idea of a binary search can be extended to dynamic data structures ➔ binary trees.

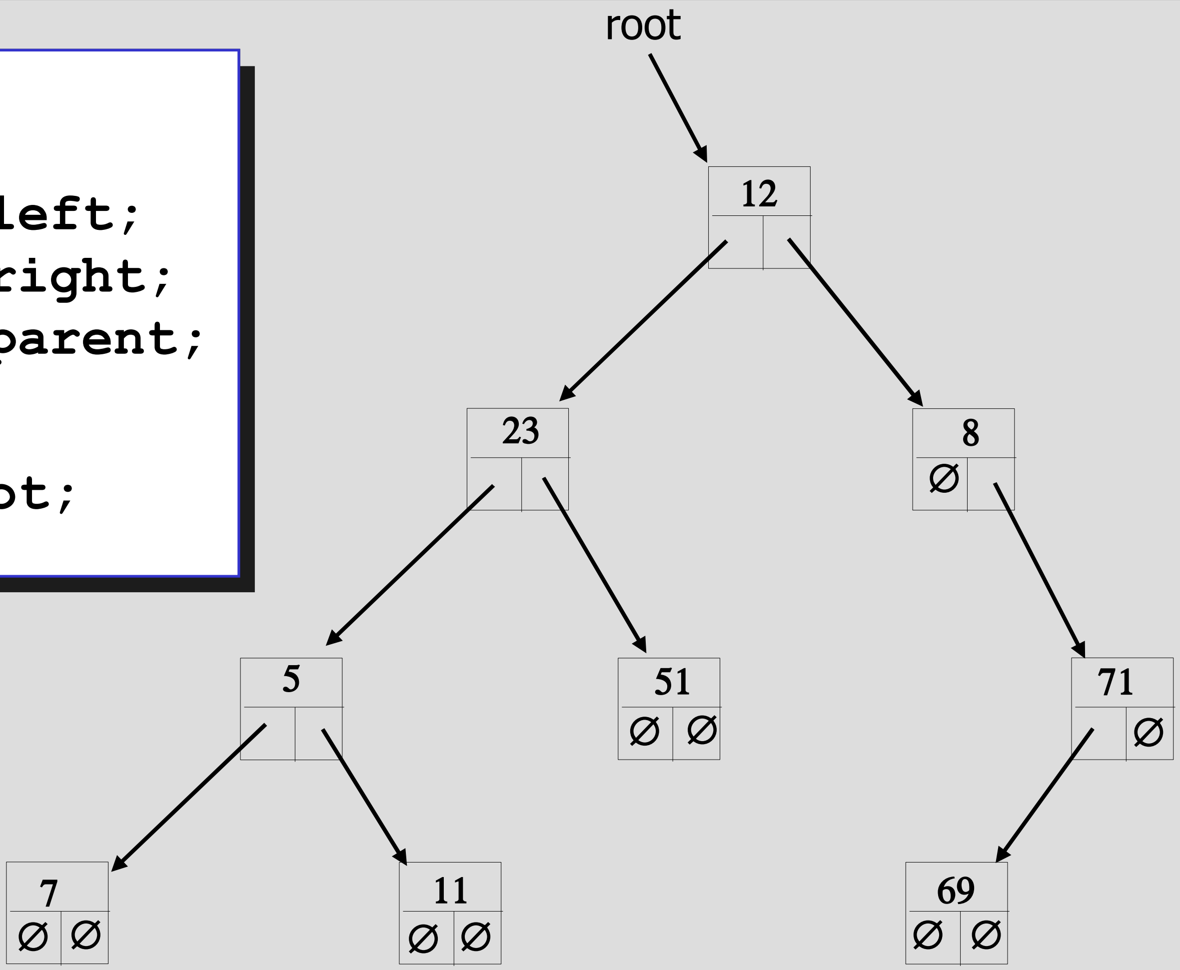
Binary Trees (Java)

```
class node {
  int key;
  node left;
  node right;
  node parent;
}

node root;
```



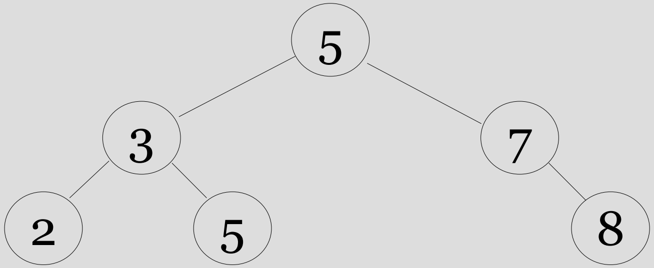
Binary Trees (C)

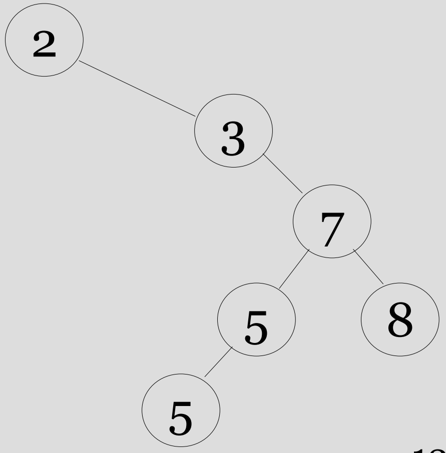
```
struct node {
  int key;
  struct node* left;
  struct node* right;
  struct node* parent;
}

struct node* root;
```



Binary Search Trees

- A **binary search tree** (BST) is a binary tree T with the following properties:
  - each internal node stores an item *(k,e)* of a dictionary
  - keys stored at nodes in the **left subtree** of *v* are **less than or equal** to *k*
  - keys stored at nodes in the **right subtree** of *v* **are greater than or equal** to *k*
- Example BSTs for 2, 3, 5, 5, 7, 8





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Tree Walks

- Keys in a BST can be printed using "tree walks"
- Keys of each node printed between keys in the left and right subtree – *inorder* tree traversal

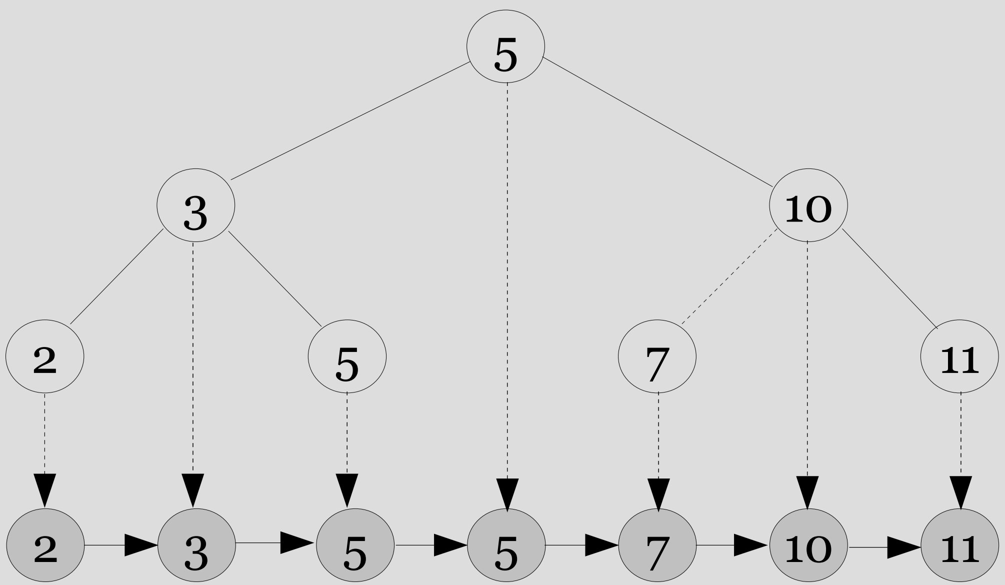
```
InorderTreeWalk(x)
    if x ≠ NIL then
        InorderTreeWalk(x.left)
        print x.key
        InorderTreeWalk(x.right)
```

Tree Walks/2

- InorderTreeWalk is a divide-and-conquer algorithm.
- It prints all elements in monotonically increasing order.
- Running time $\Theta(n)$.

Tree Walks/2

- **Inorder tree walk** can be thought of as a projection of the BST nodes onto a one dimensional interval.



Tree Walks/3

Other forms of tree walk:

- A **preorder tree walk** processes each node before processing its children.
- A **postorder tree walk** processes each node after processing its children.

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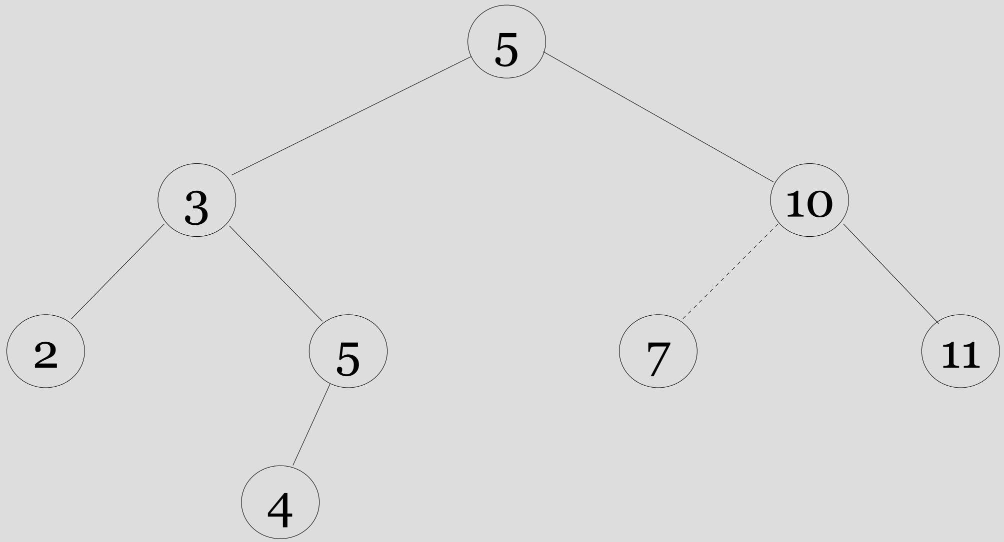
Searching a BST



- To find an element with key k in a tree T
  - compare k with $T.key$
  - if $k < T.key$, search for k in $T.left$
  - otherwise, search for k in $T.right$

Pseudocode for BST Search

- Recursive version: divide-and-conquer

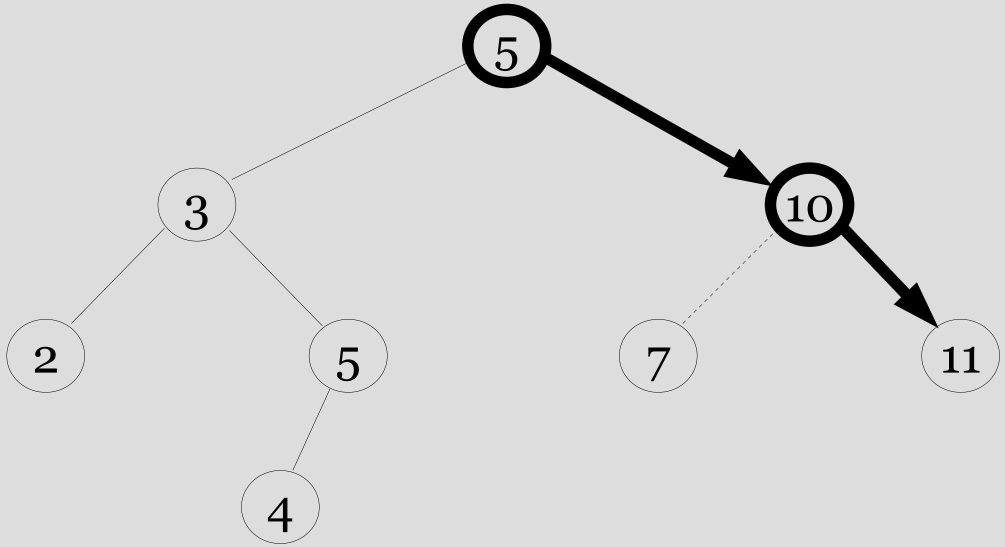
```
Search(T,k)
if T = NIL then return NIL
if k = T.key then return T
if k < T.key
  then return Search(T.left,k)
  else return Search(T.right,k)
```

- Iterative version

```
Search(T,k)
x := T
while x ≠ NIL and k ≠ x.key do
  if k < x.key
    then x := x.left
    else x := x.right
return x
```

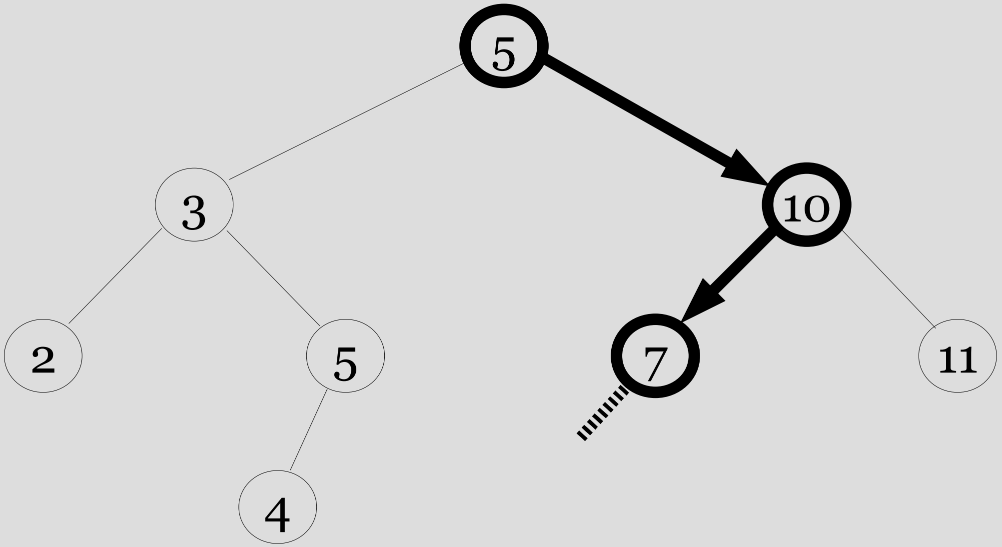
Search Examples

- Search(T, 11)



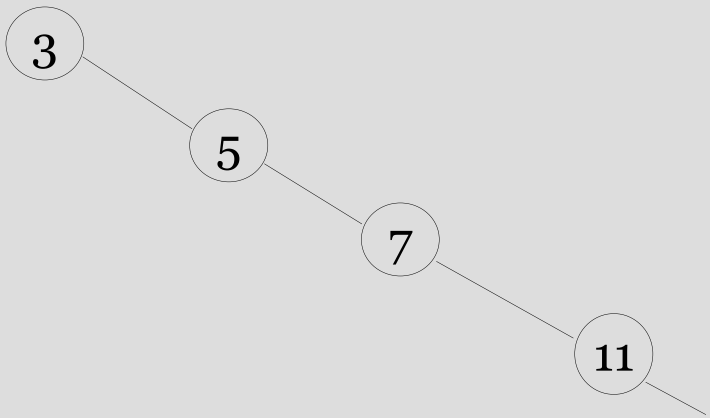
Search Examples/2

- Search(T, 6)



Analysis of Search

- Running time on tree of height *h* is *O(h)*
- After the insertion of *n* keys, the worst-case running time of searching is *O(n)*



BST Minimum (Maximum)

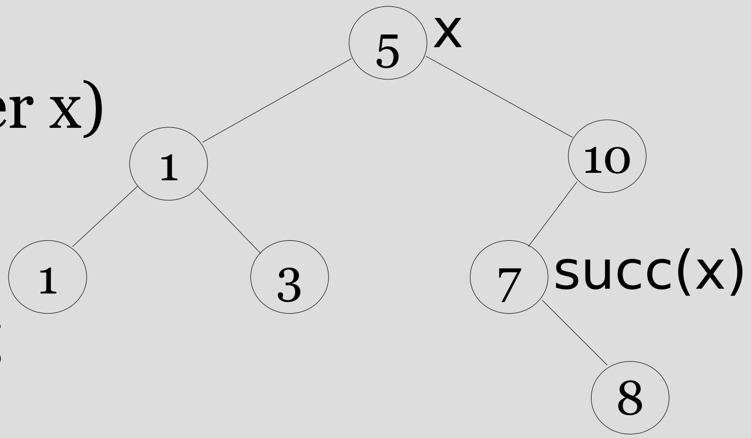
- Find the minimum key in a tree rooted at x.

```
TreeMinimum(x)
while x.left ≠ NIL do
  x := x.left
return x
```

- Maximum: same, x.right instead of x.left
- Running time *O(h)*, i.e., it is proportional to the height of the tree.

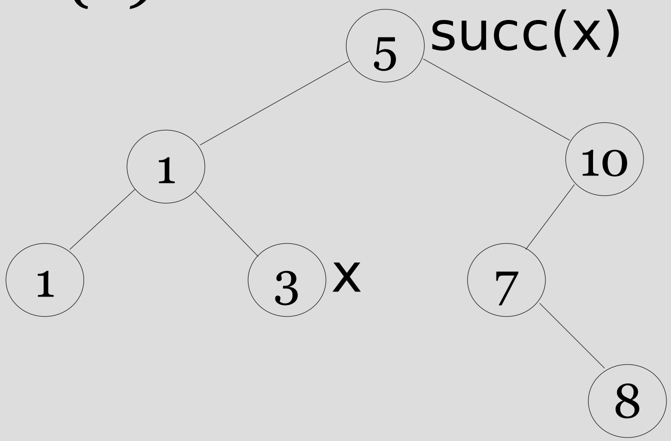
Successor

- Given *x*, find the node with the smallest key greater than *x*.key.
- We can distinguish two cases, depending on the right subtree of *x*
- Case 1: The right subtree of x is non-empty (succ(x) inserted after x)
  - successor is the leftmost node in the right subtree.
  - this can be done by returning TreeMinimum(x.right).



Successor/2

- Case 2: the right subtree of *x* is empty (succ(x), if any, was inserted before x).
  - The successor (if any) is the lowest ancestor of *x* whose left subtree contains *x*.
  - *Note: it x had a right child, then it would be smaller than succ(x)*



Successor Pseudocode

```
TreeSuccessor(x)
if x.right ≠ NIL
  then return TreeMinimum(x.right)
y := x.parent
while y ≠ NIL and x = y.right
  x := y
  y := y.parent
return y
```

- For a tree of height *h*, the running time is *O(h)*.
- *Note: no comparison among keys needed!*

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BST Insertion

- The basic idea derives from searching:
  - construct an element *p* whose left and right children are NULL and insert it into *T*
  - find location in *T* where *p* belongs to (as if searching for *p.key*),
  - add *p* there
- The running time on a tree of height *h* is *O(h)*.

BST Insertion: Pseudocode

- Notice: trailing pointer technique

```
TreeInsert(n, root)
 front:=root; rear:=NIL;
 while front ≠ NIL do
  rear:=front;
  if n.key < front.key
   then front:=front.left
   else front:=front.right
 if rear = NIL
  then n.parent:=NIL; return n;
 elsif n.key < rear.key
  then rear.left:=n;
  else rear.right:=n;
 n.parent:=rear;
 return root;
```

BST Insertion Code (java)

- Have a ”one step delayed” pointer.

```
node insert(node p, node r) { //insert p in r
  node y = NULL; node x = r;
  while (x != NULL) {
    y := x;
    if (x.key < p.key) x = x.right;
    else x = x.left;
  }
  if (y == NULL) {r = p; p.parent=null;}// r is empty
  else if (y.key < p.key) y.right = p;
  else y.left = p;
  p.parent =y;
  return r;
}
```

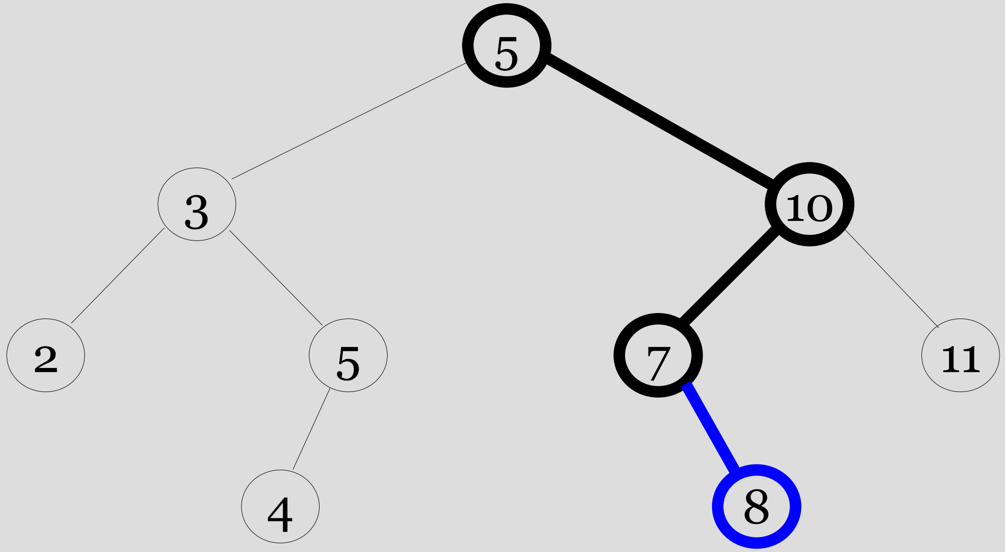
BST Insertion Code (C)

- Have a ”one step delayed” pointer.

```
struct node* insert(struct node* p, struct node* r) {
  struct node* y = NULL; struct node* x = r;
  while (x != NULL) {
    y := x;
    if (x->key < p->key) x = x->right;
    else x = x->left;
  }
  if (y == NULL) {r = p;p->partent=null}
  else if (y->key < p->key) y->right = p;
  else y->left = p;
  p->parent = u;
  return r;
}
```

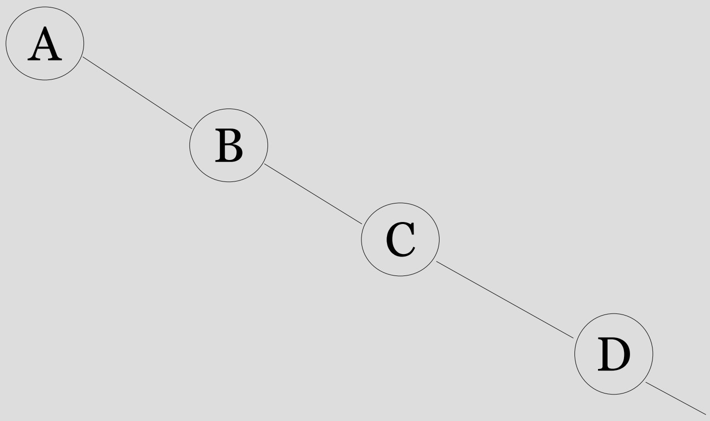
BST Insertion Example

- Insert 8



BST Insertion: Worst Case

- In what kind of sequence should the insertions be made to produce a BST of height n?



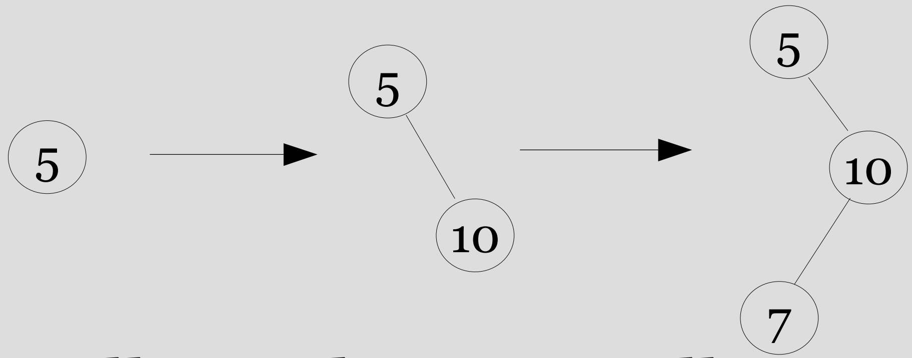
BST Sorting

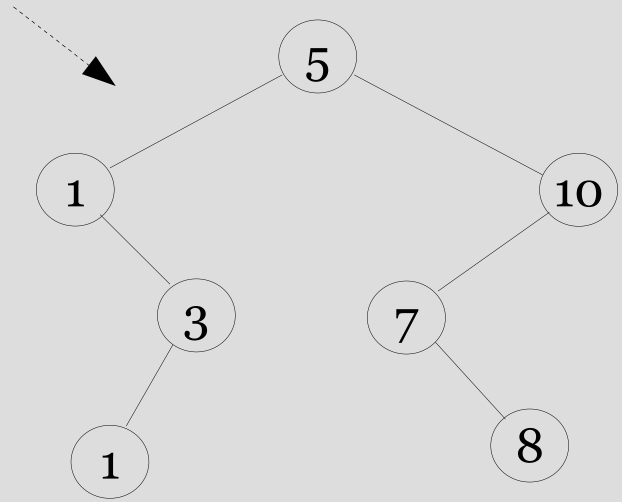
- Use TreeInsert and InorderTreeWalk to sort a list of n elements, A

```
TreeSort(A)
T := NIL
for i := 1 to n
  TreeInsert(T, BinTree(A[i]))
InorderTreeWalk(T)
```

BST Sorting/2

- Sort the following numbers
  5 10 7 1 3 1 8
- Build a binary search tree





- Call InorderTreeWalk

  1 1 3 5 7 8 10

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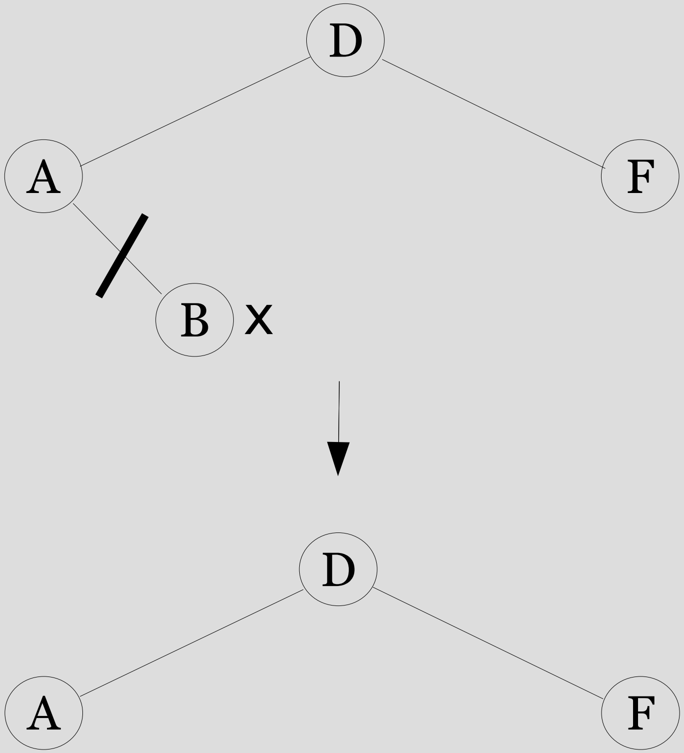
Deletion

- Delete node x from a tree T
- We can distinguish three cases
  - x has no child
  - x has one child
  - x has two children

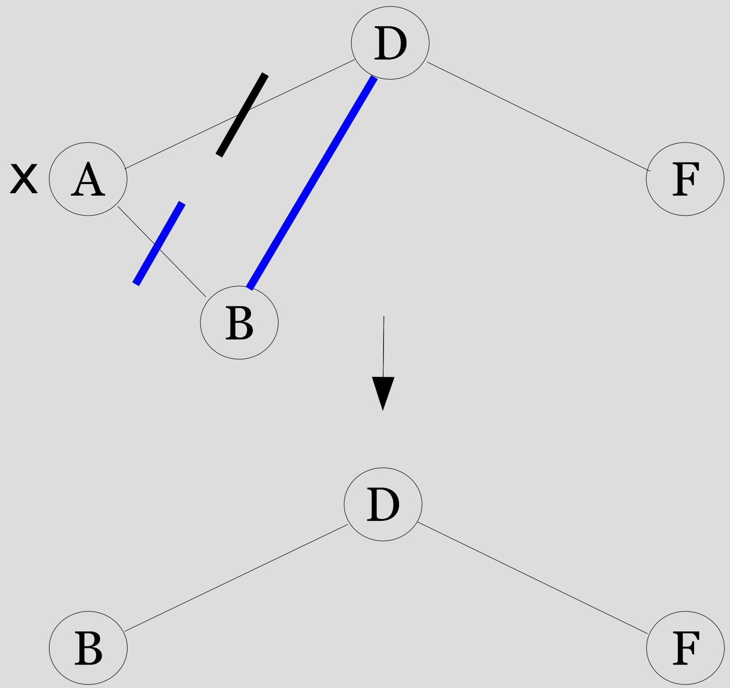
Deletion Case 1

- If *x* has no children: simply remove *x*



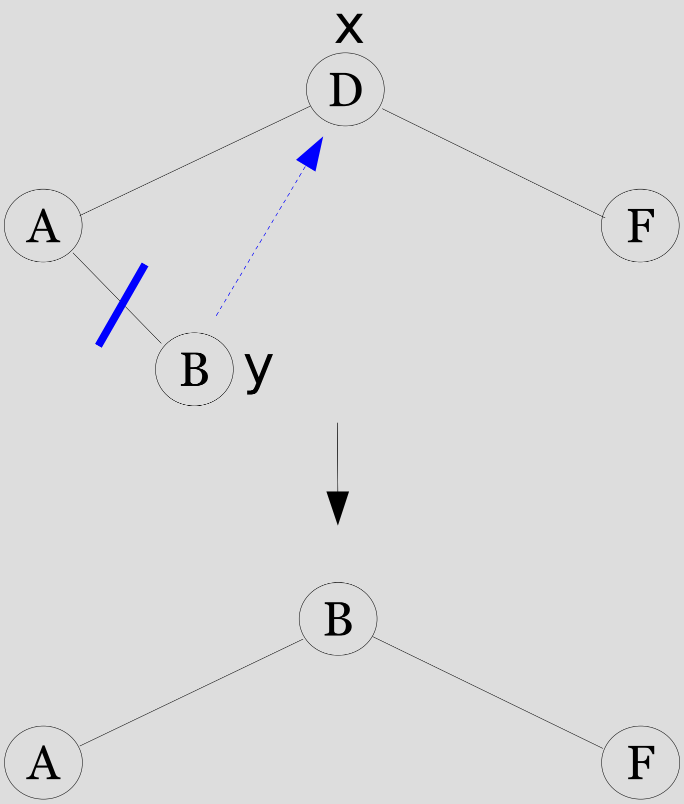
Deletion Case 2

- If *x* has exactly one child, make parent of x point to that child and delete x.



Deletion Case 3

- If *x* has two children:
  - find the largest child y in the left subtree of x (i.e. y is predecessor(x))
  - Recursively remove y (note that y has at most one child), and
  - replace x with y.
- "Specular" version with successor(x) (CLRS)



Deletion Pseudocode

```
Delete(T,x)
  if x.left = nil or x.right = nil
     then drop := x
     else drop := Succ(x)
  if drop.left ≠ nil
     then keep := drop.left
     else keep := drop.right
  if keep ≠ nil
     then keep.parent := drop.parent
  if drop.parent = nil
     then T.root := keep
     else if drop = drop.parent.left
          then drop.parent.left := keep
    else drop.parent.right := keep
  if drop ≠ x
     then x.key := drop.key
     % x.info := drop.info
```

Version with parent pointer

BST Deletion Code (java)

- Version without “parent” field
- Note again the trailing pointer technique

```
node delete(node root, node x) {

  front = root; rear = NULL;
  while (front != x) {
    rear := front;
    if (x.key < front.key) front := front.left;
    else front := front.right;
  } // rear points to a parent of x (if any)


  …
```

BST Deletion Code (java)/2

- x has less than 2 children
- Fix pointer of parent of x

```
…
   if (x.right == NULL) {
     if (rear == NULL) root = x.left;
     else if (rear.left == x) rear.left = x.left;
     else rear.right = x.left;}
   else if (x.left == NULL) {
     if (rear == NULL) root = x.right;
     else if (rear.left == x) rear.left = x.right;
     else rear.right = x.right;
   else {
…
```

BST Deletion Code (java)/3

- x has 2 children

```
succ = x.right; srear = succ;
while (succ.left != NULL)
      { srear:=succ; succ:=succ.left; }

if (rear == NULL) root = succ;
else if (rear.left == x) rear.left = succ;
else rear.right = succ;

succ.left = x.left;
if (srear != succ) {
  srear.left = succ.right;
  succ.right = x.right;
}
return root
```

BST Deletion Code (C)

- Version without “parent” field

```
struct node* delete(struct node* root,
                    struct node* x) {

  u = root; v = NULL;
  while (u != x) {
    v := u;
    if (x->key < u->key) u := u->left;
    else u := u->right;
  } // v points to a parent of x (if any)

  …
```

BST Deletion Code (C)/2

- x has less than 2 children
- Fix pointer of parent of x

```
…
  if (u->right == NULL) {
    if (v == NULL) root = u->left;
    else if (v->left == u) v->left = u->left;
    else v->right = u->left;
  else if (u->left == NULL) {
    if (v == NULL) root = u->right;
    else if (v->left == u) v->left = u->right;
    else v->right = u->right;
  else {
…
```

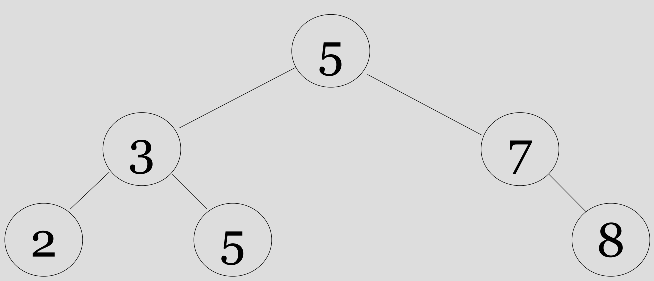
BST Deletion Code (C)/3

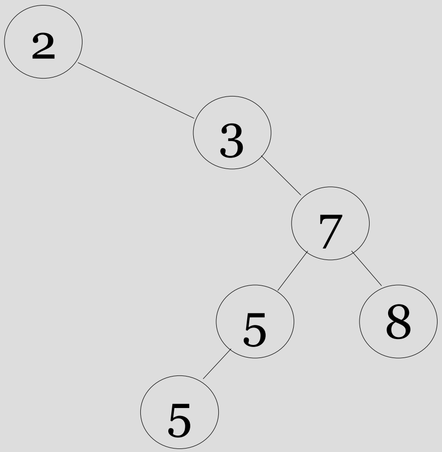
- x has 2 children

```
p = x->left; q = p;
while (p->right != NULL) { q:=p; p:=p->right; }

if (v == NULL) root = p;
else if (v->left == u) v->left = p;
else v->right = p;

p->right = u->right;
if (q != p) {
  q->right = p->left;
  p->left = u->left;
}
return root
```

Balanced Binary Search Trees

- Problem: execution time for tree operations is $\Theta(h)$, which in worst case is $\Theta(n)$.
- Solution: balanced search trees *guarantee* small height $h = O(\log n)$.





Suggested exercises

- Implement a binary search tree with the following functionalities:
  - init, max, min, successor, predecessor, search (iterative & recursive), insert, delete (both swap with succ and pred), print, print in reverse order
  - TreeSort

Suggested exercises/2

Using paper & pencil:

- draw the trees after each of the following operations, starting from an empty tree:
  1. Insert 9,5,3,7,2,4,6,8,13,11,15,10,12,16,14
  2. Delete 16, 15, 5, 7, 9 (both with succ and pred strategies)
- simulate the following operations after 1:
  - Find the max and minimum
  - Find the successor of 9, 8, 6

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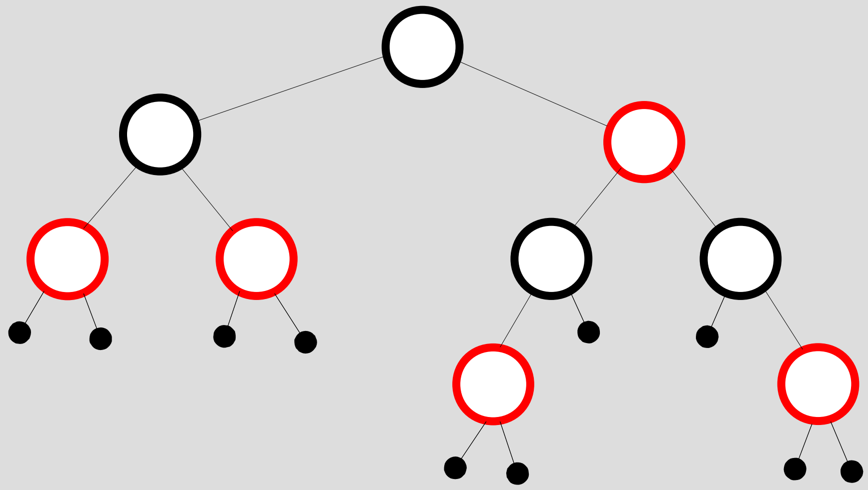
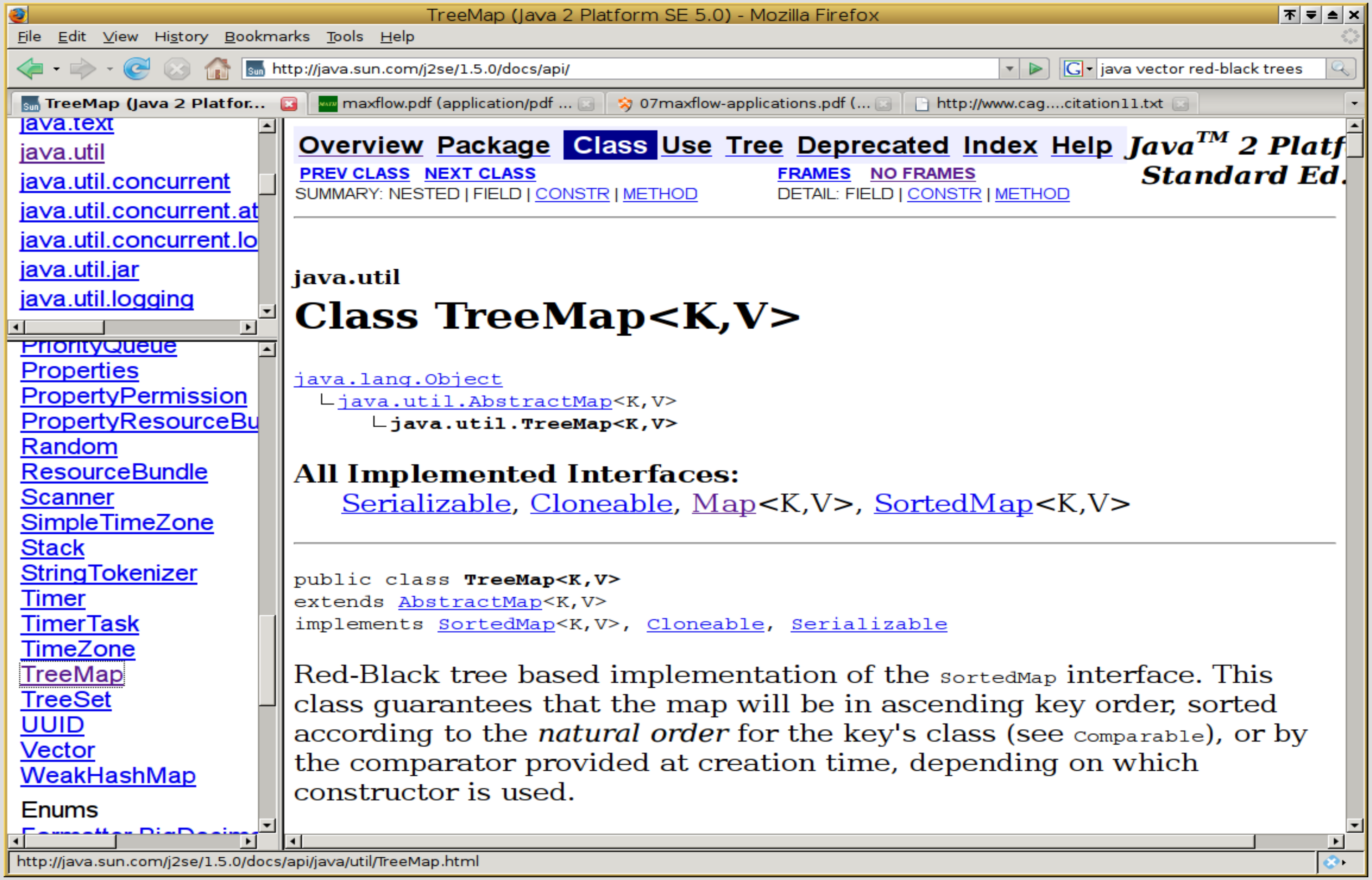
Red/Black Trees

- A **red-black** tree is a binary search tree with the following properties:
  1. Nodes (or incoming edges) are colored **red** or **black**
  2. NULL leaves are **black**
  3. The root is **black**
  4. No two consecutive **red nodes** on any root-leaf path.
  5. Same number of black nodes on any root-leaf path (called *black height* of the tree).

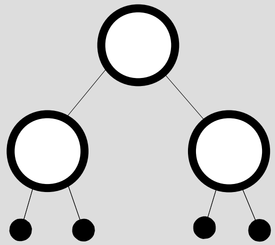


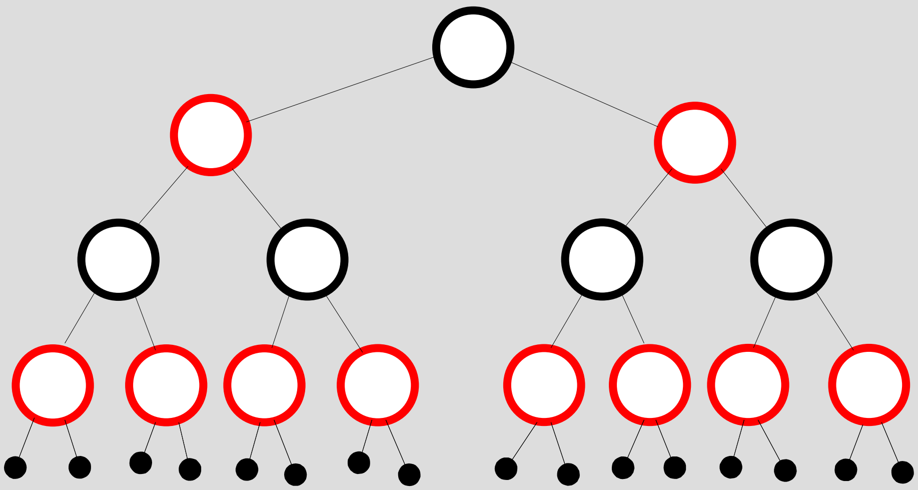
Java's TreeMap



RB-Tree Properties

- Some measures
  - n – # of internal nodes
  - h – height
  - bh – black height
- $2^{bh} - 1 \leq n$
- $bh \geq h/2$
- $2^{h/2} \leq n + 1$
- $h \leq 2 \log(n + 1)$
- BALANCED!





RB-Tree Properties/2

- Operations on a binary-search tree (search, insert, delete, ...) can be accomplished in $O(h)$ time.
- The RB-tree is a binary search tree, whose height is bounded by 2 log(n +1), thus the operations run in $O(\log n)$.
  - Provided that we can maintain red-black tree properties spending no more than $O(h)$ time on each insertion or deletion.

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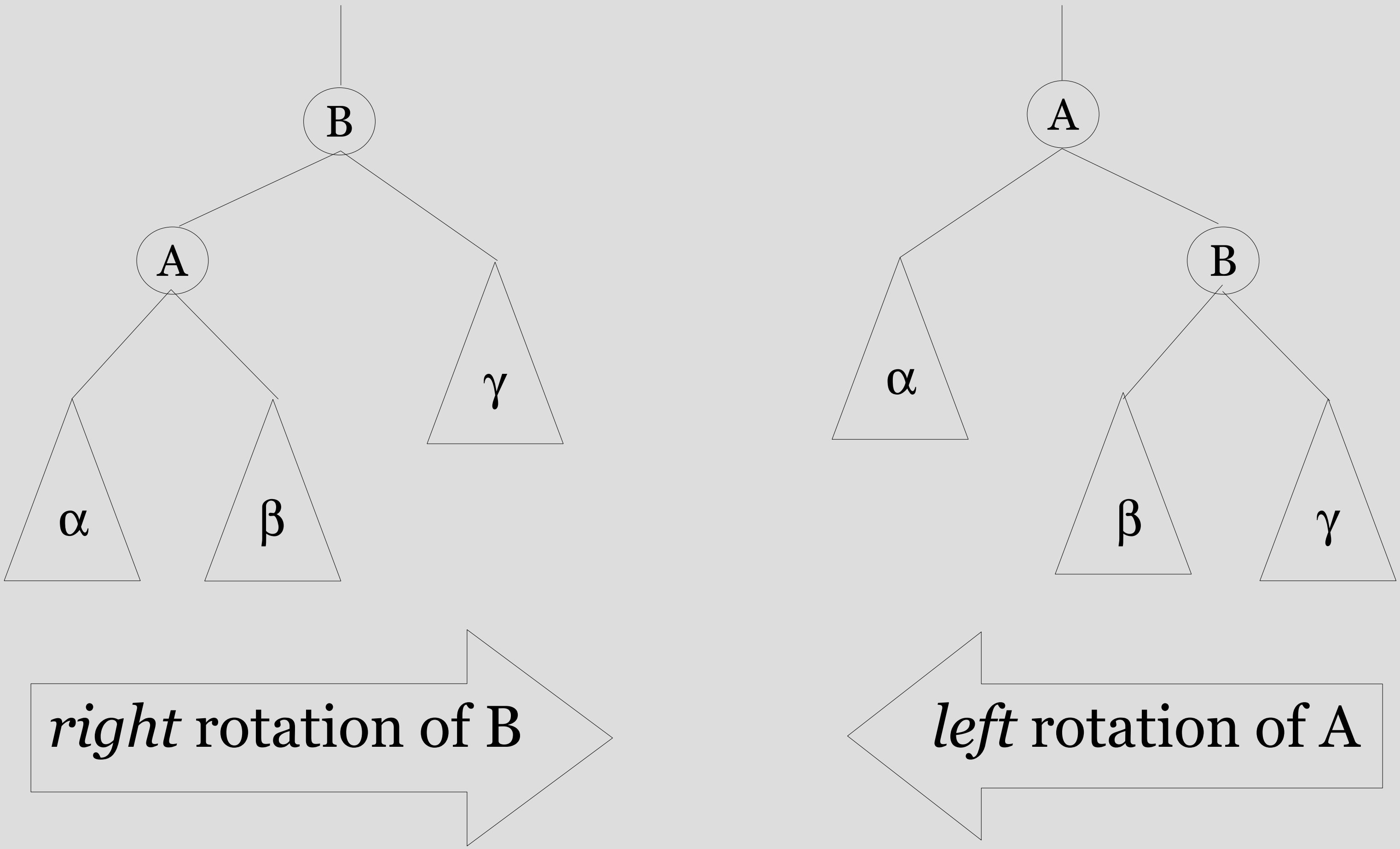
Rotation



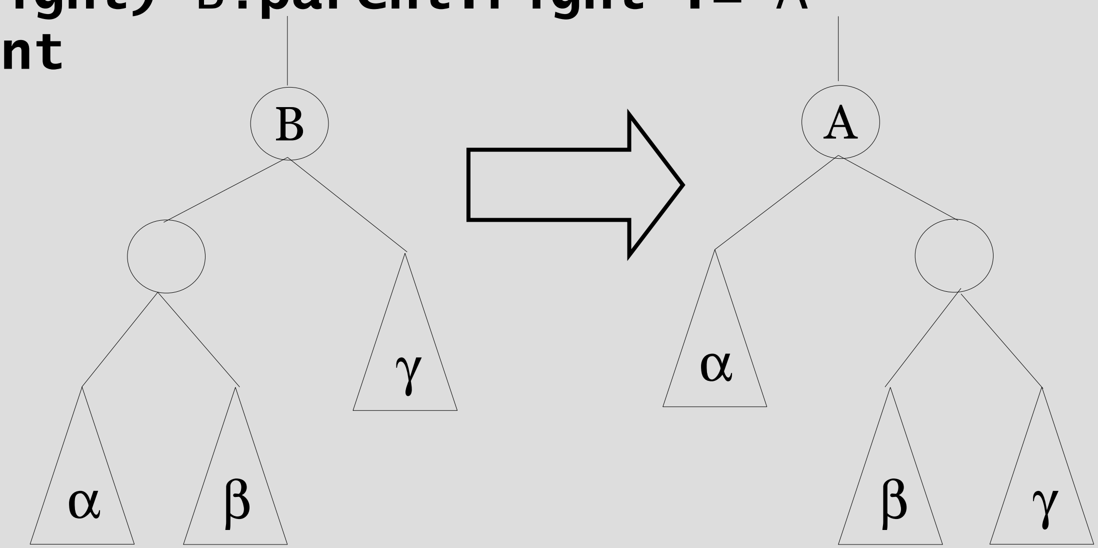
Right Rotation

```
RightRotate(B)
A := B.left

B.left := A.right
B.left.parent := B

if (B = B.parent.left) B.parent.left := A
if (B = B.parent.right) B.parent.right := A
A.parent := B.parent

A.right := B
B.parent := A
```



The Effect of a Rotation

- Maintains inorder key ordering
  - $\forall a \in \alpha$, $b \in \beta$, $c \in \gamma$
    we can state the invariant
    - $a <= A <= b <= B <= c$
- After right rotation
  - Depth(α) decreases by 1
  - Depth(β) stays the same
  - Depth(γ) increases by 1
- Left rotation: symmetric
- Rotation takes *O(1)* time

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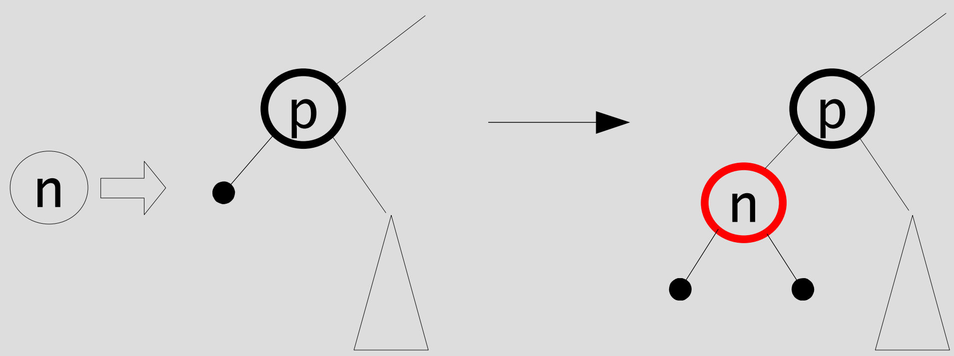
Insertion in the RB-Trees

```
RBInsert(T,n)
Insert n into T using the binary search tree
insertion procedure
n.left := NIL
n.right := NIL
n.color := red
RBInsertFixup(n)
```

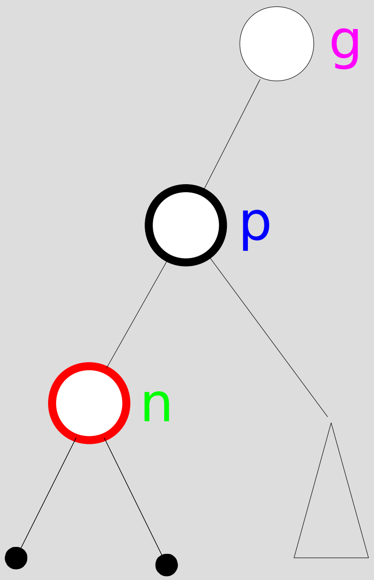


Fixing Up a Node: Intuition

- Case 0: parent is black
  => *ok*
- Case 1: both parent and uncle are red
  => change colour of parent/uncle to black
  => change colour of grandparent to red
  => *fix up the grandparent*
  Exception: grandparent is root => then keep it black
- Case 2: parent is red and uncle is black, and
  node and parent are in a straight line
  => *rotate at grandparent*
- Case 3: parent is red and uncle is black, and
  node and parent are **not** in a straight line
  => *rotate at parent* (leads to Case 2)

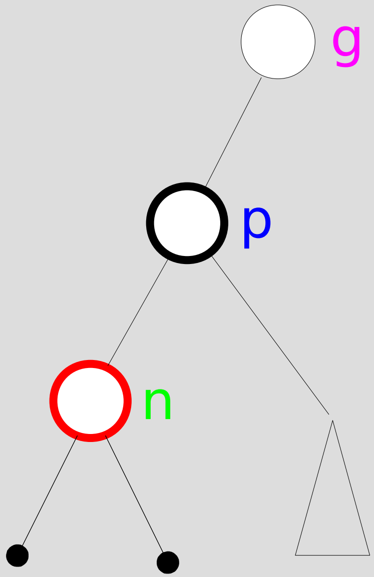
Insertion

- Let
  - `n = the new node`
  - `p = n.parent`
  - `g = p.parent`
- In the following assume:
  - `p = g.left`



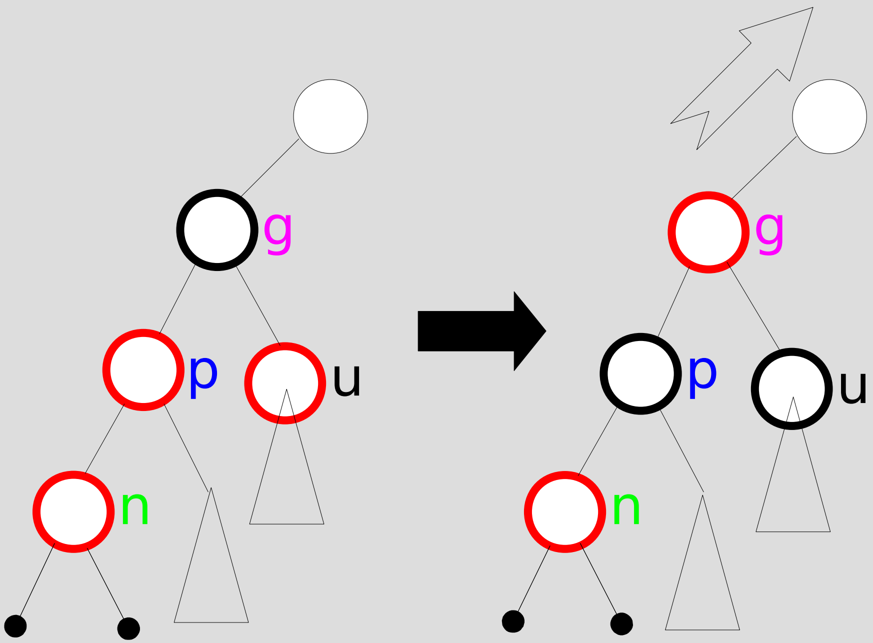
Insertion: Case 0

- p.**color** = black
  - No properties of the tree are violated
  - we are done.



Insertion: Case 1

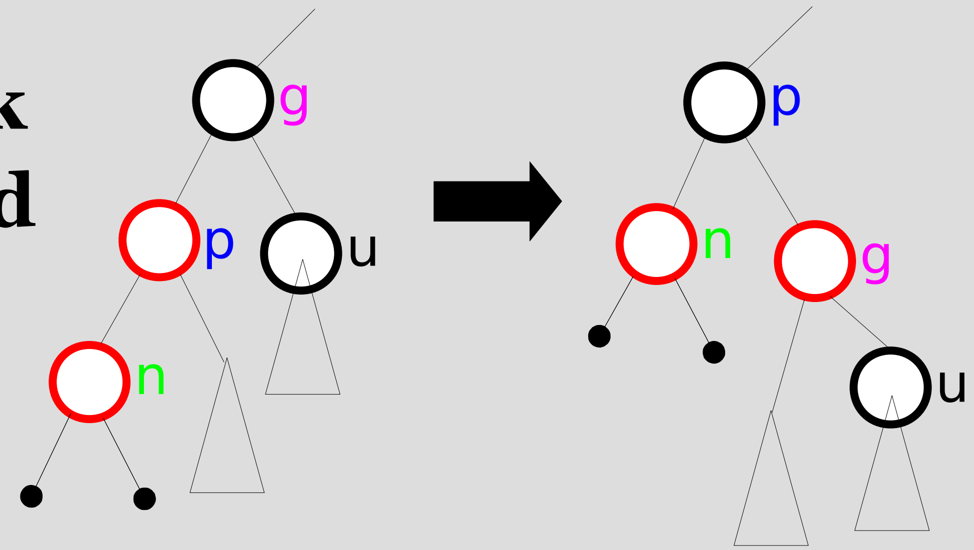
- Case 1
  - n's **uncle** u **is red**
- Action
  - `p.color := black`
  - `u.color := black`
  - `g.color := red`
  - `n := g`



- Note: the tree rooted at g is balanced enough (black depth of all descendants remains unchanged).

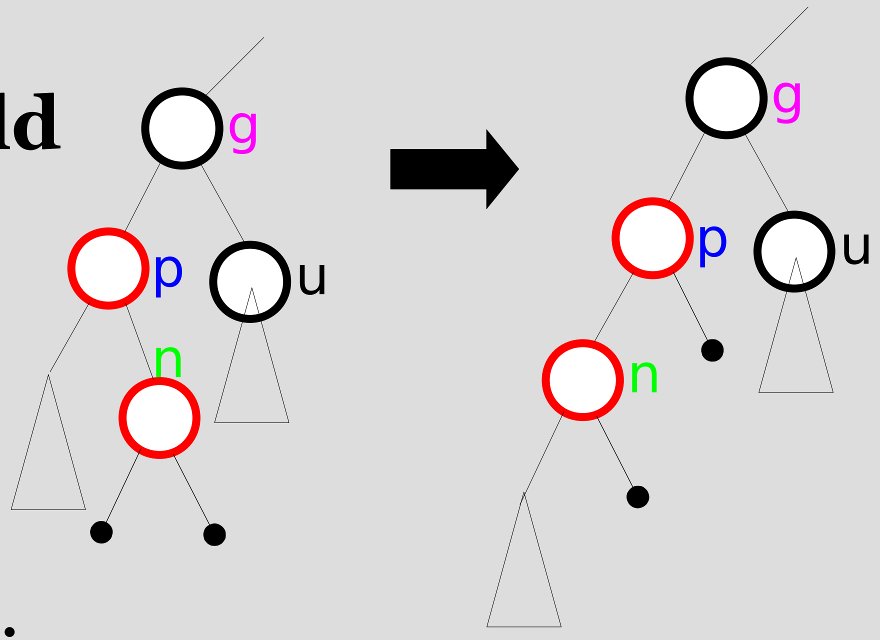
Insertion: Case 2

- Case 2
  - n's uncle u is black and n is a left child
- Action
  - `p.color := black`
  - `g.color := red`
  - `RightRotate(g)`
- Note: the tree rooted at g is balanced enough (black depth of all descendents remains unchanged).



Insertion: Case 3

- Case 3
  - **n's uncle u is black and n is a right child**
- Action
  - `LeftRotate(p)`
  - `n := p`
- Note
  - The result is a case 2.



Insertion: Mirror cases

- All three cases are handled analogously if p is a right child.
- Exchange *left* and *right* in all three cases.

Insertion: Case 2 and 3 mirrored

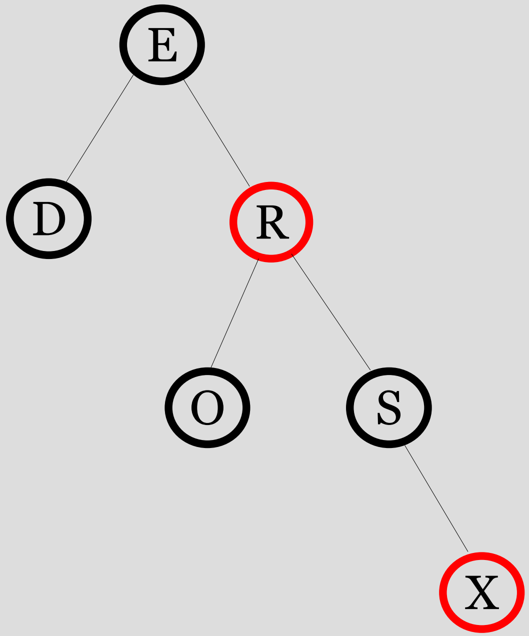
- Case 2m
  - **n's uncle u is black and n is a *right* child**
  - Action
  - `p.color := black`
  - `g.color := red`
  - *Left*Rotate(g)
- Case 3m
  - **n's uncle u is black and n is a *left* child**
  - Action
    - *Right*Rotate(p)
    - `n := p`

Insertion Summary

- If two red nodes are adjacent, we do either
  - **a restructuring** (with one or two rotations) and **stop** (cases 2 and 3), or
  - recursively **propagate** red upwards (case 1)
- A **restructuring** takes constant time and is performed at most once. It reorganizes an off-balanced section of the tree
- **Propagations** may continue up the tree and are executed *O(log n)* times (height of the tree)
- The running time of an insertion is ***O(log n)***.

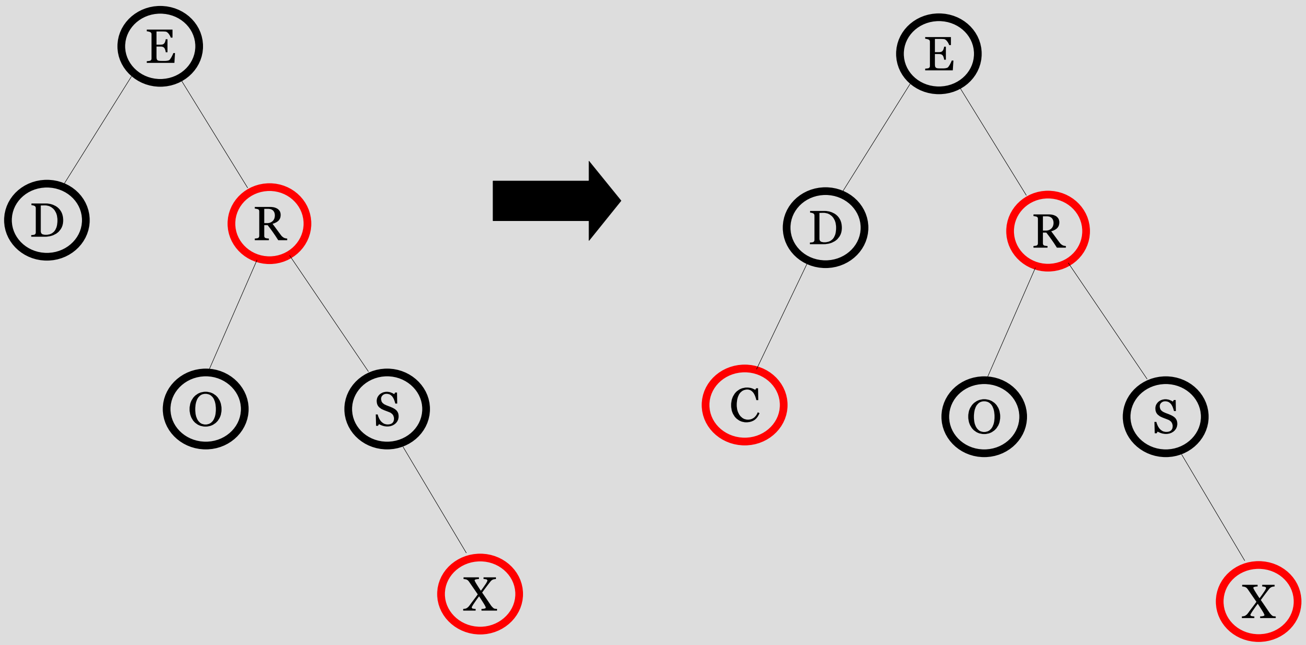
An Insertion Example

- Inserting "REDSOX" into an empty tree

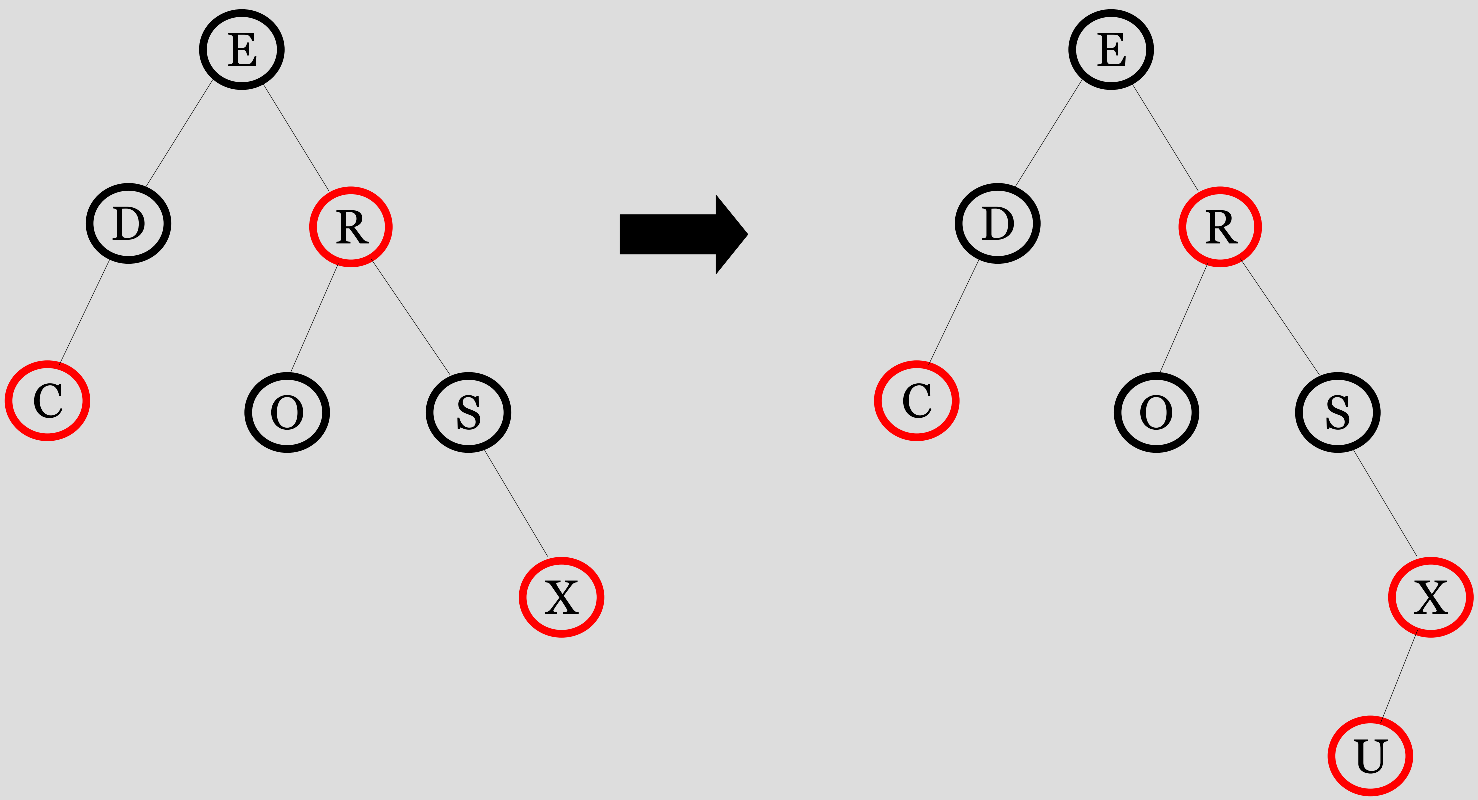


- Now, let us insert "CUBS"

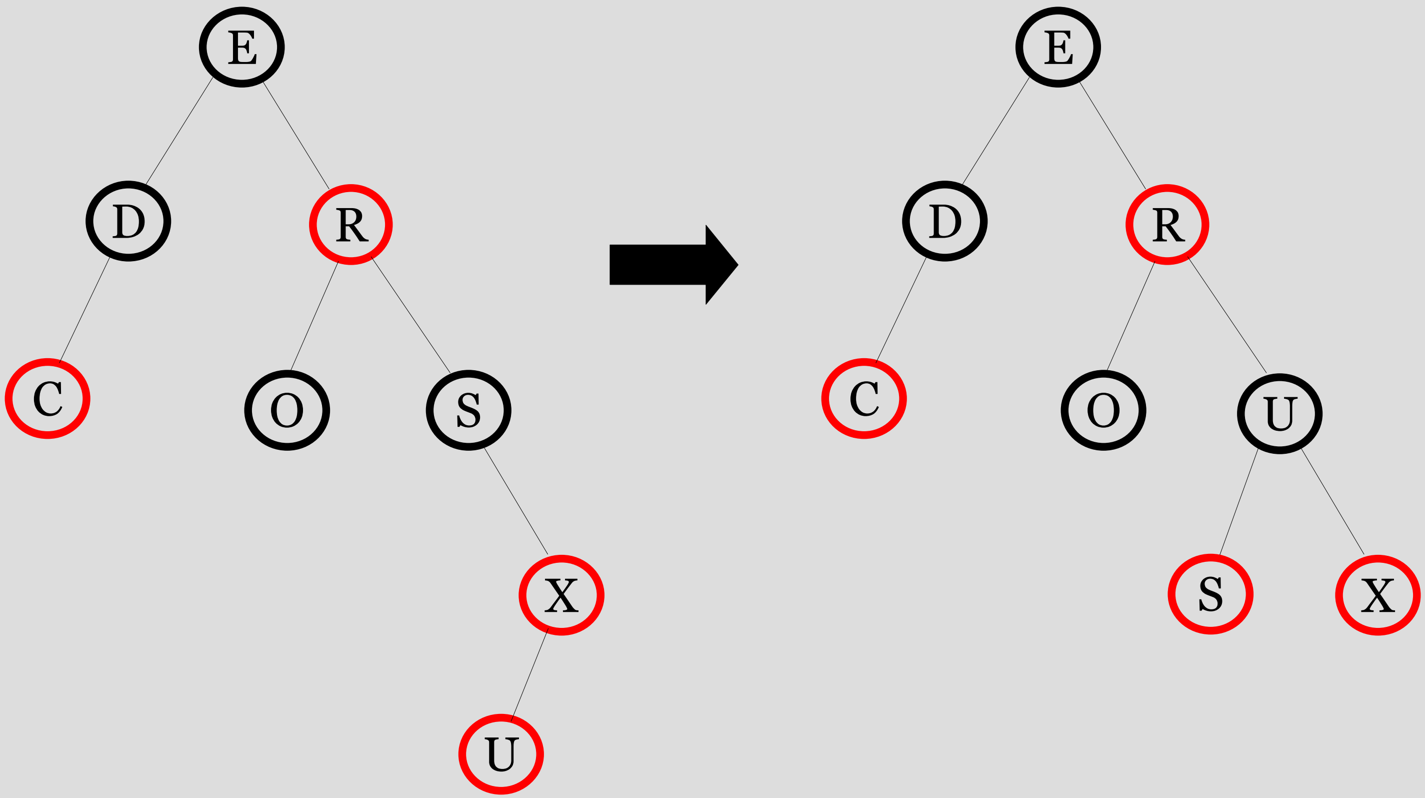
Insert C (case 0)



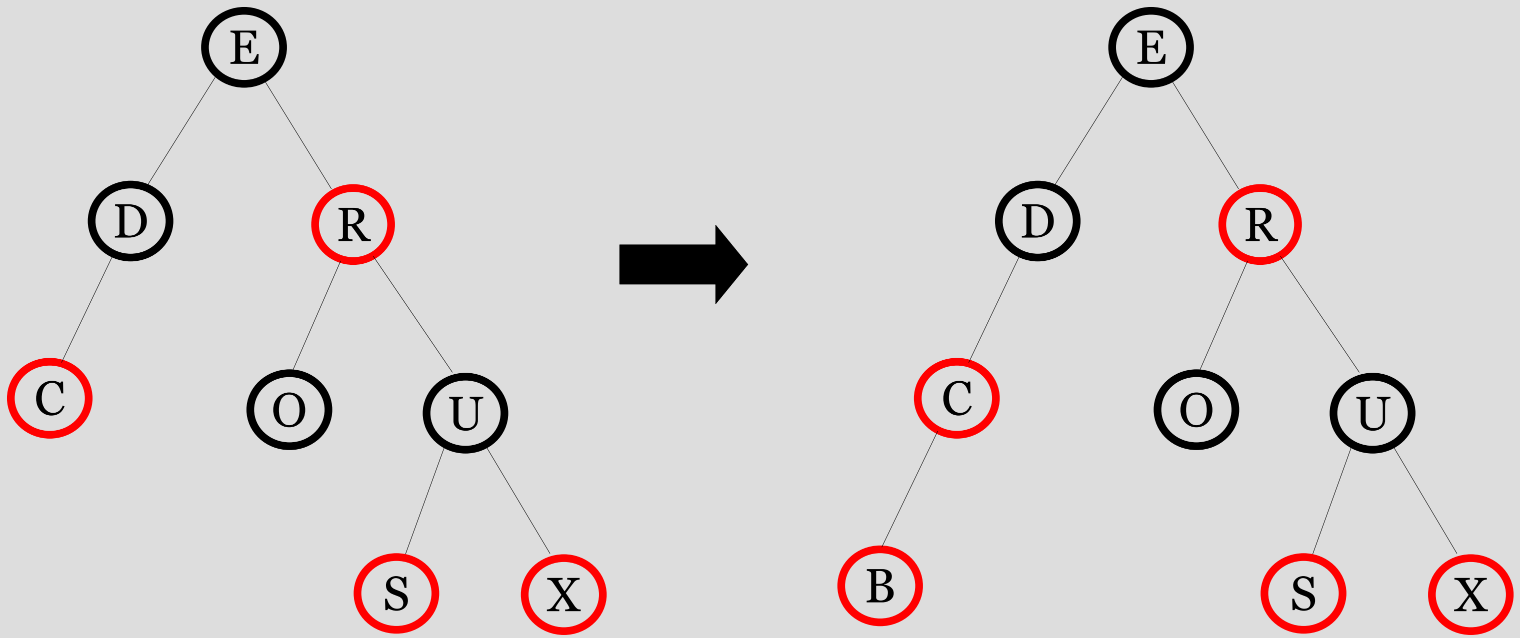
Insert U (case 3, mirror)



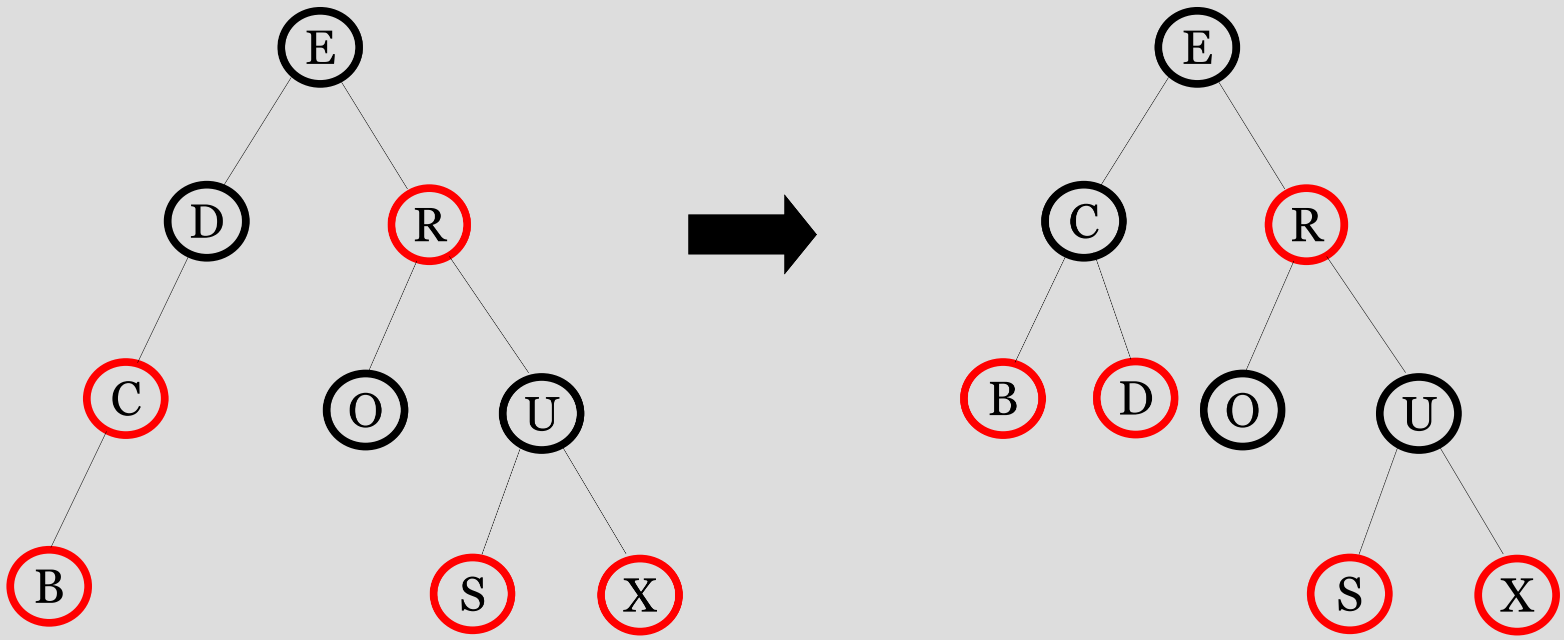
Insert U/2



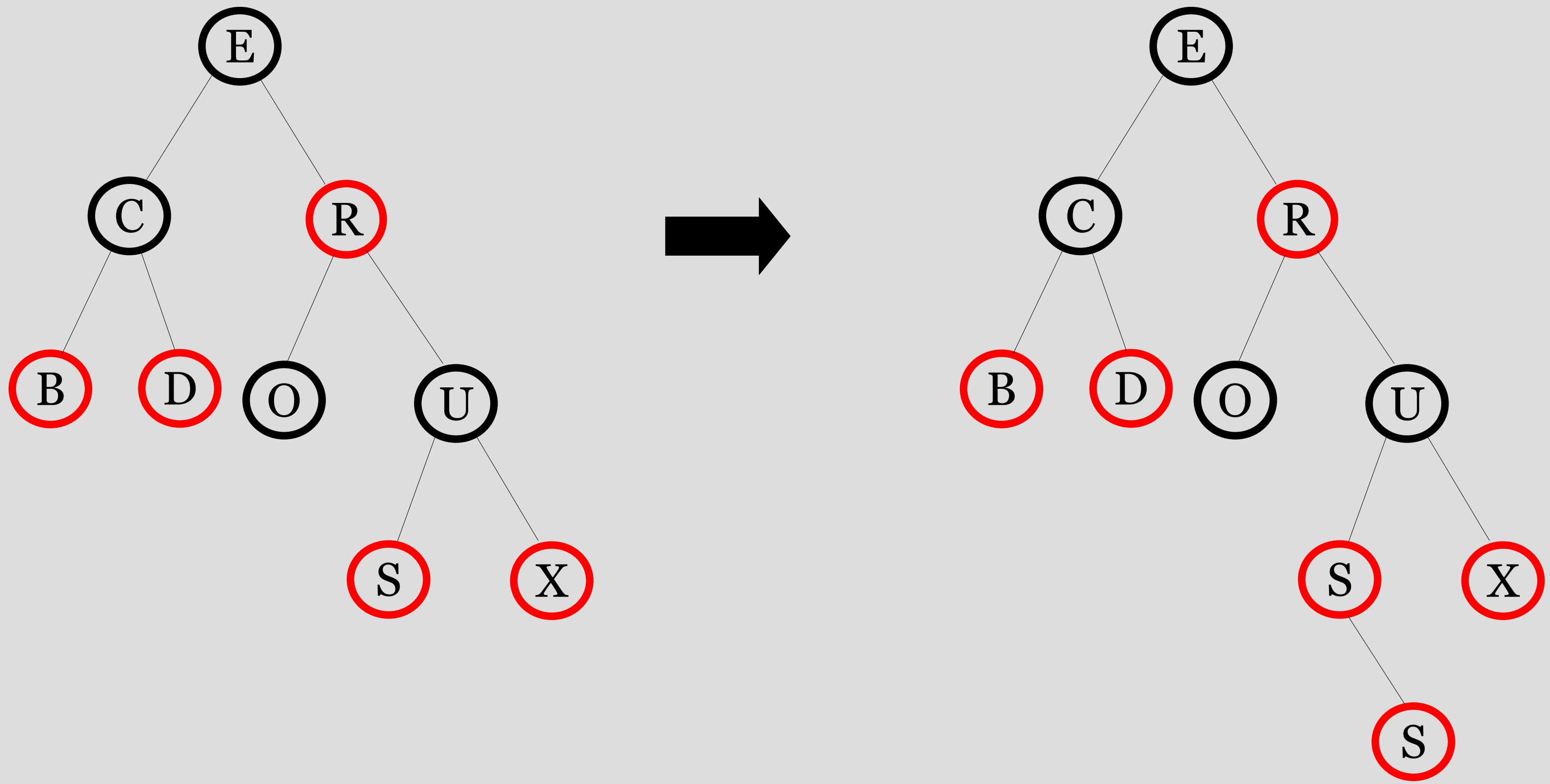
Insert B (case 2)



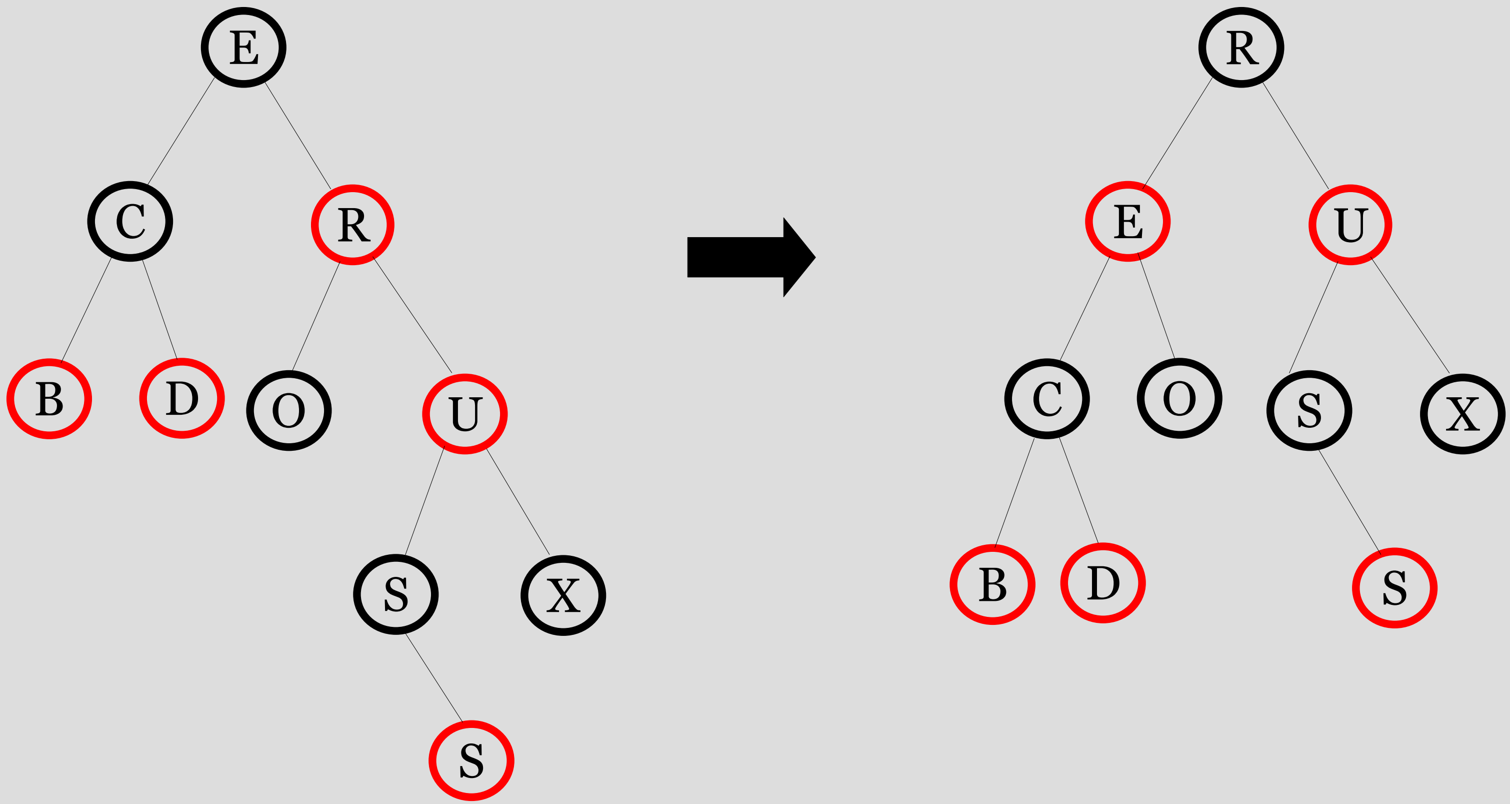
Insert B/2



Insert S (case 1)



Insert S/2 (case 2 mirror)

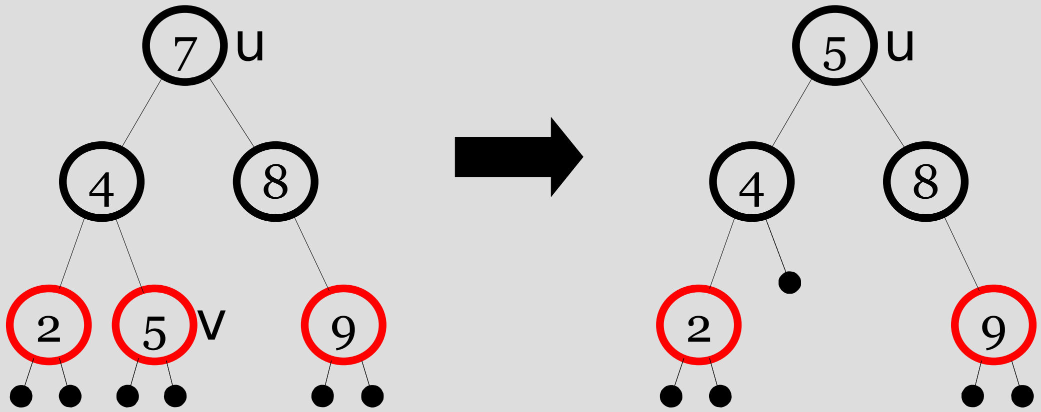


Data Structures and Algorithms
Week 6

- Binary Search Trees
  - Tree traversals
  - Searching
  - Insertion
  - Deletion
- Red-Black Trees
  - Properties
  - Rotations
  - Insertion
  - Deletion

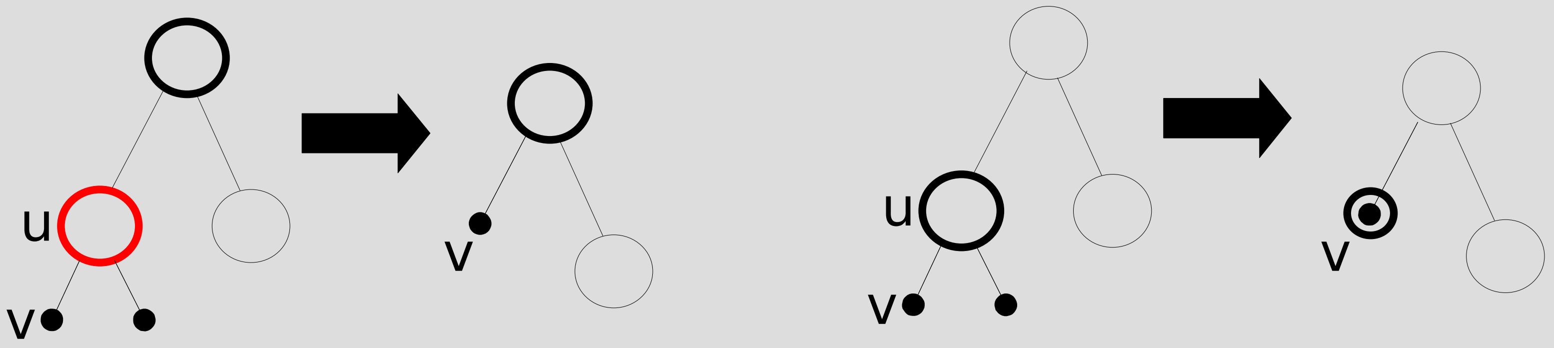
Deletion

- We first apply binary search tree deletion.
  - We can easily delete a node that has at least one *nil* child
  - If the key to be deleted is stored at a node u with two children, we replace its content with the content of the largest node v of the left subtree and delete v instead.



Deletion Algorithm

1. Remove u
2. If u.**color** = red, we are done. Else, assume that v (replacement of u) gets *additional black color*:
   – If v.**color** = red then v.**color :=** black and we are done!
   – Else v's color is **"double black"**.

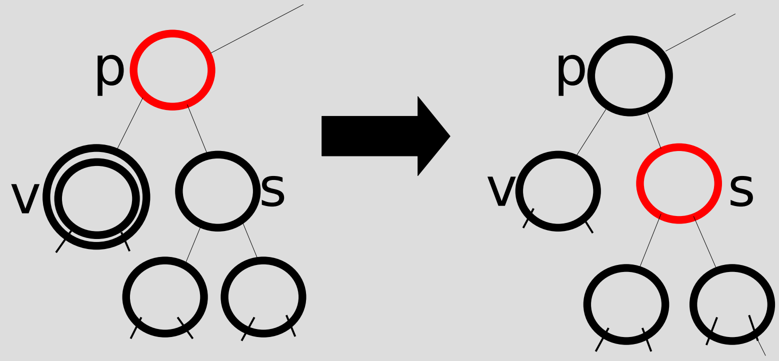


Deletion Algorithm/2

- How to eliminate double black edges?
  - The intuitive idea is to perform a **color compensation**
    - Find a red edge nearby, and change the pair (**red**, **double black**) into (**black, black**)
  - Two cases: **restructuring** and **recoloring**
  - Restructuring resolves the problem locally, while recoloring may propagate it upward.
- Hereafter we assume v is a left child (swap right and left otherwise)

Deletion Case 1

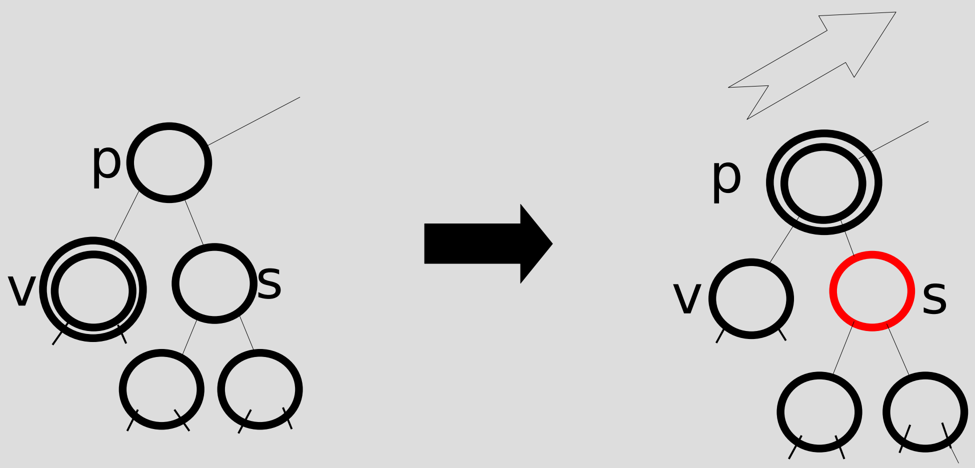
- Case 1
  - **v's sibling s is black and both children of s are black**
- Action
  - `s.color := red`
  - `v = p`



- Note
  - We reduce the black depth of both subtrees of p by 1. Parent p becomes more black.

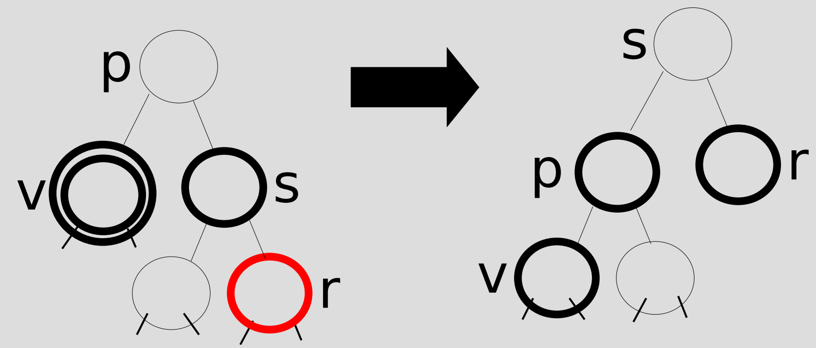
Deletion: Case 1

- If parent p becomes **double black**, continue upward.



Deletion: Case 2

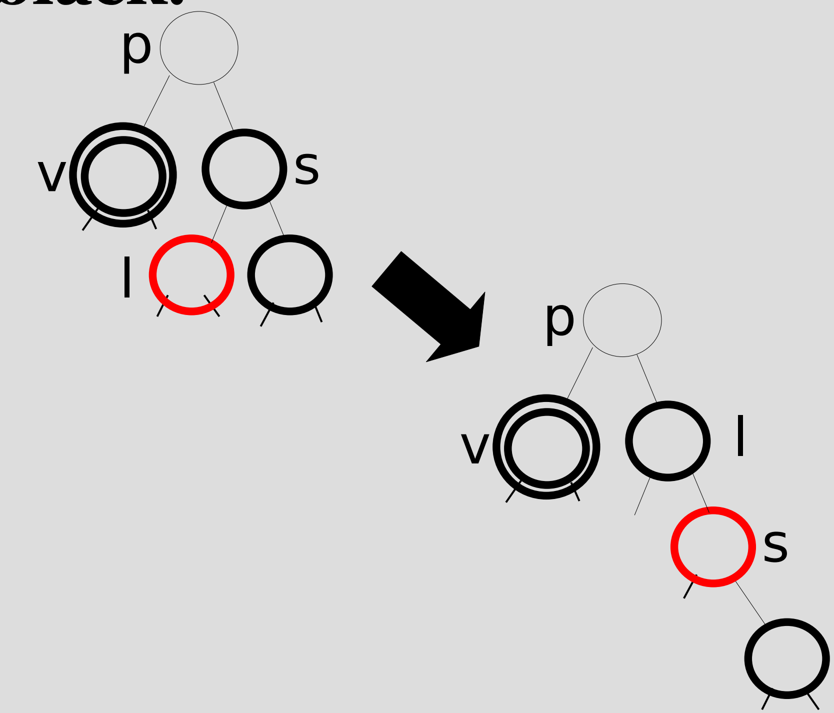
- Case 2
  - **v's sibling s is black and s's right child is red.**
- Action
  - `s.color = p.color`
  - `p.color = black`
  - `s.right.color = black`
  - `LeftRotate(p)`



- Idea: Compensate the extra black ring of v by the red of r
- Note: Terminates after restructuring.

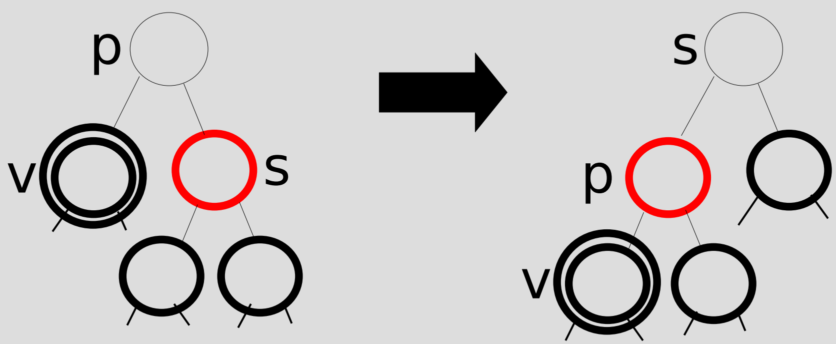
Deletion: Case 3

- Case 3
  - **v's sibling s is black, s's left child is red, and s's right child is black.**
- Idea: Reduce to case 2
- Action
  - `s.left.color = black`
  - `s.color = red`
  - `RightRotation(s)`
  - `s = p.right`
- Note:
  - This is now case 2



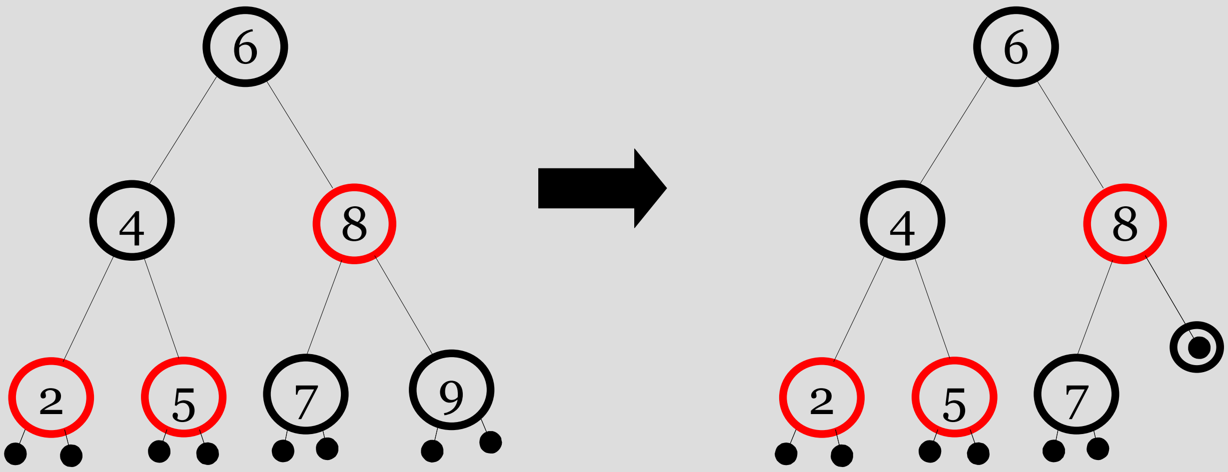
Deletion: Case 4

- Case 4
  - **v's sibling s is red**
- Idea: give v a black sibling
- Action
  - `s.color = black`
  - `p.color = red`
  - *`Left`*`Rotation(p)`
  - `s = p.right`



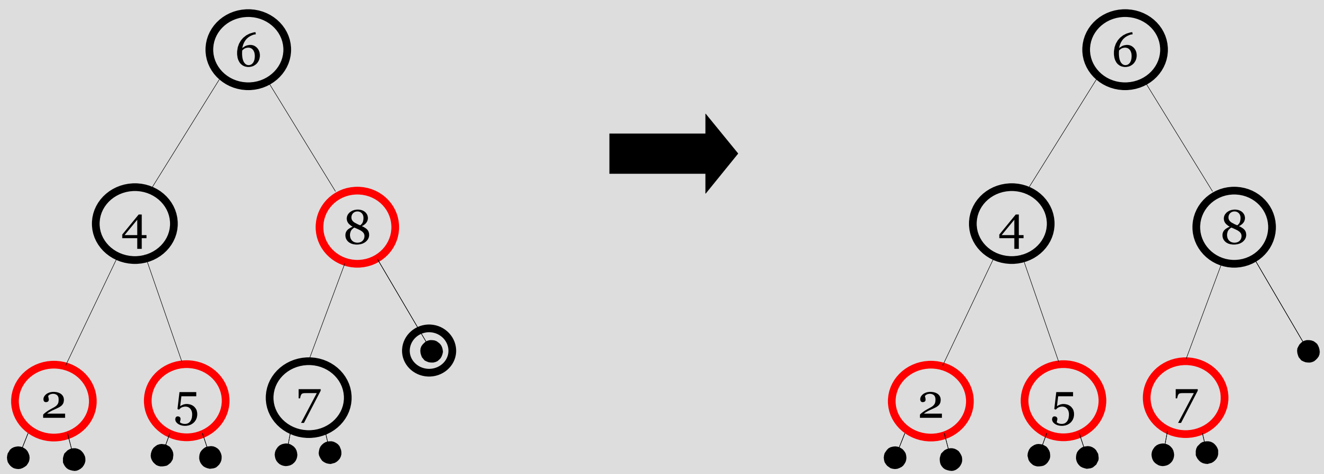
- Note
  - This is now a case 1, 2, or 3

Delete 9

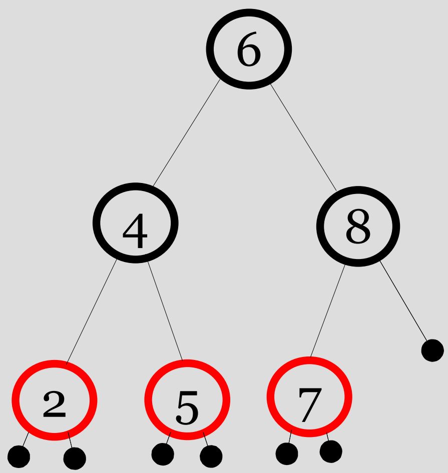


Delete 9/2

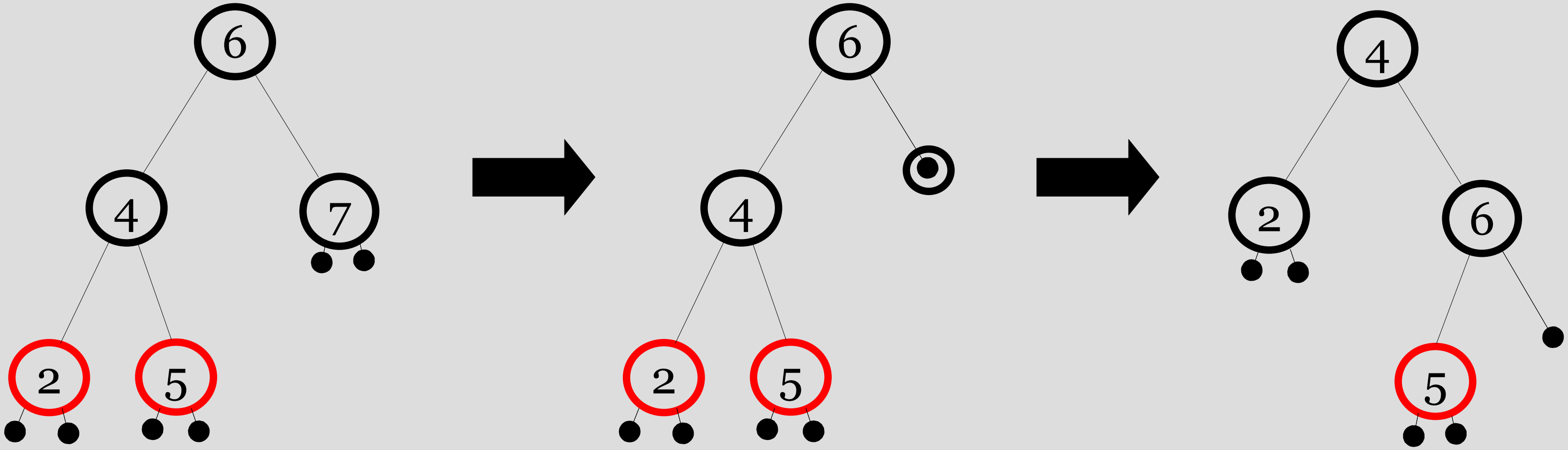
- Case 2 (sibling is black with black children) – recoloring



Delete 8



Delete 7: restructuring



How long does it take?

- Deletion in a RB-tree takes $O(\log n)$
  - Maximum three rotations and $O(\log n)$ recolorings

Suggested exercises

- Add left-rotate and right-rotate to the implementation of binary trees
- Implement a red-black search tree with the following functionalities:
  - (...), insert, delete

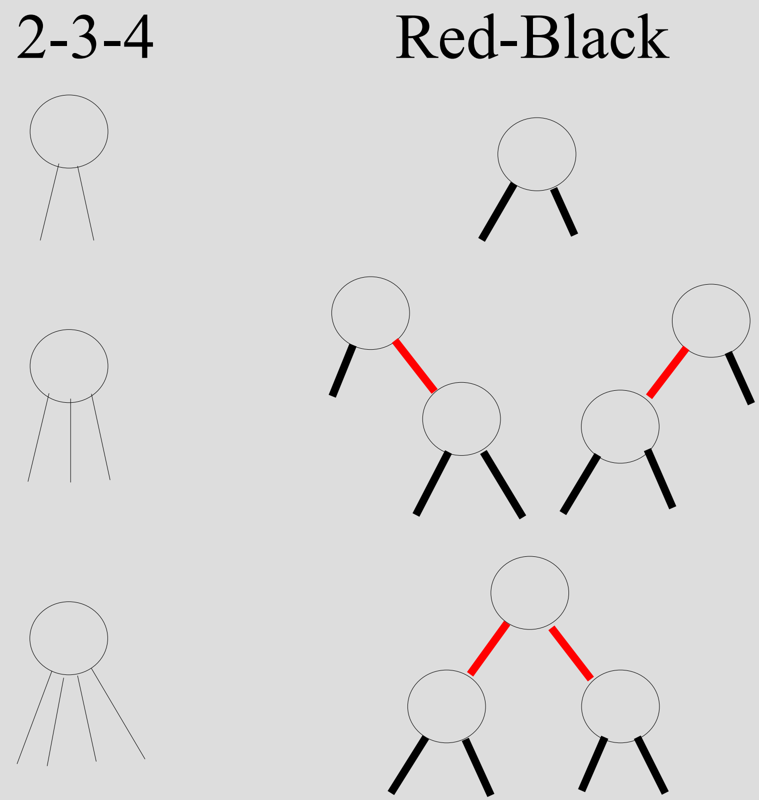
Suggested exercises/2

Using paper & pencil:

- draw the RB-trees after each of the following operations, starting from an empty tree:
  1. Insert 1,2,3,4,5,6,7,8,9,10,11,12
  2. Delete 12,11,10,9,8,7,6,5,4,3,2,1
- Try insertions and deletions at random

Other Balanced Trees

- Red-Black trees are related to 2-3-4 trees (non-binary)
- AVL-trees have simpler algo-rithms, but may perform a lot of rotations



Next Week

- Hashing