2. Finite Satisfiability

1. Satisfiable Formulas Without Finite Model

Write down a first-order logic formula that is satisfiable, but does not have any finite model.

Hint 1: Think of a chain that (i) has a starting point, (ii) has always a next member, and (iii) does not loop.

Hint 2: What can you say about a set with a function that is injective, but not surjective?

2. Axiomatizing Finiteness

Write down a second order formula φ such that

1. all models of φ are finite;
2. for every number n, there is a model I of φ such that $|\Delta^I| \geq n$.

3. Compactness of First-Order Logic

Show the following claim:

> Let Φ be a (possibly infinite) set of first-order formulas. If Φ is unsatisfiable, then there is a finite subset $\Phi_0 \subseteq \Phi$ that is unsatisfiable.

Hint: Think of the resolution calculus and the refutation completeness of resolution.

4. Finiteness Cannot be Axiomatized in FOL

Let φ be a first-order formula such that every model of φ is finite (that is, has a finite domain). Then there is a number n such that every model of φ has at most n elements.

Hint: Think of closed formulas ψ_n such that ψ_n is satisfied by exactly those interpretations that have at least n elements.

5. FOL$_{\mathit{fin}}$ Does Not Satisfy the Compactness Theorem

Show that there is an infinite set Φ of first-order formulas such that Φ has no finite model, but every finite subset of Φ has a finite model.
What can you conclude from this about a possible sound and complete calculus?