Ex. 36 — Prove the following claim.

Claim 1. Uniform substitution preserves validity.
The above Claim 1 is implied by the following claim.

Claim 2. For all $\phi$ in $ML(P)$, $ML(P)$-frame $F$,

$$\text{for all } p \text{ and } \theta \text{ in } ML(P), \text{ if } F \models \phi \text{ then } F \models \phi(p/\theta).$$

(3)

Now, proving (3) by contraposition means proving the following claim.

Claim 3. If, for all $p$ and $\theta$ in $ML(P)$, and for some $M$ based on $F$ and $w$ in $F$, $M, w \not\models \phi(p/\theta)$ then there exists $M'$ based on $F$ and $w'$ in $F$ such that $M', w' \not\models \phi$. In turn, this follows from the following more general claim:

Claim 4. If, for all $p$ and $\theta$ in $ML(P)$, and for all $M$ based on $F$ there exists $M'$ based on $F$ with the following property:

$$M, w \models \phi \iff M', w \models \phi(p/\theta).$$

(4)

Prove the latter claim and work out the logical dependencies among the above claims.

Answer (ex. 36) — Now, take any $M = (W, R, V)$ in $F$. Define its variant $M' = (W, R, V')$ as follows:

$$V'(p) = \{ w \in W \mid M, w \models \theta \},$$

$$V'(q) = V(q) \text{ for } q \neq p \text{ in } P.$$  

(4)

If one proves that

$$M', w \models \phi \iff M, w \models \phi(p/\theta)$$

(5)

then Claim 4 follows. We now prove (5) by induction on $\phi$.

Induction basis. The base case means proving (5) with $\phi$ equal to a proposition letter. This follows from (4) — students are invited to spell it out in details.

Induction step. We distinguish three cases, one for each primitive logical symbol of the basic modal language.

Case 1 means proving (5) with $\phi = \lnot \phi'$. Now, $M', w \models \lnot \phi' \iff (\text{by definition of satisfiability})$ $M', w \not\models \phi' \iff (\text{IH on } \phi')$ $M, w \not\models \phi'(p/\theta) \iff (\text{by definition of satisfiability})$ $M, w \models \lnot \phi'(p/\theta)$.

Case 2 means proving (5) with $\phi = \psi \land \psi'$. Now, $M', w \models \psi \land \psi' \iff (\text{by definition of satisfiability})$ $M', w \models \psi$ and $M, w \models \psi' \iff (\text{IH on } \psi$ and $\psi')$ $M, w \models \psi(p/\theta)$ and $M, w \models \psi'(p/\theta) \iff (\text{by definition of substitution})$ $M, w \models (\psi \land \psi')(p/\theta)$.
Case 3 means proving (5) with $\phi = \Diamond \phi'$. Now, $M', w \models \Diamond \phi'$ iff (by definition of satisfiability) there is $v$ in $W$ and $Rvw$ and $M', v \models \phi'$ iff (by IH on $\phi'$ and the fact that $M$ and $M'$ are based on the same frame $(W, R)$) there is $v$ in $W$ and $Rvw$ and $M, v \models \phi'(p/\theta)$ iff (by definition of satisfiability) $M, w \models \Diamond \psi(p/\theta)$.

Ex. 37 — Definability of a class of frames.

1. Prove that $\Box p$ defines the class of completely disconnected frames: 
   $$\forall x \forall y \neg Rxy.$$ 

2. Prove that $\Box (\Box p \rightarrow q) \lor \Box (\Box q \rightarrow p)$ defines the class of piecewise connected frames: 
   $$\forall x \forall y (Rxy \land Ryz \rightarrow Rxz \lor Rzx).$$

3. Conclude Example 3.6 of your textbook.

Answer (ex. 37) — 

3. We prove by contraposition that if $F$ validates the Löb formula then $F$ is transitive. Non transitivity yields that $Rvu$, $Rvu$ and $\neg Rwu$ for some $w, v, u$ of $F$. Let us consider the following formula, equivalent to the Löb formula: $\Diamond p \rightarrow \Diamond (\Box \neg p \land p)$ (why can we consider this instead of the Löb formula?). Next, let us define $M$ with $V(p) = \{v, u\}$ over $F$. Now, $M, v \models p$ hence $M, w \models \Diamond p$. If $M, w \models \Diamond (\Box \neg p \land p)$ then there exists $z$ with $M, z \models p$ and $Rwz$ (1), and $M, z \models \Box \neg p$. By definition of $V$ and $R$, (1) gives $z = v$. But $M, v \not\models \Box \neg p$ since $M, u \models p$. Therefore $M, w \not\models \Diamond (\Box \neg p \land p)$, and hence $F$ does not validate the Löb formula.

Ex. 39 — Definability properties. Let $ML(P)$ be the basic modal language over $P$, $\mathcal{F}_1$ and $\mathcal{F}_2$ be two classes of frames for it.

1. Assume that $\Sigma_1$ defines $\mathcal{F}_1$ and $\Sigma_2$ defines $\mathcal{F}_2$. Then, what class of frames does $\Sigma_1 \cup \Sigma_2$ define? Prove your statement.

2. What is the set of $ML(P)$ formulas which defines the class of reflexive and transitive frames?

Answer (ex. 39) — (1) follows from this:

$F \models \phi$ for all $\phi \in \Sigma_1 \cup \Sigma_2$

iff

$F \models \phi$ for all $\phi \in \Sigma_1$ and for all $\phi \in \Sigma_2$

iff

$F \in \mathcal{F}_1 \cap \mathcal{F}_2$. 

2
Ex. 41 — Non definability. Let $ML(P)$ be the basic modal language. Prove the following claims.

1. The class of frames with precisely $n \geq 1$ states is not definable in $ML(P)$.
2. The class of frames each state of which has at most one $R$-successor, that is,
   \[
   \forall x \forall y \forall z (Rxy \land Rxz \rightarrow z = y),
   \]
   is not definable in $ML(P)$.
3. The class of non-reflexive frames (i.e., $\exists x \neg Rxx$) is not definable in $ML(P)$.

Answer (ex. 41) —

1. Consider a frame $F$, and $F \sqcup F$.
2. Take $F = (\{0, 1, 0', 1'\}, \{(0, 1), (0', 1')\})$ and $G = (\{a, b, b'\}, \{(a, b), (a, b')\})$.
   Now, $f(0) = f(0') = a$, $f(1) = b$ and $f(1') = b'$ is a surjective bounded morphism.
   Clearly, it is a hom. Let us check the back condition:
   - if $R_G f(0)b$ then $R_F 01$ with $f(1) = b$;
   - if $R_G f(0')b$ then $R_F 01$ with $f(1) = b$;
   - if $R_G f(0)b'$ then $R_F 01$ with $f(1') = b'$;
   - if $R_G f(0')b'$ then $R_F 01$ with $f(1') = b'$.

Now, $F$ validates the given property but $G$ does not ($a$ has precisely 2 $R$-successors).
Theorem 3.14 yields that the property is not definable in $ML(P)$.
3. Take $N = (\mathbb{N}, S)$. It validates the given property. However, the reflexive state is a bounded morphic image of $N$.

Ex. 42 — Local entailment. Prove that, if $\psi$ is a local semantic consequence over the class all models of $\phi$ (that is, $\phi \models_M \psi$) then $\models \phi \rightarrow \psi$, and vice versa.

Answer (ex. 42) — This follows immediately from the definition of $\models_M$ as students are invited to check.

Ex. 54 — Theorems. Prove the following closure property of the set of theorems of a modal logic $\Omega$.

Claim 6. Let $\Omega$ be a modal logic. If $\vdash_\Omega \phi_0 \rightarrow \phi_1$ and $\vdash_\Omega \phi_1 \rightarrow \phi_2$ then $\vdash_\Omega \phi_0 \rightarrow \phi_2$.
(The rule is derived in $\Omega$.)


**Answer (ex. 54)** — The set of $\Omega$-theorems contains $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$ as a propositional tautology. The set is closed for uniform substitution, and hence $\vdash_\Omega (\phi_0 \rightarrow \phi_1) \rightarrow ((\phi_1 \rightarrow \phi_2) \rightarrow (\phi_0 \rightarrow \phi_2)) \ (\ast)$. Now, assume that $\vdash_\Omega \phi_0 \rightarrow \phi_1$. From this and $(\ast)$, we obtain $\vdash_\Omega (\phi_1 \rightarrow \phi_2) \rightarrow (\phi_0 \rightarrow \phi_2) \ (\odot)$ by modus ponens. Next, assume that $\vdash_\Omega \phi_1 \rightarrow \phi_2$. From this and $(\odot)$, we obtain $\vdash_\Omega \phi_0 \rightarrow \phi_2$ by modus ponens.

**Ex. 55 — Consistency and negation.** Let $\Lambda$ be a modal logic. Suppose that $\Sigma$ is $\Lambda$-consistent. Then prove the following: $\Sigma \cup \{\phi\} \vdash_\Lambda \bot \iff \Sigma \vdash_\Lambda \neg \phi$.

**Answer (ex. 55)** — $\Sigma \cup \{\phi\} \vdash_\Lambda \bot \iff \Sigma \vdash_\Lambda \phi \rightarrow \bot$ by the deduction theorem. Let us now show that if $\Sigma \vdash_\Lambda \phi \rightarrow \bot$ then $\Sigma \vdash_\Lambda \neg \phi$. Assume that $\Sigma \vdash_\Lambda \phi \rightarrow \bot$, that is, $\vdash_\Lambda \bigwedge_{i=1}^{n} \psi_i \rightarrow (\phi \rightarrow \bot)$ where $\psi_i \in \Sigma$ for each $i = 1, \ldots, n$. Now, $\vdash_\Lambda (\phi \rightarrow \bot) \rightarrow \neg \phi$ (propositional tautology and uniform substitution). This and Claim 6 yield that $\Sigma \vdash_\Lambda \neg \phi$. A similar argument proves that $\Sigma \vdash_\Lambda \phi \rightarrow \bot$ only if $\Sigma \vdash_\Lambda \neg \phi$.

**Ex. 56 — Inconsistency.** Prove that the following statements are equivalent, where $\Sigma$ is any set of modal formulas and $\Lambda$ a modal logic:

1. $\Sigma \vdash_\Lambda \bot$;
2. there exists $\psi$ such that $\Sigma \vdash_\Lambda \psi \land \neg \psi$;
3. $\Sigma \vdash_\Lambda \phi$ for all modal formulas $\phi$.

**Answer (ex. 56)** —

1 $\Rightarrow$ 2. Take any formula $\psi$ of $ML(P)$. The tautology instance $\bot \rightarrow \psi \land \neg \psi$ and Claim 6 yield $\Sigma \vdash_\Lambda \psi \land \neg \psi$.

2 $\Rightarrow$ 3. Let be $\psi$ be as in (2), that is, $\Sigma \vdash_\Lambda \psi \land \neg \psi$. Then, for any $\phi$, $\psi \land \neg \psi \rightarrow \phi$ is a tautology instance. This and Claim 6 yield $\Sigma \vdash_\Lambda \phi$.

3 $\Rightarrow$ 1. Take $\phi = \bot$.

**Ex. 57 — Compactness of $\vdash$.** Let $\Lambda$ be any modal logic. Prove that a set of formulas $\Sigma$ is $\Lambda$-consistent iff every subset of $\Sigma$ is such.

**Answer (ex. 57)** — Prove it by contraposition: $\Sigma \vdash_\Lambda \bot \iff$ there exists a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\Sigma_0 \vdash_\Lambda \bot$.

**Ex. 59 — Soundness.** Prove that $S5$ is not sound w.r.t. the class of reflexive
frames. Is $S_5$ sound w.r.t. the class of universal frames (namely, those for which $\forall x \forall y Rxy$ holds)?

**Answer (ex. 59)** — We leave the first statement to students. As for the second statement, observe that $S_5$ is sound for the class of equivalence frames, which strictly includes that of universal frames. Students now should apply the definition of soundness and conclude the proof.

**Ex. 60** — MCS’s. Prove Proposition 4.16 of [BRV] as follows. Let $\Gamma$ be a maximal $\Omega$-consistent set.

i. If $\phi \in \Gamma$ and $\phi \rightarrow \psi \in \Gamma$ then $\psi \in \Gamma$.

ii. If $Th(\Omega)$ is the set of $\Omega$-theorems then $Th(\Omega) \subseteq \Gamma$.

iii. For every $\phi \in ML(P)$, either $\phi \in \Gamma$ or $\neg \phi \in \Gamma$.

iv. For every $\phi \vee \psi \in ML(P)$, $\phi \vee \psi \in \Gamma$ iff $\phi \in \Gamma$ or $\psi \in \Gamma$.

**Answer (ex. 60)** — The first two items of that proposition immediately follow from this claim:

**Claim 7.** Let $\Gamma$ be a maximally $\Omega$-consistent set. $\Gamma \vdash_\Omega \phi$ iff $\phi \in \Gamma$.

**Proof.**

*Right-to-left:* $\vdash_\Omega \phi \rightarrow \phi$ (1) because $p \rightarrow p$ is a tautology and the set of $\Omega$-theorems is closed for uniform substitution. The definition of $\Omega$-theorem with premises from $\Gamma$, to which $\phi$ belongs, yields $\Gamma \vdash_\Omega \phi$ (note that we did not need maximality here).

*Left-to-right:* we prove it by reductio ad absurdum. If $\Gamma \vdash_\Omega \phi$ then $\vdash_\Omega \bigwedge_{i=1}^{n} \psi'_i \rightarrow \phi$ where the $\psi'_i$ are $\Gamma$ formulas. If $\phi \notin \Gamma$ then maximality yields $\Gamma \cup \{\phi\} \vdash_\Omega \bot$, and hence $\vdash_\Omega \phi \rightarrow \bigwedge_{i=1}^{n} \psi'_i \rightarrow \bot$ where the $\psi_i$ are $\Gamma$ formulas. Claim 6 now yields $\vdash_\Omega \bigwedge_{i=1}^{n} \psi'_i \rightarrow \bigwedge_{i=1}^{n} \psi_i \rightarrow \bot$, that is, $\Gamma$ is $\Omega$-inconsistent. 

As for Item (i), note that $\phi \rightarrow \psi \in \Gamma$ yields (by Claim 7) that $\Gamma \vdash_\Omega \phi \rightarrow \psi$. This is true iff there exists $\bigwedge_{i=1}^{n} \psi_i$ with $\psi_i \in \Gamma$ for all $i = 1 \ldots n$ and $\vdash_\Omega \bigwedge_{i=1}^{n} \psi_i \rightarrow (\phi \rightarrow \psi)$, that is, $\vdash_\Omega \bigwedge_{i=1}^{n} \psi_i \land \phi \rightarrow \psi$ (*). If $\phi \in \Gamma$ then (*) yields that $\Gamma \vdash_\Omega \psi$, and hence (by Claim 7) $\psi \in \Gamma$.

Item (ii) follows from Claim 7 and the fact that if $\vdash_\Omega \phi$ then $\Gamma \vdash_\Omega \phi$.

As for Item (iii), consistency and Item (i) (with $\psi = \bot$) yield that $\phi$ and $\neg \phi$ cannot both belong to $\Gamma$. Assume that $\phi \notin \Gamma$. Then maximality yields $\Gamma \cup \{\phi\} \vdash_\Omega \bot$, and hence (Consistency and Negation Exercise) $\Gamma \vdash_\Omega \neg \phi$. The above claim yields that $\neg \phi \in \Gamma$. A similar argument holds that if $\neg \phi \notin \Gamma$ then $\phi \in \Gamma$. 

5
Item (iv) follows similarly. The propositional tautology \( p \rightarrow p \lor q \) and uniform substitution yield \( \vdash_\Omega \phi \rightarrow \phi \lor \psi \). Item (ii) gives \( \phi \rightarrow \phi \lor \psi \in \Gamma \). If \( \phi \in \Gamma \) then Item (i) yields \( \phi \lor \psi \in \Gamma \). A similar argument gives the same conclusion under the assumption \( \psi \in \Gamma \). Vice versa, assume that \( \phi \lor \psi \in \Gamma \) and \( \phi \not\in \Gamma \), that is, \( \neg \phi \in \Gamma \) by Item (iii). The propositional tautology \( p \lor q \rightarrow (\neg p \rightarrow q) \) and uniform substitution yield \( \vdash_\Omega \phi \lor \psi \rightarrow (\neg \phi \rightarrow \psi) \). Item (ii) and two applications of Item (i) give \( \psi \in \Gamma \).

**Ex. 61 — Strong Completeness.** Prove that \( S5 \) is strongly complete w.r.t. the class of universal frames.

**Answer (ex. 61) —** Let \( ML(P) \) be the basic modal language. \( S5 \) is strongly complete w.r.t. the class of frames the accessibility relation of which is of equivalence. In particular, the canonical model \( M^{S5} \) is based on such a frame. Take any \( \Gamma \)-consistent set of \( ML(P) \) formulas and its \( S5 \)-MCS extension (which exists due to the Lindebaum lemma). Take the submodel \( M^{S5}_{\Gamma^+} \) of \( M^{S5} \) generated by \( \Gamma^+ \); since \( R^{S5} \) is of equivalence, \( M^{S5}_{\Gamma^+} \) is based on a universal frame. By the satisfiability preservation theorem on generated submodels, we know that \( M^{S5}_{\Gamma^+} \models \Gamma \) iff \( M^{S5}_{\Gamma^+} \models \Gamma \). The canonical model theorem states that \( M^{S5}_{\Gamma^+} \models \Gamma \); thus \( M^{S5}_{\Gamma^+} \models \Gamma \).