Modal Logic: Exercises

KRDB FUB stream course

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Ex. 36 — Prove the following claim.
Claim 1. Uniform substitution preserves validity.
The above Claim 1 is implied by the following claim.
Claim 2. For all ϕ in $ML(P)$, $ML(P)$-frame F,

$$\text{for all } p \text{ and } \theta \text{ in } ML(P), \text{ if } F \models \phi \text{ then } F \models \phi(p/\theta). \tag{3}$$

Now, proving (3) by contraposition means proving the following claim.
Claim 3. If, for all p and θ in $ML(P)$, and for some M based on F and w in F, $M, w \not\models \phi(p/\theta)$ then there exists M' based on F and w' in F such that $M', w' \not\models \phi$.
In turn, this follows from the following more general claim:
Claim 4. If, for all p and θ in $ML(P)$, and for all M based on F there exists M' based on F with the following property:

$$M, w \models \phi \text{ iff } M', w \models \phi(p/\theta).$$

Prove the latter claim and work out the logical dependencies among the above claims.

Answer (ex. 36) — Now, take any $M = (W, R, V)$ in F. Define its variant $M' = (W, R, V')$ as follows:

$$V'(p) = \{w \in W \mid M, w \models \theta\}, \tag{4}$$
$$V'(q) = V(q) \text{ for } q \neq p \text{ in } P.$$

If one proves that

$$M', w \models \phi \text{ iff } M, w \models \phi(p/\theta) \tag{5}$$

then Claim 4 follows. We now prove (5) by induction on ϕ.

Induction basis. The base case means proving (5) with ϕ equal to a proposition letter. This follows from (4) — students are invited to spell it out in details.

Induction step. We distinguish three cases, one for each primitive logical symbol of the basic modal language.

Case 1 means proving (5) with $\phi = \neg\phi'$. Now, $M', w \models \neg\phi'$ iff (by definition of satisfiability) $M', w \not\models \phi'$ iff (IH on ϕ') $M, w \not\models \phi'(p/\theta)$ iff (by definition of satisfiability) $M, w \models \neg\phi'(p/\theta)$.

Case 2 means proving (5) with $\phi = \psi \wedge \psi'$. Now, $M', w \models \psi \wedge \psi'$ iff (by definition of satisfiability) $M', w \models \psi$ and $M', w \models \psi'$ iff (IH on ψ and ψ') $M, w \models \psi(p/\theta)$ and $M, w \models \psi'(p/\theta)$ iff (by definition of satisfiability) $M, w \models \psi(p/\theta) \wedge \psi'(p/\theta)$ iff (by definition of substitution) $M, w \models (\psi \wedge \psi')(p/\theta)$.

Case 3 means proving (5) with $\phi = \Diamond\phi'$. Now, $M', w \models \Diamond\phi'$ iff (by definition of satisfiability) there is v in W and Rwv and $M', v \models \phi'$ iff (by IH on ϕ' and the fact that M and M' are based on the same frame (W, R)) there is v in W and Rwv and $M, v \models \phi'(p/\theta)$ iff (by definition of satisfiability) $M, w \models \Diamond\psi(p/\theta)$.

Ex. 37 — Definability of a class of frames.

1. Prove that $\Box p$ defines the class of completely disconnected frames: $\forall x \forall y \neg Rxy$.
2. Prove that $\Box(\Box p \to q) \vee \Box(\Box q \to p)$ defines the class of piecewise connected frames: $\forall x \forall y (Rxy \wedge Ryz \;\to\; Rxz \vee Rzx)$.
3. Conclude Example 3.6 of your textbook.

Answer (ex. 37) —
3. We prove by contraposition that if F validates the Löb formula then F is transitive. Non transitivity yields that Rwv, Rvu and $\neg Rwu$ for some w, v, u of F. Let us consider the following formula, equivalent to the Löb formula: $\Diamond p \to \Diamond(\Box\neg p \wedge p)$ (why can we consider this instead of the Löb formula?). Next, let us define M with $V(p) = \{v, u\}$ over F. Now, $M, v \models p$ hence $M, w \models \Diamond p$. If $M, w \models \Diamond(\Box\neg p \wedge p)$ then there exists z with $M, z \models p$ and Rwz (1), and $M, z \models \Box\neg p$. By definition of V and R, (1) gives $z = v$. But $M, v \not\models \Box\neg p$ since $M, u \models p$. Therefore $M, w \not\models \Diamond(\Box\neg p \wedge p)$, and hence F does not validate the Löb formula.

Ex. 39 — Definability properties. Let $ML(P)$ be the basic modal language over P, $\mathcal{F}_1$ and $\mathcal{F}_2$ be two classes of frames for it.

1. Assume that Σ_1 defines $\mathcal{F}_1$ and Σ_2 defines $\mathcal{F}_2$. Then, what class of frames does $\Sigma_1 \cup \Sigma_2$ define? Prove your statement.
2. What is the set of $ML(P)$ formulas which defines the class of reflexive and transitive frames?

Answer (ex. 39) — (1) follows from this:

$$F \models \phi \text{ for all } \phi \in \Sigma_1 \cup \Sigma_2$$

iff

$$F \models \phi \text{ for all } \phi \in \Sigma_1 \text{ and for all } \phi \in \Sigma_2$$

iff

$$F \in \mathcal{F}_1 \cap \mathcal{F}_2.$$

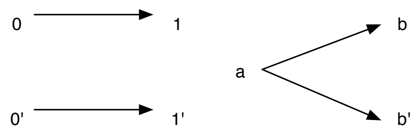
Students are now asked to answer (2).

Ex. 41 — Non definability. Let $ML(P)$ be the basic modal language. Prove the following claims.

1. The class of frames with precisely $n \geq 1$ states is not definable in $ML(P)$.
2. The class of frames each state of which has *at most one* R-successor, that is,
$$\forall x \forall y \forall z (Rxy \wedge Rxz \rightarrow z = y),$$
is not definable in $ML(P)$.
3. The class of non-reflexive frames (i.e., $\exists x \neg Rxx$) is not definable in $ML(P)$.

Answer (ex. 41) —
1. Consider a frame F, and $F \uplus F \ldots$
2. Take $F = (\{0, 1, 0', 1'\}, \{(0, 1), (0', 1')\})$ and $G = (\{a, b, b'\}, \{(a, b), (a, b')\})$. Now, $f(0) = f(0') = a$, $f(1) = b$ and $f(1') = b'$ is a surjective bounded morphism. Clearly, it is a hom. Let us check the back condition:

•if $R_G f(0) b$ then $R_F 01$ with $f(1) = b$;
•if $R_G f(0') b$ then $R_F 01$ with $f(1) = b$;
•if $R_G f(0) b'$ then $R_F 01$ with $f(1') = b'$;
•if $R_G f(0') b'$ then $R_F 01$ with $f(1') = b'$.



Now, F validates the given property but G does not (a has precisely 2 R-successors). Theorem 3.14 yields that the property is not definable in $ML(P)$.
3. Take $N = (\mathbb{N}, S)$. It validates the given property. However, the reflexive state is a bounded morphic image of $N \ldots$

Ex. 42 — Local entailment. Prove that, if ψ is a local semantic consequence over the class all models of ϕ (that is, $\phi \models_{\mathcal{M}} \psi$) then $\models \phi \rightarrow \psi$, and vice versa.

Answer (ex. 42) — This follows immediately from the definition of $\models_{\mathcal{M}}$ as students are invited to check.

Ex. 54 — Theorems. Prove the following closure property of the set of theorems of a modal logic Ω.
Claim 6. Let Ω be a modal logic. If $\vdash_\Omega \phi_0 \rightarrow \phi_1$ and $\vdash_\Omega \phi_1 \rightarrow \phi_2$ then $\vdash_\Omega \phi_0 \rightarrow \phi_2$. (The rule is *derived* in Ω).

Answer (ex. 54) — The set of Ω-theorems contains $(p \to q) \to ((q \to r) \to (p \to r))$ as a propositional tautology. The set is closed for uniform substitution, and hence $\vdash_\Omega (\phi_0 \to \phi_1) \to ((\phi_1 \to \phi_2) \to (\phi_0 \to \phi_2))$ $(*)$. Now, assume that $\vdash_\Omega \phi_0 \to \phi_1$. From this and $(*)$, we obtain $\vdash_\Omega (\phi_1 \to \phi_2) \to (\phi_0 \to \phi_2)$ $(\odot)$ by modus ponens. Next, assume that $\vdash_\Omega \phi_1 \to \phi_2$. From this and $(\odot)$, we obtain $\vdash_\Omega \phi_0 \to \phi_2$ by modus ponens.

Ex. 55 — Consistency and negation. Let Λ be a modal logic. Suppose that Σ is Λ-consistent. Then prove the following: $\Sigma \cup \{\phi\} \vdash_\Lambda \bot$ iff $\Sigma \vdash_\Lambda \neg\phi$.

Answer (ex. 55) — $\Sigma \cup \{\phi\} \vdash_\Lambda \bot$ iff $\Sigma \vdash_\Lambda \phi \to \bot$ by the deduction theorem. Let us now show that if $\Sigma \vdash_\Lambda \phi \to \bot$ then $\Sigma \vdash_\Lambda \neg\phi$. Assume that $\Sigma \vdash_\Lambda \phi \to \bot$, that is, $\vdash_\Lambda \bigwedge_{i=1}^{n} \psi_i \to (\phi \to \bot)$ where $\psi_i \in \Sigma$ for each $i = 1, \dots, n$. Now, $\vdash_\Lambda (\phi \to \bot) \to \neg\phi$ (propositional tautology and uniform substitution). This and Claim 6 yield that $\Sigma \vdash_\Lambda \neg\phi$. A similar argument proves that $\Sigma \vdash_\Lambda \phi \to \bot$ only if $\Sigma \vdash_\Lambda \neg\phi$.

Ex. 56 — Inconsistency. Prove that the following statements are equivalent, where Σ is any set of modal formulas and Λ a modal logic:

1. $\Sigma \vdash_\Lambda \bot$;
2. there exists ψ such that $\Sigma \vdash_\Lambda \psi \wedge \neg\psi$;
3. $\Sigma \vdash_\Lambda \phi$ for all modal formulas ϕ.

Answer (ex. 56) —
$1 \Rightarrow 2$. Take any formula ψ of $ML(P)$. The tautology instance $\bot \to \psi \wedge \neg\psi$ and Claim 6 yield $\Sigma \vdash_\Lambda \psi \wedge \neg\psi$.
$2 \Rightarrow 3$. Let be ψ be as in (2), that is, $\Sigma \vdash_\Lambda \psi \wedge \neg\psi$. Then, for any ϕ, $\psi \wedge \neg\psi \to \phi$ is a tautology instance. This and Claim 6 yield $\Sigma \vdash_\Lambda \phi$.
$3 \Rightarrow 1$. Take $\phi = \bot$.

Ex. 57 — Compactness of $\vdash$. Let Λ be *any* modal logic. Prove that a set of formulas Σ is Λ-consistent iff every subset of Σ is such.

Answer (ex. 57) — Prove it by contraposition: $\Sigma \vdash_\Lambda \bot$ iff there exists a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\Sigma_0 \vdash_\Lambda \bot$.

Ex. 59 — Soundness. Prove that **S5** is not sound w.r.t. the class of reflexive frames. Is **S5** sound w.r.t. the class of universal frames (namely, those for which $\forall x \forall y Rxy$ holds)?

Answer (ex. 59) — We leave the first statement to students. As for the second statement, observe that **S5** is sound for the class of equivalence frames, which strictly includes that of universal frames. Students now should apply the definition of soundness and conclude the proof.

Ex. 60 — **MCS's.** Prove Proposition 4.16 of [BRV] as follows. Let Γ be a maximal Ω-consistent set.

i. If $\phi \in \Gamma$ and $\phi \to \psi \in \Gamma$ then $\psi \in \Gamma$.

ii. If $Th(\Omega)$ is the set of Ω-theorems then $Th(\Omega) \subseteq \Gamma$.

iii. For every $\phi \in ML(P)$, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.

iv. For every $\phi \vee \psi \in ML(P)$, $\phi \vee \psi \in \Gamma$ iff $\phi \in \Gamma$ or $\psi \in \Gamma$.

Answer (ex. 60) — The first two items of that proposition immediately follow from this claim:
Claim 7. Let Γ be a maximally Ω-consistent set. $\Gamma \vdash_\Omega \phi$ iff $\phi \in \Gamma$.

Proof.
Right-to-left: $\vdash_\Omega \phi \to \phi$ (1) because $p \to p$ is a tautology and the set of Ω-theorems is closed for uniform substitution. The definition of Ω-theorem with premises from Γ, to which ϕ belongs, yields $\Gamma \vdash_\Omega \phi$ (note that we did not need maximality here).
Left-to-right: we prove it by reductio ad absurdum. If $\Gamma \vdash_\Omega \phi$ then $\vdash_\Omega \bigwedge_{i=1}^n \psi'_i \to \phi$ where the ψ'_i are Γ formulas. If $\phi \notin \Gamma$ then maximality yields $\Gamma \cup \{\phi\} \vdash_\Omega \bot$, and hence $\vdash_\Omega \phi \to (\bigwedge_{i=1}^n \psi_i \to \bot)$ where the ψ_i are Γ formulas. Claim 6 now yields $\vdash_\Omega \bigwedge_{i=1}^n \psi'_i \to (\bigwedge_{i=1}^n \psi_i \to \bot)$, that is, Γ is Ω-inconsistent. □

As for Item (i), note that $\phi \to \psi \in \Gamma$ yields (by Claim 7) that $\Gamma \vdash_\Omega \phi \to \psi$. This is true iff there exists $\bigwedge_i^n \psi_i$ with $\psi_i \in \Gamma$ for all $i = 1 \ldots n$ and $\vdash_\Omega \bigwedge_i^n \psi_i \to (\phi \to \psi)$, that is, $\vdash_\Omega \bigwedge_i^n \psi_i \wedge \phi \to \psi$ $(*)$. If $\phi \in \Gamma$ then $(*)$ yields that $\Gamma \vdash_\Omega \psi$, and hence (by Claim 7) $\psi \in \Gamma$.

Item (ii) follows from Claim 7 and the fact that if $\vdash_\Omega \phi$ then $\Gamma \vdash_\Omega \phi$.

As for Item (iii), consistency and Item (i) (with $\psi = \bot$) yield that ϕ and $\neg\phi$ cannot *both* belong to Γ. Assume that $\phi \notin \Gamma$. Then maximality yields $\Gamma \cup \{\phi\} \vdash_\Omega \bot$, and hence (Consistency and Negation Exercise) $\Gamma \vdash_\Omega \neg\phi$. The above claim yields that $\neg\phi \in \Gamma$. A similar argument holds that if $\neg\phi \notin \Gamma$ then $\phi \in \Gamma$.

Item (iv) follows similarly. The propositional tautology $p \to p \vee q$ and uniform substitution yield $\vdash_\Omega \phi \to \phi \vee \psi$. Item (ii) gives $\phi \to \phi \vee \psi \in \Gamma$. If $\phi \in \Gamma$ then Item (i) yields $\phi \vee \psi \in \Gamma$. A similar argument gives the same conclusion under the assumption $\psi \in \Gamma$. Vice versa, assume that $\phi \vee \psi \in \Gamma$ and $\phi \notin \Gamma$, that is, $\neg\phi \in \Gamma$ by Item (iii). The propositional tautology $p \vee q \to (\neg p \to q)$ and uniform substitution yield $\vdash_\Omega \phi \vee \psi \to (\neg\phi \to \psi)$. Item (ii) and two applications of Item (i) give $\psi \in \Gamma$.

Ex. 61 — Strong Completeness. Prove that **S5** is strongly complete w.r.t. the class of universal frames.

Answer (ex. 61) — Let $ML(P)$ be the basic modal language. **S5** is strongly complete w.r.t. the class of frames the accessibility relation of which is of equivalence. In particular, the canonical model $M^{\mathbf{S5}}$ is based on such a frame. Take any Γ-consistent set of $ML(P)$ formulas and its **S5**-MCS extension (which exists due to the Lindebaum lemma). Take the submodel $M^{\mathbf{S5}}_{\Gamma^+}$ of $M^{\mathbf{S5}}$ generated by Γ^+; since $R^{\mathbf{S5}}$ is of equivalence, $M^{\mathbf{S5}}_{\Gamma^+}$ is based on a universal frame. By the satisfiability preservation theorem on generated submodels, we know that $M^{\mathbf{S5}}_{\Gamma^+}, \Gamma^+ \models \Gamma$ iff $M^{\mathbf{S5}}, \Gamma^+ \models \Gamma$. The canonical model theorem states that $M^{\mathbf{S5}}, \Gamma^+ \models \Gamma$; thus $M^{\mathbf{S5}}_{\Gamma^+}, \Gamma^+ \models \Gamma$.