Exercises for Discrete Maths

Discrete Maths

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Week 4

Computer Science

Free University of Bozen-Bolzano

Exercise Set 7.4, p. 440: Cardinality and Computability

Exercise 26. Prove that any infinite set A contains a countably infinite subset.

Proof. We construct inductively a function $f : \mathbb{N} \mapsto A$.

Basis Step: Pick an arbitrary[1] element $a_1 \in A$. Let $f(1) = a_1$.

Inductive Step: Assume that $f(n)$ has been defined for $n \geq 1$. Now, $A - \{f(1), \ldots, f(n)\} \neq \emptyset$ because A is infinite. Pick an arbitrary $b \in A - \{f(1), \ldots, f(n)\}$. Define $f(n+1) = b$.

Next we prove that f is injective. If $1 \leq m < n$ then $f(m) \in \{f(1), \ldots, f(n-1)\}$ whereas $f(n) \in A - \{f(1), \ldots, f(n-1)\}$. Thus $f(n) \neq f(m)$, that is, f is bijective from $\mathbb{N}$ to $f(\mathbb{N})$. Thus $f(\mathbb{N})$ is countable by definition of countable set.

Exercise Set 8.1, p. 449: Relations

Exercise 17. Draw the graph, after realizing that xRy iff $x - y = 3n$ for some $n \in \mathbb{Z}$, and hence

$$\begin{aligned} R = \{&(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),\\ &(8,5),(8,2),(7,4),(6,3),(5,2)\\ &(5,8),(2,8),(4,7),(3,6),(2,5)\}\,. \end{aligned}$$

Exercise Set 8.2, pp. 458–459: Properties of Relations

Exercise 1. Draw the graph. R is not reflexive because $(2,2) \notin R$. It is not symmetric because $(3,2) \notin R$. It is not transitive because $(1,0)$ and $(0,3)$ are in R but $(1,3) \notin R$.

Exercise 20. Show that R is an equivalence relation—it is reflexive, symmetric and transitive.

Exercise 25. As above, show that R is an equivalence relation.

Exercise 37. If R and S are reflexive, then $R \cap S$ is so. Explain why.

Exercise 38. If R and S are symmetric, then $R \cap S$ is so. Explain why.

Exercise 39. If R and S are transitive, then $R \cap S$ is so. Explain why.

Exercise 40. If R and S are reflexive, then $R \cup S$ is so. Explain why.

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[1] Formally, the Axiom of Choice allows us to do so.

Exercise 41. If R and S are symmetric, then $R \cup S$ is so. **Proof.** Let $(x, y) \in R \cup S$. Then either $(x, y) \in R$ and then $(y, x) \in R$, or $(x, y) \in S$ and then $(y, x) \in S$. Thus, $(y, x) \in R \cup S$.

Exercise 42. If R and S are transitive, then $R \cup S$ is not necessarily so. Counter-example: $R = \{(a, b)\}$ and $S = \{(b, c)\}$.

Exercise 51. Let $R = \{(0, 1), (0, 2), (1, 1), (1, 3), (2, 2), (3, 0)\}$. Find its transitive closure R^t, after drawing the directed graph of R.

Exercise Set 8.3, p. 475–477: Equivalence Relations

Exercise 2. A relation R induced by a partition is of equivalence—reflexive, symmetric, transitive. See Theorem 8.3.1.

a) The partition is $P = \{\{0, 2\}, \{1\}, \{3, 4\}\}$. Then

$$\begin{aligned} \{0, 2\} &= [0] = [2] \\ \{1\} &= [1] \\ \{3, 4\} &= [3] = [4] \end{aligned}$$

and hence

$$R = \{(0, 0), (2, 2), (0, 2), (2, 0), (1, 1), (3, 3), (4, 4), (3, 4), (4, 3)\}.$$

b) The partition is $P = \{\{0\}, \{1, 3, 4\}, \{2\}\}$. Then, reasoning as above,

$$R = \{(0, 0), (1, 1), (3, 3), (4, 4), (1, 3), (3, 1), (1, 4), (4, 1), (3, 4), (4, 3), (2, 2)\}.$$

c) The partition is $P = \{\{0\}, \{1, 2, 3, 4\}\}$. Then, reasoning as above,

$$\begin{aligned} R = \{&(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (1, 2), \\ &(2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 2)\}. \end{aligned}$$

Exercise 8. Consider the powerset of $X = \{a, b, c\}$ and define R on the powerset as follows: URV iff U and V have the same cardinality. Find the equivalence classes of R.

The equivalence classes are: $[\{\emptyset\}] = \{\emptyset\}$; $[\{a\}] = \{\{a\}, \{b\}, \{c\}\}$; $[\{a, b\}] = \{\{a, b\}, \{a, c\}, \{b, c\}\}$; $[\{a, b, c\}] = \{\{a, b, c\}\}$.

Exercise 28. Consider the following relation I over reals: xIy iff $(x - y) \in \mathbb{Z}$. Prove that it is of equivalence and characterize its equivalence classes. See the book resolution.

Exercise 46. Let R be a relation on a set A and suppose R is symmetric and transitive. Prove the following: If for every x in A there exists a y in A such that xRy, then R is an equivalence relation.

Proof. For every x in A there is a y in A such that xRy, then, by symmetry, yRx, and by transivity, xRx. Thus R is also reflexive and so it is an equivalence relation.