Belief network inference

Four main approaches to determine posterior distributions in belief networks:

- Variable Elimination: exploit the structure of the network to eliminate (sum out) the non-observed, non-query variables one at a time.
- Search-based approaches: enumerate some of the possible worlds, and estimate posterior probabilities from the worlds generated.
- Stochastic simulation: random cases are generated according to the probability distributions.
- Variational methods: find the closest tractable distribution to the (posterior) distribution we are interested in.

Factors

A factor is a representation of a function from a tuple of random variables into a number.

We will write factor f on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$.

We can assign some or all of the variables of a factor:

- $f(X_1 = v_1, X_2, \ldots, X_j)$, where $v_1 \in dom(X_1)$, is a factor on $X_2, \ldots, X_j$.
- $f(X_1 = v_1, X_2 = v_2, \ldots, X_j = v_j)$ is a number that is the value of f when each X_i has value v_i.

The former is also written as $f(X_1, X_2, \ldots, X_j)_{X_1 = v_1}$, etc.

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	0.9
f	0.8

$$r(X{=}t, Y{=}f, Z{=}f) = 0.8$$

Multiplying factors

The product of factor $f_1(\overline{X}, \overline{Y})$ and $f_2(\overline{Y}, \overline{Z})$, where $\overline{Y}$ are the variables in common, is the factor $(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z})$ defined by:

$$(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z}) \quad = \quad f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z}).$$

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 \times f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

Summing out variables

We can sum out a variable, say X_1 with domain $\{v_1, \ldots, v_k\}$, from factor $f(X_1, \ldots, X_j)$, resulting in a factor on $X_2, \ldots, X_j$ defined by:

$$
\begin{aligned}
& (\sum_{X_1} f)(X_2, \ldots, X_j) \\
= \quad & f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)
\end{aligned}
$$

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	0.46

Evidence

If we want to compute the posterior probability of Z given evidence $Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j$:

$$\begin{aligned} & P(Z|Y_1 = v_1, \ldots, Y_j = v_j) \\ = & \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{P(Y_1 = v_1, \ldots, Y_j = v_j)} \\ = & \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{\sum_Z P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}. \end{aligned}$$

So the computation reduces to the probability of $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$.
We normalize at the end.

Probability of a conjunction

Suppose the variables of the belief network are $X_1, \ldots, X_n$.
To compute $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$, we sum out the other variables, $Z_1, \ldots, Z_k = \{X_1, \ldots, X_n\} - \{Z\} - \{Y_1, \ldots, Y_j\}$.
We order the Z_i into an elimination ordering.

$$
\begin{aligned}
&P(Z, Y_1 = v_1, \ldots, Y_j = v_j) \\
&= \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}. \\
&= \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^{n} P(X_i | parents(X_i))_{Y_1 = v_1, \ldots, Y_j = v_j}.
\end{aligned}
$$

Computing sums of products

Computation in belief networks reduces to computing the sums of products.

- How can we compute $ab + ac$ efficiently?

Computing sums of products

Computation in belief networks reduces to computing the sums of products.

- How can we compute $ab + ac$ efficiently?
- Distribute out the a giving $a(b + c)$

Computing sums of products

Computation in belief networks reduces to computing the sums of products.

- How can we compute $ab + ac$ efficiently?
- Distribute out the a giving $a(b + c)$
- How can we compute $\sum_{Z_1} \prod_{i=1}^{n} P(X_i|parents(X_i))$ efficiently?

Computing sums of products

Computation in belief networks reduces to computing the sums of products.

- How can we compute $ab + ac$ efficiently?
- Distribute out the a giving $a(b + c)$
- How can we compute $\sum_{Z_1} \prod_{i=1}^{n} P(X_i|parents(X_i))$ efficiently?
- Distribute out those factors that don't involve Z_1.

Variable elimination algorithm

To compute $P(Z | Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j)$:

- Construct a factor for each conditional probability.
- Set the observed variables to their observed values.
- Sum out each of the other variables (the $\{Z_1, \ldots, Z_k\}$) according to some elimination ordering.
- Multiply the remaining factors. Normalize by dividing the resulting factor $f(Z)$ by $\sum_Z f(Z)$.

Summing out a variable

To sum out a variable Z_j from a product $f_1, \ldots, f_k$ of factors:

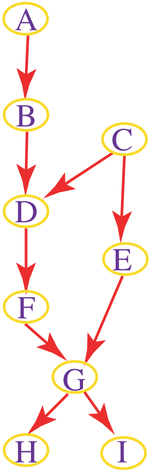
- Partition the factors into
  - those that don't contain Z_j, say $f_1, \ldots, f_i$,
  - those that contain Z_j, say $f_{i+1}, \ldots, f_k$

We know:

$$\sum_{Z_j} f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \times \left(\sum_{Z_j} f_{i+1} \times \cdots \times f_k \right).$$

- Explicitly construct a representation of the rightmost factor. Replace the factors $f_{i+1}, \ldots, f_k$ by the new factor.

Variable elimination example



$$\left.\begin{array}{l} P(A) \\ P(B|A) \end{array}\right\} \xrightarrow{\text{elim } A} f_1(B)$$

$$\left.\begin{array}{l} P(C) \\ P(D|BC) \\ P(E|C) \end{array}\right\} \xrightarrow{\text{elim } C} f_2(BDE)$$

$$P(F|D)$$

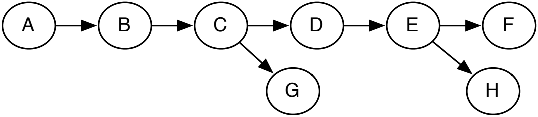
$$P(G|FE)$$

$$P(H|G) \; \} \xrightarrow{\text{obs } H} f_3(G)$$

$$P(I|G) \; \} \xrightarrow{\text{elim } I} f_4(G)$$

$$P(D, h) = \ldots (\textstyle\sum_A P(A)P(B|A))(\textstyle\sum_I P(I|G))$$

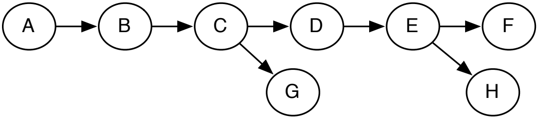
Variable Elimination example



Query: $P(G|f)$; elimination ordering: A, H, E, D, B, C

$$P(G|f) \propto \sum_C \sum_B \sum_D \sum_E \sum_H \sum_A P(A)P(B|A)P(C|B)(D|C)$$
$$P(E|D)P(f|E)P(G|C)P(H|E)$$

Variable Elimination example



Query: $P(G|f)$; elimination ordering: A, H, E, D, B, C

$$P(G|f) \propto \sum_C \sum_B \sum_D \sum_E \sum_H \sum_A P(A)P(B|A)P(C|B)(D|C) \\ P(E|D)P(f|E)P(G|C)P(H|E)$$

$$= \sum_C \left(\sum_B \left(\sum_A P(A)P(B|A) \right) P(C|B) \right) P(G|C) \\ \left(\sum_D P(D|C) \left(\sum_E P(E|D)P(f|E) \sum_H P(H|E) \right) \right)$$

