Advanced Knowledge Based Systems
CS3411

Deduction in Propositional Logic

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Decision Procedures in Logic

A **decision procedure** solves a problem with YES or NO answers:

$$\boxed{KB \vdash_i \alpha}$$

- Sentence α can be derived from the set of sentences *KB* by procedure i.
- *Soundness*: procedure i is sound if
  whenever procedure i proves that a sentence α can be derived from a set of sentences *KB* ($KB \vdash_i \alpha$), then it is also true that *KB* entails α ($KB \models \alpha$).
  - "no wrong inferences are drawn"
  - A sound procedure may fail to find the solution in some cases, when there is actually one.
- *Completeness*: procedure i is complete if
  whenever a set of sentences *KB* entails a sentence α ($KB \models \alpha$), then procedure i proves that α can be derived from *KB* ($KB \vdash_i \alpha$).
  - "all the correct inferences are drawn"
  - A complete procedure may claim to have found a solution in some cases, when there is actually no solution.

Sound and Incomplete Algorithms

- Sound and incomplete algorithms are very popular: they are considered *good* approximations of problem solving procedures.
- Sound and incomplete algorithms may reduce the algorithm complexity.
- Sound and incomplete algorithms are often used due to the inability of programmers to find sound and complete algorithms.

Good Decision procedures

- If an incomplete reasoning mechanism is provided, we can conclude either that the semantics of the representation language does not really capture the meaning of the "world" and of "*what should follow*", or that the algorithms can not infer all the things we would expect.
- Having **sound and complete** reasoning procedures is important!
- Sound and complete decision procedures are good candidates for implementing reasoning modules within larger applications.

An extreme example

Let's consider two decision procedures:

- F, which always returns the result NO independently from its input
- T, which always returns the result YES independently from its input

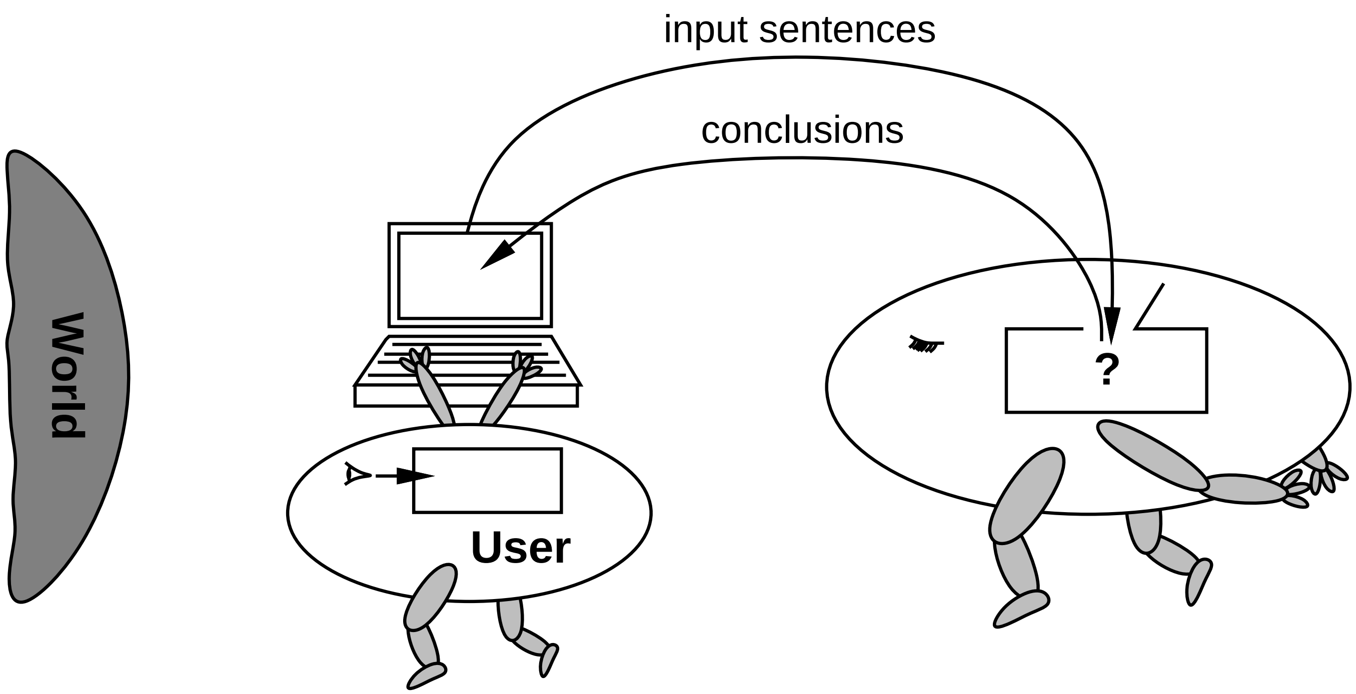
Let's consider the problem of computing entailment between formulas;

- F is a sound algorithm for computing entailment.
- T is a complete algorithm for computing entailment.

Dual problems

Can we use a sound but incomplete decision procedure for a problem to solve the **dual problem** by inverting the answers?

T is an unsound procedure for computing non-entailment between formulas. *(Why?)*



Incompleteness of the reasoning procedures of the reasoning agent leads to *unsound* reasoning of the whole agent, if the main system relies on *negative conclusions* of the reasoning agent module.

Propositional Decision Procedures

- Truth tables provide a sound and complete decision procedure for testing satisfiability, validity, and entailment in propositional logic.
  - The proof is based on the observation that truth tables enumerate all possible models.
- Satisfiability, validity, and entailment in propositional logic are thus *decidable* problems.
- For problems involving a large number of atomic propositions the amount of calculation required by using truth tables may be prohibitive (always 2^n, where n is the number of atomic proposition involved in the formulas).

Reduction to satisfiability

- A formula ϕ is satisfiable iff there is some interpretation $\mathcal{I}$ (i.e., a truth value assignment) that satisfies ϕ (i.e., ϕ is true under $\mathcal{I}$: $\mathcal{I} \models \phi$).
- Validity, equivalence, and entailment can be reduced to satisfiability:
  - ϕ is a valid (i.e., a tautology) iff $\neg\phi$ is not satisfiable.
  - ϕ entails ψ ($\phi \models \psi$) iff $\phi \rightarrow \psi$ is valid (*deduction theorem*).
    * $\boxed{\phi \models \psi \text{ iff } \phi \wedge \neg\psi \text{ is not satisfiable}}$.
  - ϕ is equivalent to ψ ($\phi \equiv \psi$) iff $\phi \leftrightarrow \psi$ is valid.
    * $\phi \equiv \psi$ iff $\phi \models \psi$ and $\psi \models \phi$
- A *sound and complete* procedure deciding satisfiability is all we need, and the *tableaux method* is a decision procedure which checks the existence of a model.

Tableaux Calculus

- The Tableaux Calculus is a decision procedure solving the problem of satisfiability.
- If a formula is satisfiable, the procedure will constructively exhibit a model of the formula.
- The basic idea is to incrementally build the model by looking at the formula, by decomposing it in a top/down fashion. The procedure exhaustively looks at all the possibilities, so that it can eventually prove that no model could be found for unsatisfiable formulas.

Simple examples (I)

$KB = ManUn \wedge ManCity, \neg ManUn$ $KB = Chelsea \wedge ManCity, \neg ManUn$

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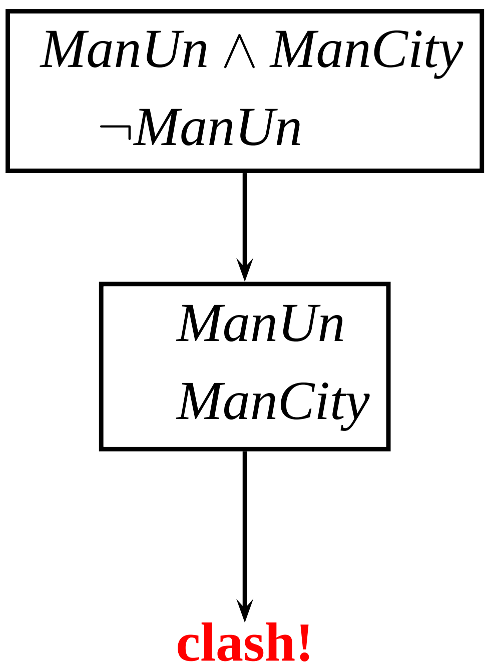

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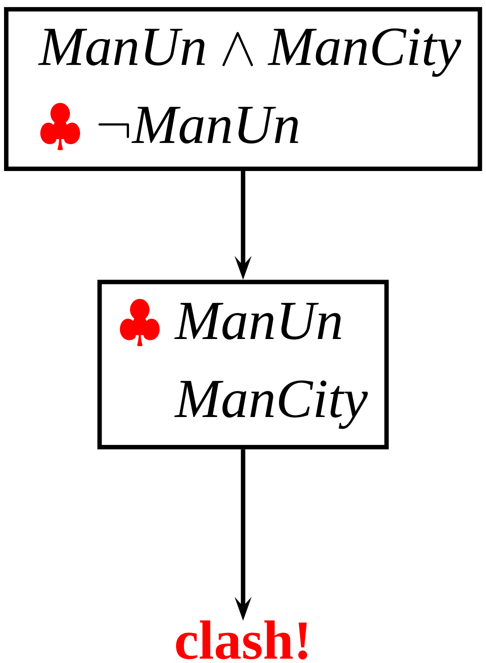

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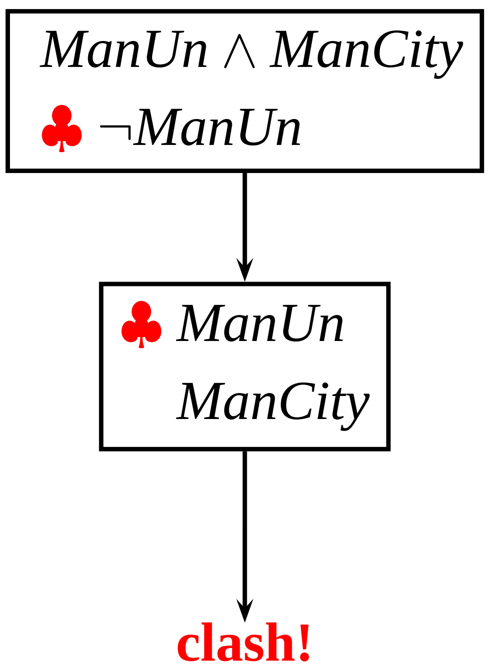

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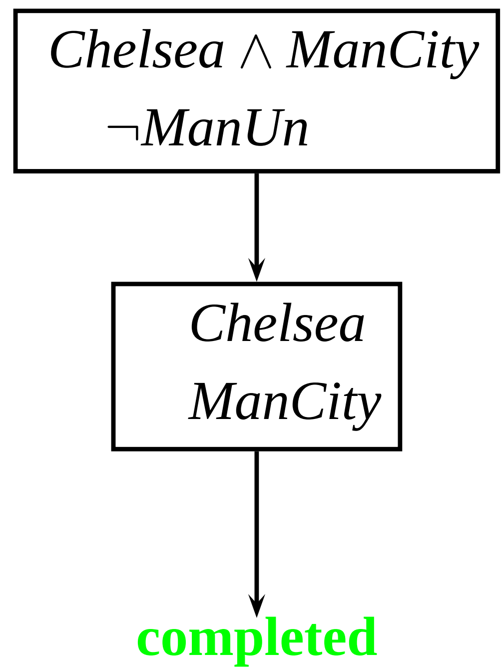

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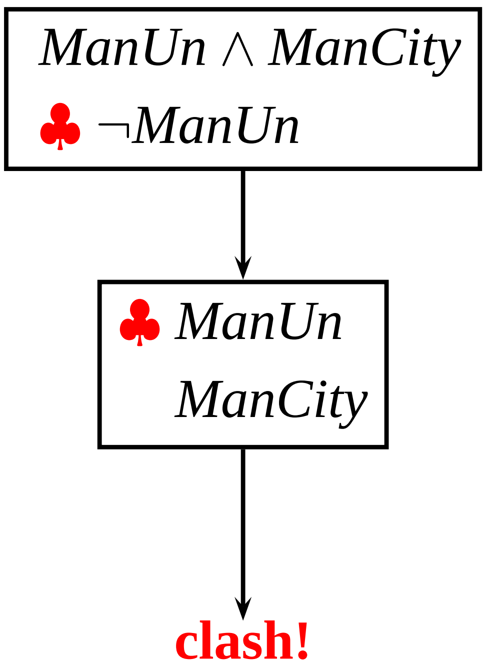

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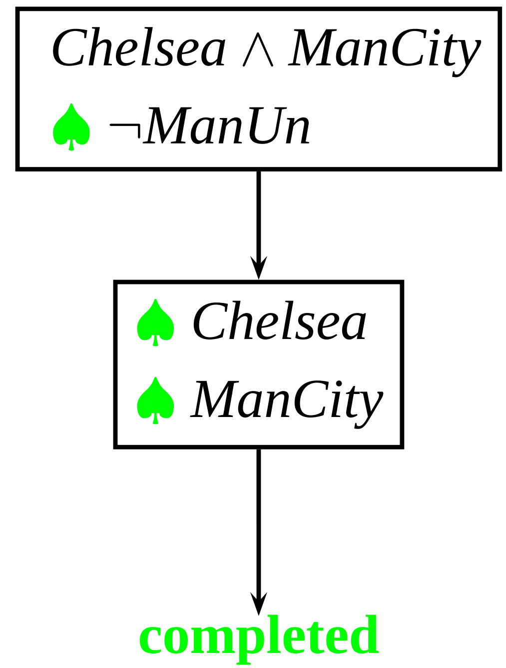

Simple examples (II)

$KB = Chelsea \lor ManUn, \neg Chelsea, \neg ManUn$

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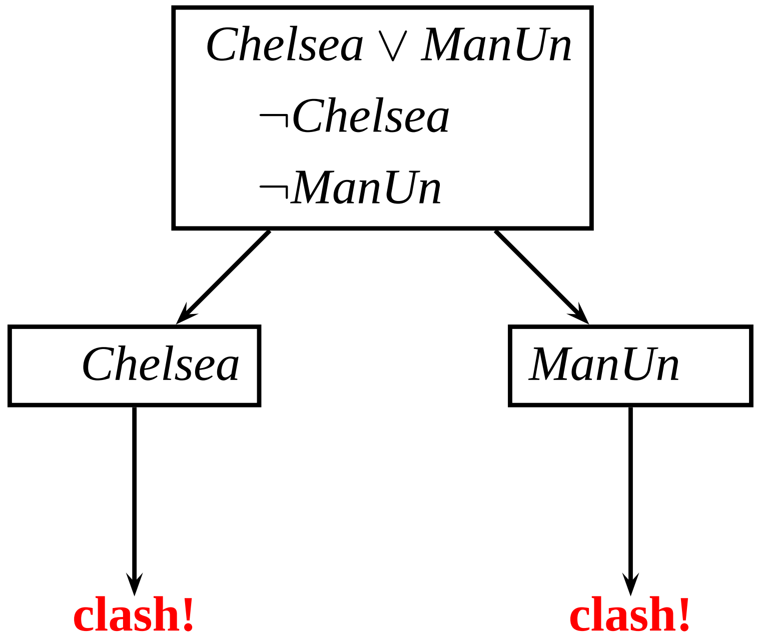

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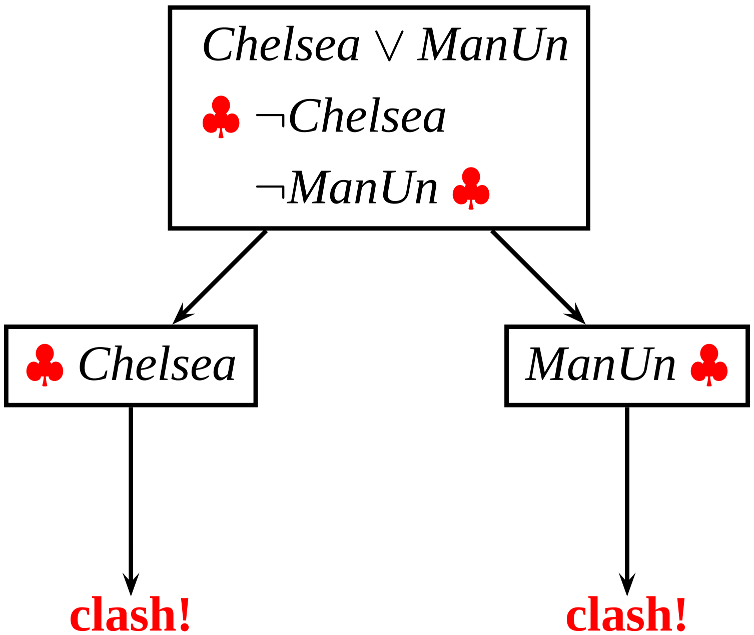

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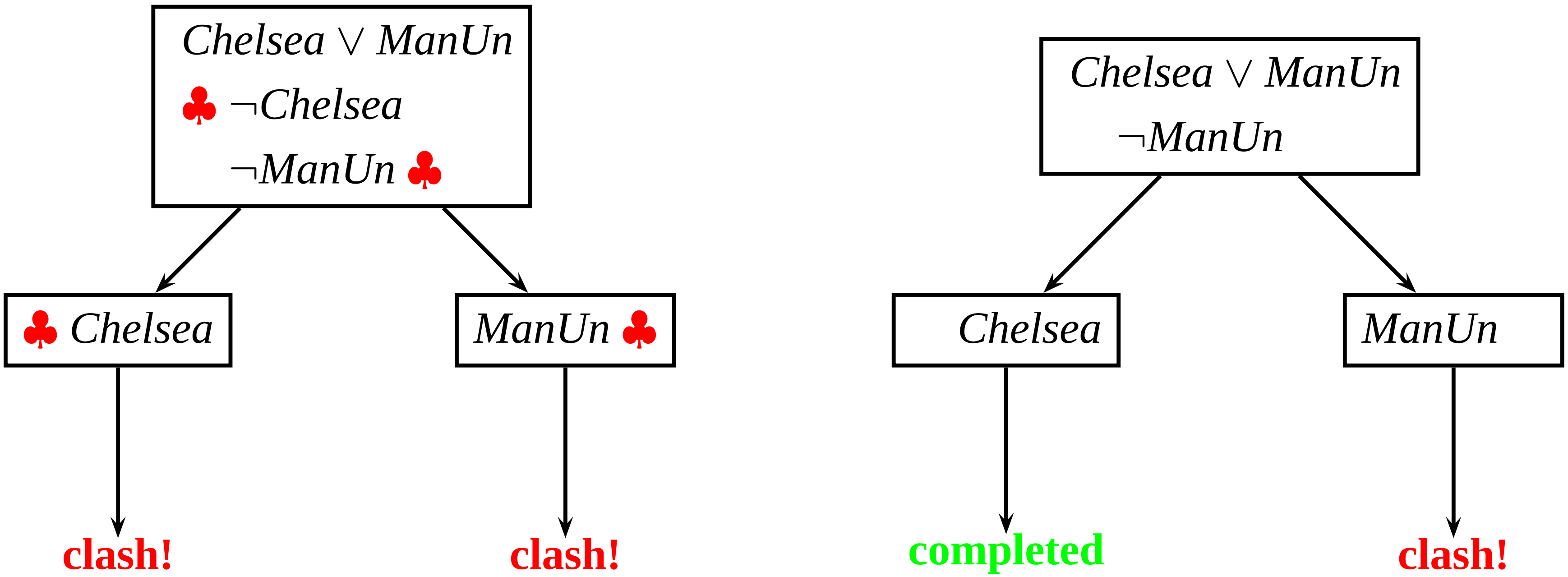

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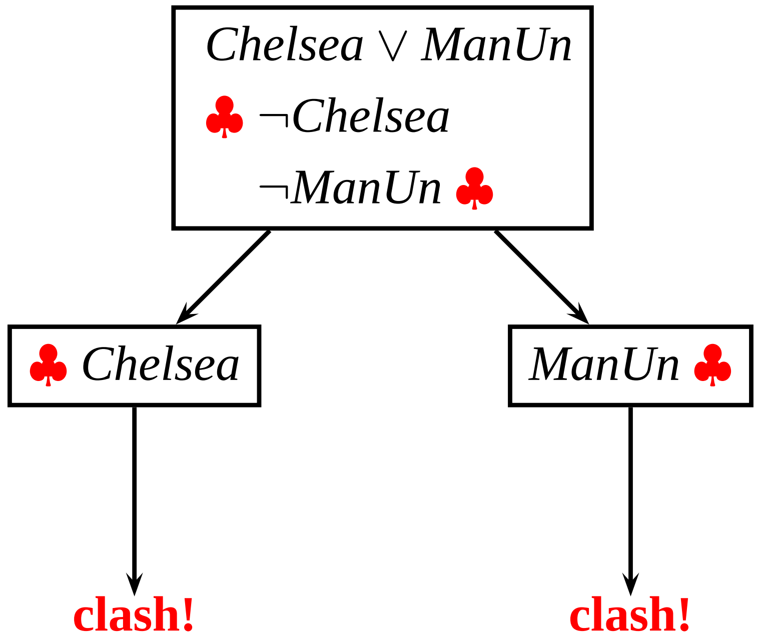

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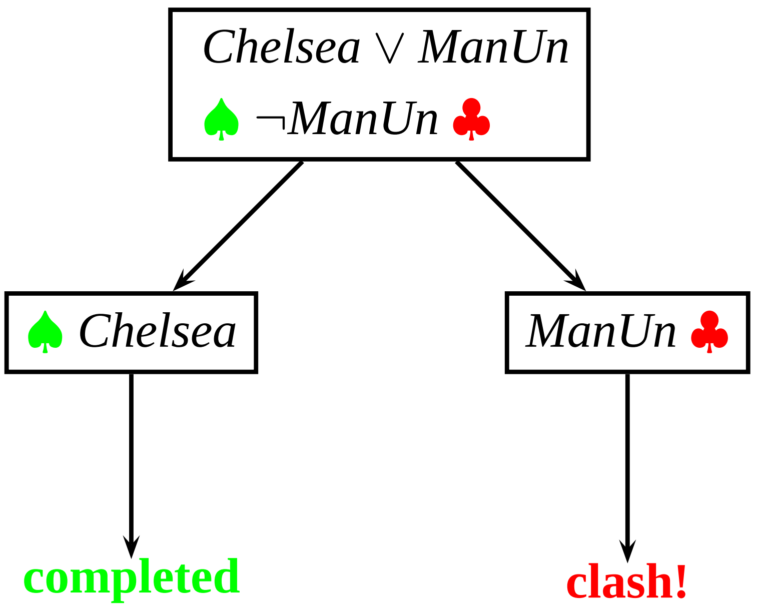

Tableaux Calculus

Finds a model for a given collection of sentences *KB* in negation normal form.

1. Consider the knowledge base *KB* as the root node of a *refutation tree*. A node in a refutation tree is called *tableaux*.

2. Starting from the root, add new formulas to the tableaux, applying the *completion rules*.

3. Completion rules are either deterministic – they yield a uniquely determined successor node – or nondeterministic – yielding several possible alternative successor nodes (*branches*).

4. Apply the completion rules until either

   (a) an explicit contradiction due to the presence of two opposite literals in a node (a *clash*) is generated in each branch, or

   (b) there is a *completed* branch where no more rule is applicable.

Models

- The completed branch of the refutation tree gives a model of *KB*: the *KB* is satisfiable. Since all formulas have been reduced to literals (i.e., either positive or negative atomic propositions), it is possible to find an assignment of truth and falsity to atomic sentences which make all the sentences in the branch true.
- If there is no completed branch (i.e., every branch has a clash), then it is not possible to find an assignment making the original *KB* true: the *KB* is unsatisfiable. In fact, the original formulas from which the tree is constructed can not be true simultaneously.

The Calculus

$$\frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}}$$

If a model satisfies a conjunction, then it also satisfies *each of* the conjuncts

$$\frac{\phi \vee \psi}{\phi \mid \psi}$$

If a model satisfies a disjunction, then it also satisfies *one of* the disjuncts. It is a non-deterministic rule, and it generates two *alternative* branches of the tableaux.

Negation Normal Form

The given tableaux calculus works only if the formula has been translated into Negation Normal Form, i.e., all the negations have been pushed down.

Example::

$\neg(A \vee (B \wedge \neg C))$

becomes

$(\neg A \wedge (\neg B \vee C))$

Entailment and Refutation

$\phi \models \psi$ iff $\phi \wedge \neg\psi$ is not satisfiable. The tableaux may exhibit a counter-example (why?).

$Chelsea \vee ManUn, \neg ManUn \models Chelsea$
(true)

$Chelsea \vee ManUn \models ManUn$
(false)

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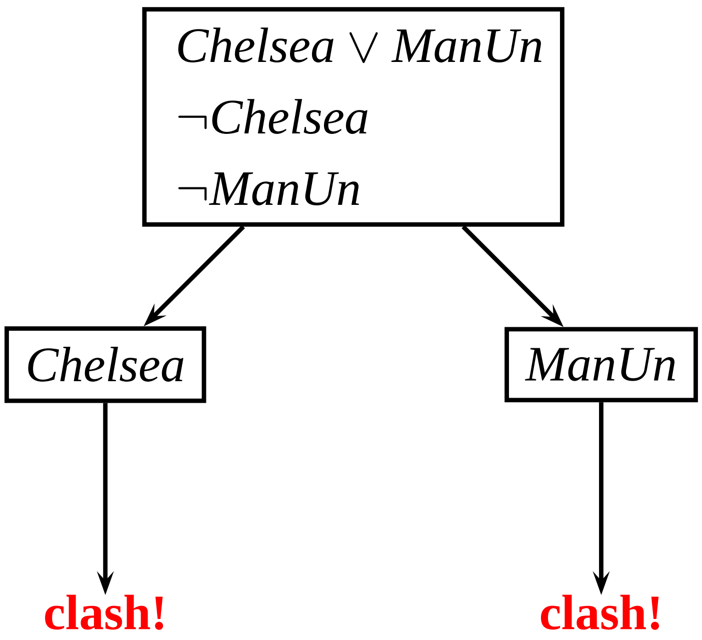

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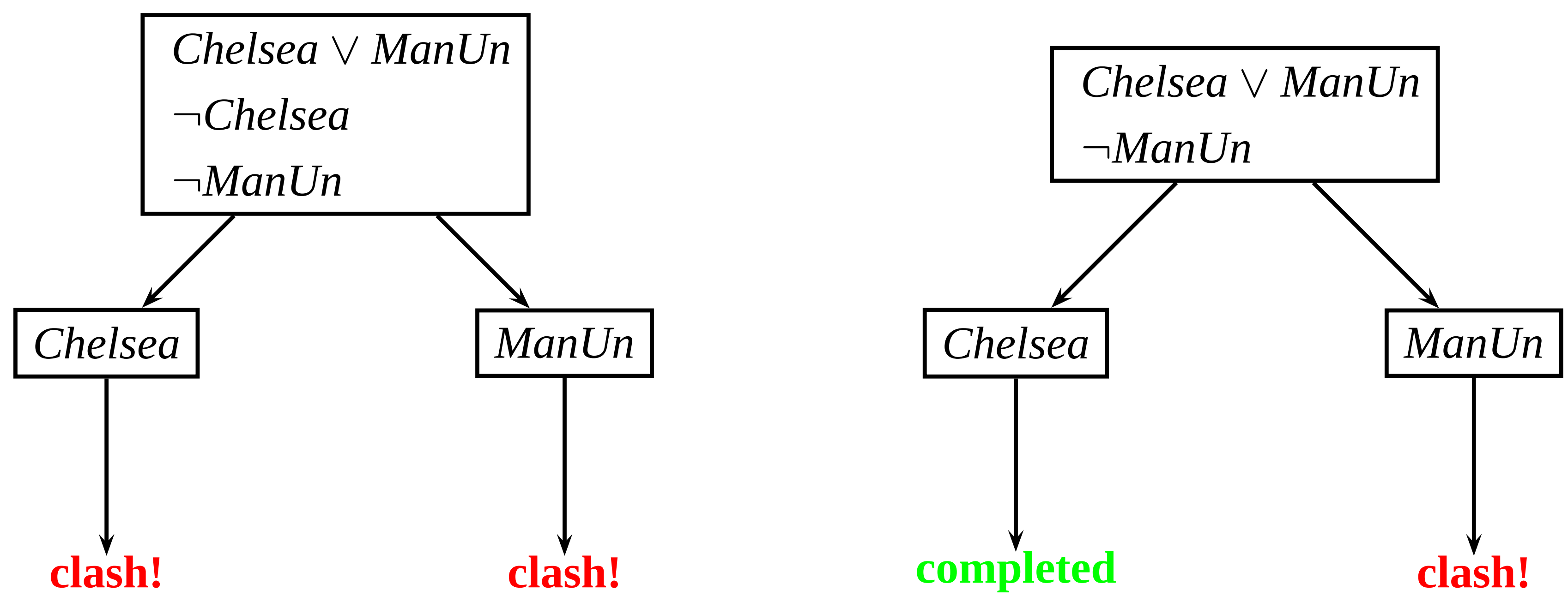

The efficiency of Tableaux: order of rule application

$KB = p \wedge q, \neg p, a \wedge b \wedge c$

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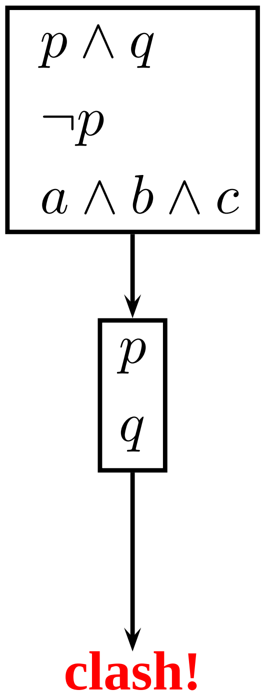

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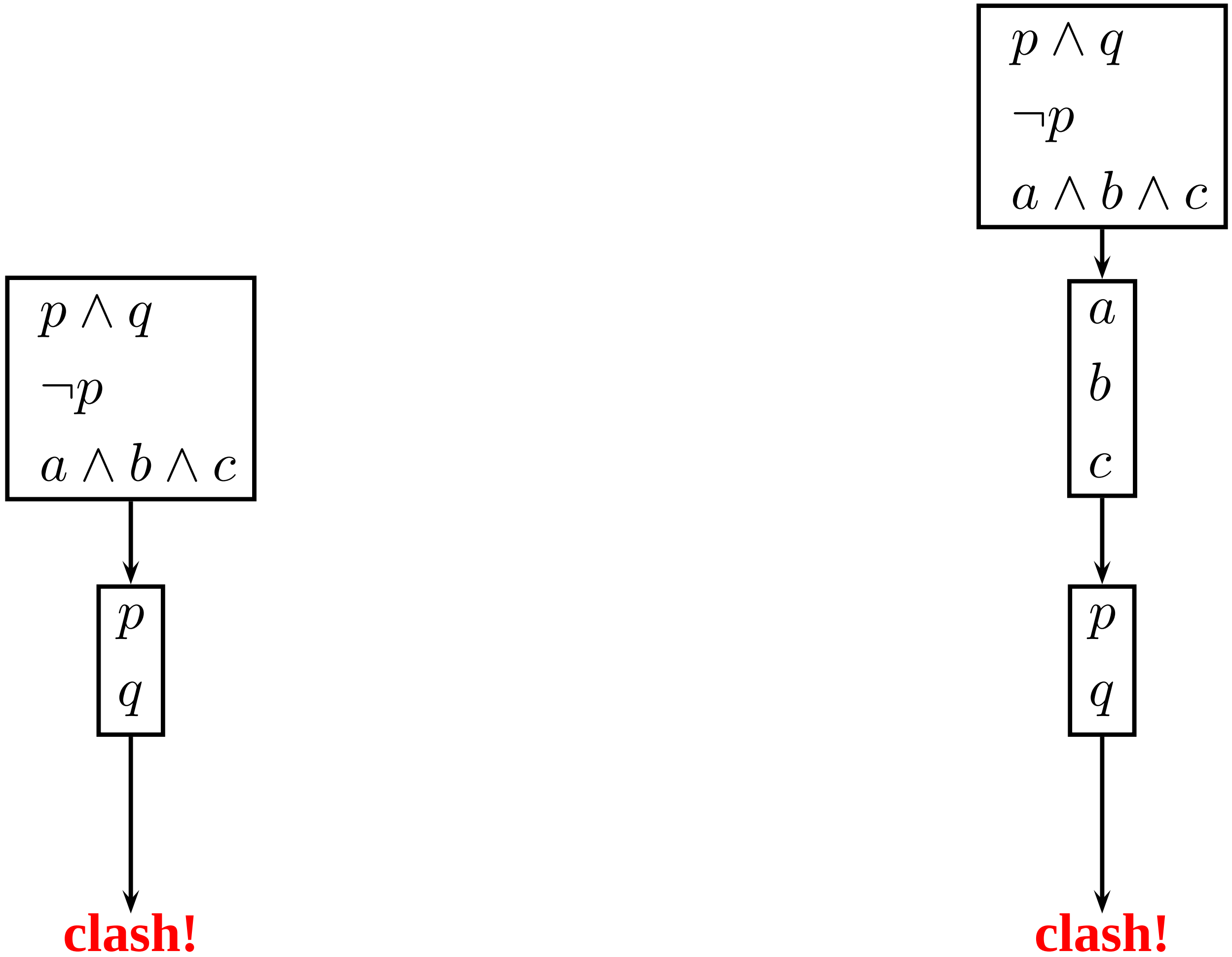


The efficiency of Tableaux: comparison with truth tables

- The complexity of truth tables depends on the number of atomic formulas appearing in the *KB*,
- the complexity of tableaux depends on the syntactic structure of the formulas in *KB*.

Try:

$KB = ((p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r))$

Tableaux as a Decision Procedure

Tableaux is a decision procedure for computing satisfiability, validity, and entailment in propositional logics:

- it is a sound algorithm
- it is a complete algorithm
- it is a terminating algorithm

Testing the hardness of satisfiability

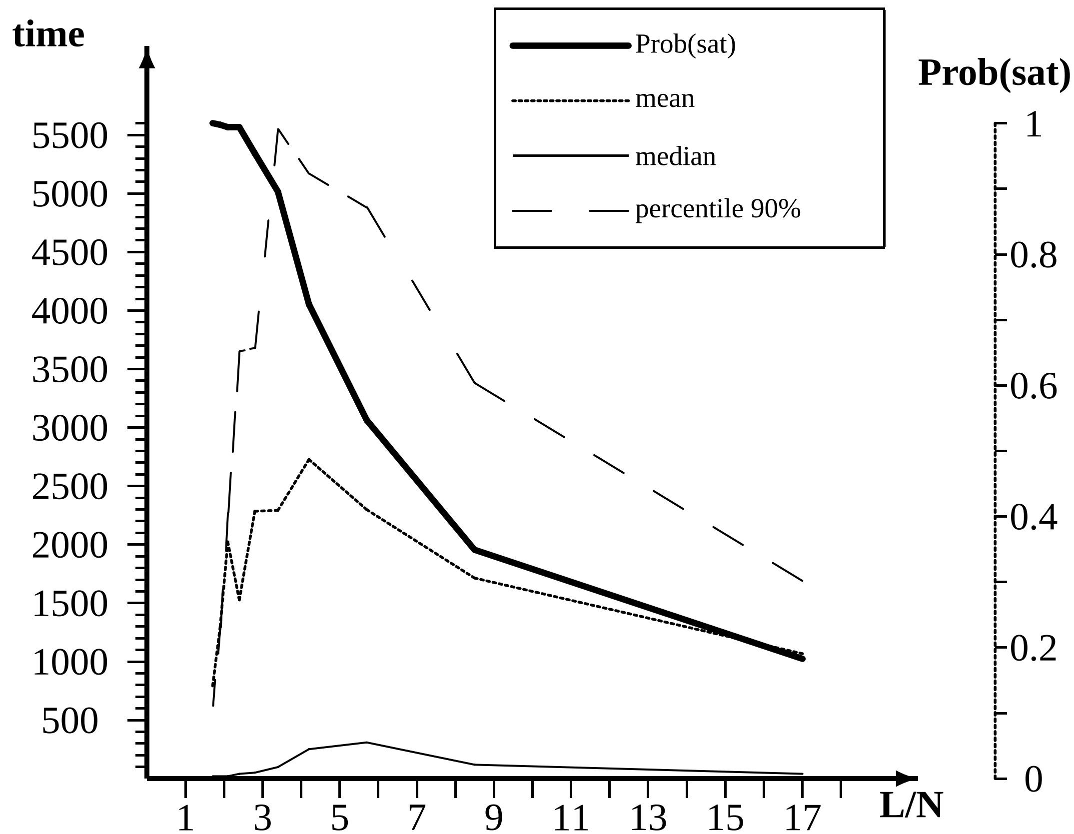
- Find a reasonable way of generating *random* formulas to be used in the test.
- Random formulas to be characterized in terms of an *order parameter* with respect to *satisfiability*, and showing neat difficult cases in correspondence to a *phase transition* in the satisfiability probability space.
- The phase transition separates the over-constrained from the under-constrained regions of the problem space.

Propositional 3-SAT

The easy-hard-easy pattern for satisfiability, within a sat/unsat transition in propositional 3-SAT problems:

$$\boxed{(P_{1,1} \vee P_{2,1} \vee P_{3,1}) \wedge \ldots \wedge (P_{1,L} \vee P_{2,L} \vee P_{3,L})}$$

$P_{i,j} \in \wp \qquad \|\wp\| = N$



Propositional Logic at work: the Graph Colouring Problem

The Graph Colouring problem is a well-known combinatorial problem from graph theory:

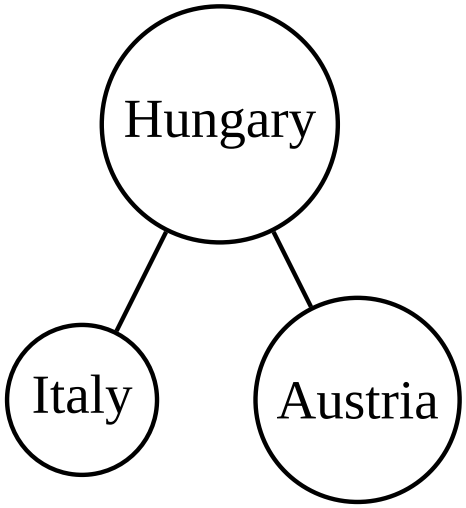
- A graph is defined as $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of *vertices* and $E = \{(v_i, v_j), \ldots, (v_k, v_l)\}$ the set of *edges* connecting pairs of vertices.
- Find a colouring function $C : V \to \mathcal{N}$, such that connected vertices always have different colours.
- There is a *decision* variant of this problem: the question is to decide whether for a particular number of colours, a couloring of the given graph exists.

Encoding as satisfiability problem

A straightforward strategy for encoding the Graph Colouring Decision Problem into a satisfiability problem in propositional logic:

- Each assignment of a colour to a single vertex is represented by a propositional variable;
- each colouring constraint (edge of the graph) is represented by a set of clauses ensuring that the corresponding vertices have different colours,
- and two additional sets of clauses ensure that valid assignments assign exactly one colour to each vertex.

Example



2 colors (Black and White): Satisfiable

Edge axioms (the dual are redundant):

$(B_I \leftrightarrow \neg B_H) \wedge (W_I \leftrightarrow \neg W_H)$

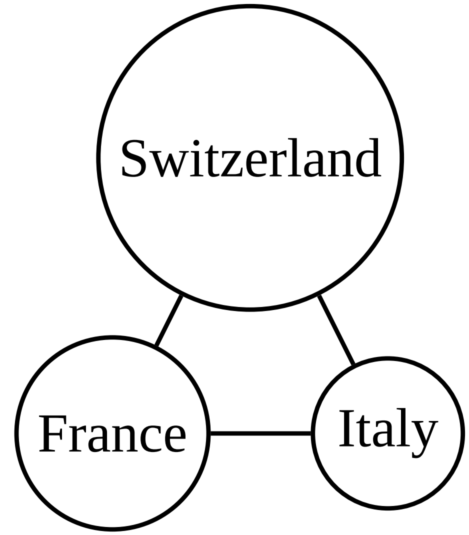
$(B_H \leftrightarrow \neg B_A) \wedge (W_H \leftrightarrow \neg W_A)$

Node axioms (the dual are redundant):

$(B_I \rightarrow \neg W_I)$

$(B_H \rightarrow \neg W_H)$

$(B_A \rightarrow \neg W_A)$



2 colors (Black and White): Unsatisfiable

Edge axioms (the dual are redundant):

$(B_F \leftrightarrow \neg B_S) \wedge (W_F \leftrightarrow \neg W_S)$

$(B_S \leftrightarrow \neg B_I) \wedge (W_S \leftrightarrow \neg W_I)$

$(B_F \leftrightarrow \neg B_I) \wedge (W_F \leftrightarrow \neg W_I)$

Node axioms (the dual are redundant):

$(B_F \rightarrow \neg W_F)$

$(B_S \rightarrow \neg W_S)$

$(B_I \rightarrow \neg W_I)$

Complexity of the problem

$vertices$	$edges$	$colours$	$vars$	$clauses$
30	60	3	90	300
50	115	3	150	545
75	180	3	225	840
100	239	3	300	1117
125	301	3	375	1403
150	360	3	450	1680
175	417	3	525	1951
200	479	3	600	2237

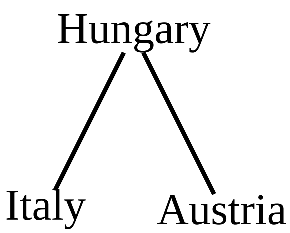
Other hot problems for propositional logic

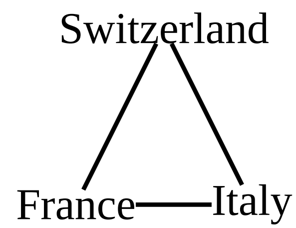
- The general scenario in **Blocks World** planning comprises a number of blocks and a table. The blocks can be piled onto each other, where the downmost block of a pile is always on the table. Given an initial and a goal configuration of blocks, the problem is to find a sequence of single block move operations which, when applied to the initial configuration, leads to the goal situation.
- In the **Logistics** planning domain, packages have to be moved between different locations in different cities. Within cities, packages are carried by trucks while between cities they are flown in planes. Both, trucks and airplanes are of limited capacity.
- **Circuit fault** analysis, **Scheduling**, $\cdots$

Exercise

Check the graph colouring problems, using the enriched tableaux calculus:

$$\frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}} \qquad \frac{\phi \vee \psi}{\begin{array}{c|c}\phi & \psi\end{array}} \qquad \frac{\phi \rightarrow \psi}{\begin{array}{c|c}\neg\phi & \psi\end{array}} \qquad \frac{\phi \leftrightarrow \psi}{\begin{array}{c|c}\phi & \neg\phi \\ \psi & \neg\psi\end{array}}$$





Edge axioms:

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Node axioms:

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