Advanced Knowledge Based Systems
CS3411

Foundations of First Order Logic

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Motivation

- We can already do a lot with propositional logic.
- But it is unpleasant that we cannot access the *structure* of atomic sentences.
- Atomic formulas of propositional logic are *too atomic* – they are just statement which my be true or false but which have no internal structure.
- In *First Order Logic* (FOL) the atomic formulas are interpreted as statements about *relationships between objects*.

Predicates and Constants

Let's consider the statements:

- *Mary is female*
- *John is male*
- *Mary and John are siblings*

In propositional logic the above statements are atomic propositions:

- `Mary-is-female`
- `John-is-male`
- `Mary-and-John-are-siblings`

In FOL atomic statements are built using predicates, which may have constants as argument:

- `Female(mary)`
- `Male(john)`
- `Siblings(mary,john)`

Variables and Quantifiers

Let's consider the statements:

- *Everybody is male or female*
- *A male is not a female*

In FOL predicates may have variables as arguments, whose value is bounded by quantifiers:

- $\forall \mathtt{x}.\ \mathtt{Male}(\mathtt{x}) \lor \mathtt{Female}(\mathtt{x})$
- $\forall \mathtt{x}.\ \mathtt{Male}(\mathtt{x}) \rightarrow \neg\mathtt{Female}(\mathtt{x})$

Deduction (why?):

- *Mary is not male*
- $\neg\mathtt{Male}(\mathtt{mary})$

Functions

Let's consider the statement:

- *The father of a person is male*

In FOL objects of the domain may be denoted by functions applied to (other) objects:

- $\forall \texttt{x}.\ \texttt{Male}(\texttt{father}(\texttt{x}))$

Syntax of FOL: atomic sentences

Countably infinity **supply of symbols** (*signature*):

- variable symbols: $x, y, z, \ldots$
- n-ary function symbols: $f, g, h, \ldots$
- individual constants: $a, b, c, \ldots$
- n-are predicate symbols: $P, Q, R, \ldots$

Terms:

t	$\rightarrow$	x	variable
	$\mid$	a	constant
	$\mid$	$f(t_1, \ldots, t_n)$	function application

Ground terms: terms that do not contain variables

Formulas: $\phi \quad \rightarrow P(t_1, \ldots, t_n)$ atomic formulas

E.g., $Brother(kingJohn, richardTheLionheart)$

$> (length(leftLegOf(richard)), length(leftLegOf(kingJohn)))$

Syntax of FOL: propositional sentences

Formulas: $\phi, \psi \to$	$P(t_1, \ldots, t_n)$	atomic formulas
$\mid$	$\bot$	false
$\mid$	$\top$	true
$\mid$	$\neg\phi$	negation
$\mid$	$\phi \wedge \psi$	conjunction
$\mid$	$\phi \vee \psi$	disjunction
$\mid$	$\phi \to \psi$	implication
$\mid$	$\phi \leftrightarrow \psi$	equivalence

- Precedence of operators: $\neg > \wedge > \vee > \to = \leftrightarrow$.
- (Ground) **atoms** and (ground) **literals**.

E.g. $Sibling(kingJohn, richard) \to Sibling(richard, kingJohn)$

$>(1, 2) \vee \leq(1, 2)$

$>(1, 2) \wedge \neg{>}(1, 2)$

Syntax of full FOL

Formulas:	$\phi, \psi \to P(t_1, \ldots, t_n)$	atomic formulas
	$\mid \quad \bot$	false
	$\mid \quad \top$	true
	$\mid \quad \neg\phi$	negation
	$\mid \quad \phi \land \psi$	conjunction
	$\mid \quad \phi \lor \psi$	disjunction
	$\mid \quad \phi \to \psi$	implication
	$\mid \quad \phi \leftrightarrow \psi$	equivalence
	$\mid \quad \forall x.\ \phi$	*universal quantification*
	$\mid \quad \exists x.\ \phi$	*existential quantification*

E.g. Everyone in England is smart:

$$\forall x.\ In(x, england) \to Smart(x)$$

Someone in France is smart:

$$\exists x.\ In(x, france) \land Smart(x)$$

Summary of Syntax of FOL

- Terms
  - variables
  - constants
  - functions
- Literals
  - atomic formula
    - relation (predicate)
  - negation
- Well formed formulas
  - truth-functional connectives
  - existential and universal quantifiers

Examples

- Brothers are siblings
- “Sibling” is reflexive
- One’s mother is one’s female parent
- A first cousin is a child of a parent’s sibling

Examples

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  $\forall x, y.\, Brother(x, y) \leftrightarrow Sibling(x, y)$
- "Sibling" is reflexive
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Examples

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  $\forall x, y.\, Brother(x, y) \leftrightarrow Sibling(x, y)$
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- "Sibling" is reflexive
  $\forall x, y.\, Sibling(x, y) \leftrightarrow Sibling(y, x)$
- One's mother is one's female parent
  $\forall x, y.\, Mother(x, y) \leftrightarrow (Female(x) \wedge Parent(x, y))$
- A first cousin is a child of a parent's sibling

Examples

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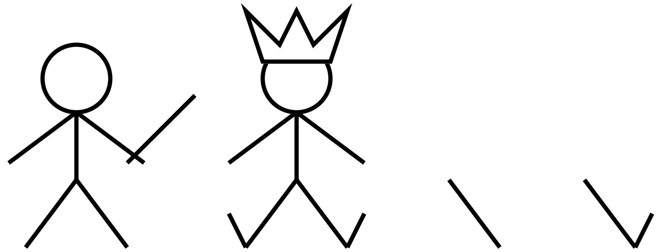
- A first cousin is a child of a parent's sibling
  $\forall x, y.\, FirstCousin(x, y) \leftrightarrow \exists p, ps.\, Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$

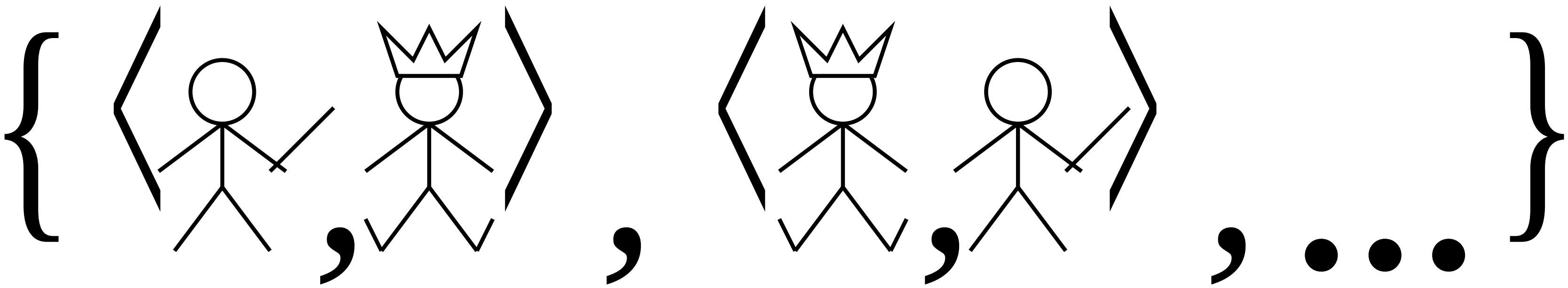
Semantics of FOL: intuition

- Just like in propositional logic, a (complex) FOL formula may be true (or false) with respect to a given interpretation.
- An interpretation specifies referents for
  constant symbols $\longrightarrow$ **objects**
  predicate symbols $\longrightarrow$ **relations**
  function symbols $\longrightarrow$ **functional relations**
- An atomic sentence $P(t_1, \ldots, t_n)$ is true in a given interpretation
  iff the *objects* referred to by $t_1, \ldots, t_n$
  are in the *relation* referred to by the predicate P.
- An interpretation in which a formula is true is called a *model* for the formula.

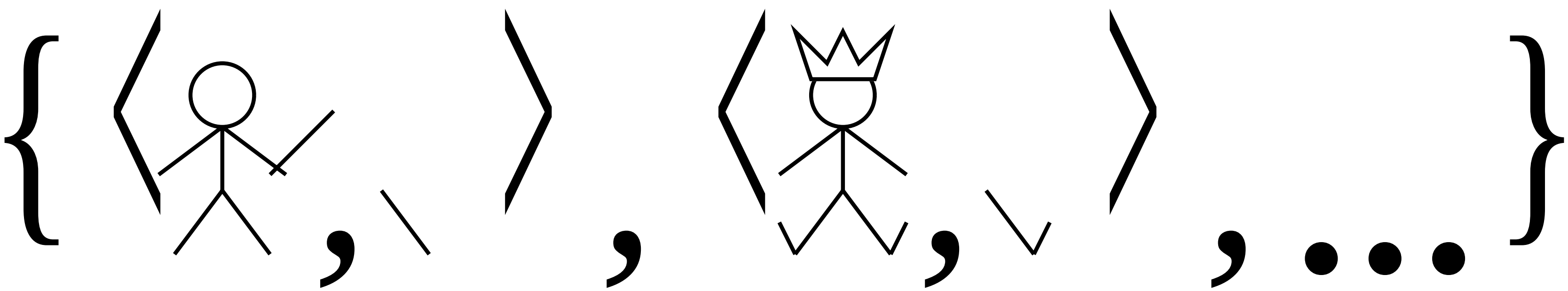
Models for FOL: Example

objects 

relations: sets of tuples of objects



functional relations: all tuples of objects + "value" object



Semantic of FOL: Interpretations

Interpretation: $\mathcal{I} = \langle \Delta, \cdot^{\mathcal{I}} \rangle$ where Δ is an arbitrary non-empty set and $\mathcal{I}$ is a function that maps

- n-ary function symbols to functions over Δ:
  $f^{\mathcal{I}} \in [\Delta^n \rightarrow \Delta]$
- individual constants to elements of Δ:
  $a^{\mathcal{I}} \in \Delta$
- n-ary predicate symbols to relation over Δ:
  $P^{\mathcal{I}} \subseteq \Delta^n$

Interpretation of ground terms:

$$(f(t_1, \ldots, t_n))^{\mathcal{I}} \quad = \quad f^{\mathcal{I}}({t_1}^{\mathcal{I}}, \ldots, {t_n}^{\mathcal{I}}) \ (\in \Delta)$$

Satisfaction of ground atoms $P(t_1, \ldots, t_n)$:

$$\mathcal{I} \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle {t_1}^{\mathcal{I}}, \ldots, {t_n}^{\mathcal{I}} \rangle \in P^{\mathcal{I}}$$

Examples

$$
\begin{aligned}
\Delta &= \{d_1, \ldots, d_n, n > 1\} \\
\mathtt{a}^{\mathcal{I}} &= d_1 \\
\mathtt{b}^{\mathcal{I}} &= d_2 \\
\mathtt{Block}^{\mathcal{I}} &= \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} &= \Delta
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= \{1, 2, 3, \ldots\} \\
\mathtt{1}^{\mathcal{I}} &= 1 \\
\mathtt{2}^{\mathcal{I}} &= 2 \\
&\vdots \\
\mathtt{Even}^{\mathcal{I}} &= \{2, 4, 6, \ldots\} \\
\mathtt{succ}^{\mathcal{I}} &= \{(1 \mapsto 2), (2 \mapsto 3), \ldots\}
\end{aligned}
$$

Examples

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\texttt{a}^{\mathcal{I}} &= d_1 \\
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\texttt{Red}^{\mathcal{I}} &= \Delta \\
\mathcal{I} &\models \texttt{Red(b)} \\
\mathcal{I} &\not\models \texttt{Block(b)}
\end{aligned}$$

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\mathtt{1}^{\mathcal{I}} &= 1 \\
\mathtt{2}^{\mathcal{I}} &= 2 \\
&\vdots \\
\mathtt{Even}^{\mathcal{I}} &= \{2, 4, 6, \ldots\} \\
\mathtt{succ}^{\mathcal{I}} &= \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \\
\mathcal{I} &\not\models \mathtt{Even}(3) \\
\mathcal{I} &\models \mathtt{Even}(\mathtt{succ}(3))
\end{aligned}
$$

Semantics of FOL: Variable Assignments

V set of all variables. Function $\alpha\colon V \to \Delta$.

Notation: $\alpha[x/d]$ is identical to α except for the variable x.

Interpretation of terms *under* $\mathcal{I}, \alpha$:

$$\begin{array}{rcl} x^{\mathcal{I},\alpha} & = & \alpha(x) \\ a^{\mathcal{I},\alpha} & = & a^{\mathcal{I}} \\ (f(t_1,\ldots,t_n))^{\mathcal{I},\alpha} & = & f^{\mathcal{I}}({t_1}^{\mathcal{I},\alpha},\ldots,{t_n}^{\mathcal{I},\alpha}) \end{array}$$

Satisfiability of atomic formulas:

$$\mathcal{I},\alpha \models P(t_1,\ldots,t_n) \quad \text{gdw.} \quad \langle {t_1}^{\mathcal{I},\alpha},\ldots,{t_n}^{\mathcal{I},\alpha}\rangle \in P^{\mathcal{I}}$$

Example:

$$\begin{array}{rcl} \alpha & = & \{(\mathsf{x} \mapsto d_1),(\mathsf{y} \mapsto d_2)\} \\ \mathcal{I},\alpha & \models & \texttt{Red(x)} \\ \mathcal{I},\alpha[\mathsf{y}/d_1] & \models & \texttt{Block(y)} \end{array}$$

Semantics of FOL: Satisfiability of formulas

A formula ϕ is satisfied by (*is true in*) an interpretation $\mathcal{I}$ under a variable assignment α, $\mathcal{I}, \alpha \models \phi$:

$$
\begin{array}{rcl}
\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) & \text{iff} & \langle {t_1}^{\mathcal{I},\alpha}, \ldots, {t_n}^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}} \\
\mathcal{I}, \alpha \models \neg\phi & \text{iff} & \mathcal{I}, \alpha \not\models \phi \\
\mathcal{I}, \alpha \models \phi \wedge \psi & \text{iff} & \mathcal{I}, \alpha \models \phi \text{ and } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \phi \vee \psi & \text{iff} & \mathcal{I}, \alpha \models \phi \text{ or } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \phi \rightarrow \psi & \text{iff} & \text{if } \mathcal{I}, \alpha \models \phi, \text{ then } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \phi \leftrightarrow \psi & \text{iff} & \mathcal{I}, \alpha \models \phi, \text{ if and only if } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \forall x.\ \phi & \text{iff} & \text{for all } d \in \Delta : \\
 & & \mathcal{I}, \alpha[x/d] \models \phi \\
\mathcal{I}, \alpha \models \exists x.\ \phi & \text{iff} & \text{there exists a } d \in \Delta : \\
 & & \mathcal{I}, \alpha[x/d] \models \phi
\end{array}
$$

Examples

$$\begin{aligned} \Delta &= \{d_1, \ldots, d_n,\}\ n > 1 \\ \mathtt{a}^{\mathcal{I}} &= d_1 \\ \mathtt{b}^{\mathcal{I}} &= d_1 \\ \mathtt{Block}^{\mathcal{I}} &= \{d_1\} \\ \mathtt{Red}^{\mathcal{I}} &= \Delta \\ \alpha &= \{(\mathtt{x} \mapsto d_1), (\mathtt{y} \mapsto d_2)\} \end{aligned}$$

1. $\mathcal{I}, \alpha \models \mathtt{Block(c)} \vee \neg\mathtt{Block(c)}$?
2. $\mathcal{I}, \alpha \models \mathtt{Block(x)} \rightarrow \mathtt{Block(x)} \vee \mathtt{Block(y)}$?
3. $\mathcal{I}, \alpha \models \mathtt{Block(a)} \wedge \mathtt{Block(b)}$?
4. $\mathcal{I}, \alpha \models \forall.\ \mathtt{x}\ \mathtt{Block(x)} \rightarrow \mathtt{Red(x)}$?
5. $\Theta = \left\{ \begin{array}{l} \mathtt{Block(a),\ Block(b)} \\ \forall \mathtt{x}\ (\mathtt{Block(x)} \rightarrow \mathtt{Red(x)}) \end{array} \right\}$

   $\mathcal{I}, \alpha \models \Theta$?

Example

Find a model of the formula:

$$\exists y.\ [\ P(y) \land \neg Q(y)\ ] \land \forall z.\ [\ P(z) \lor Q(z)\ ]$$

Example

Find a model of the formula:

$$\exists y \textbf{.} \; [\, P(y) \land \neg Q(y) \,] \land \forall z \textbf{.} [\, P(z) \lor Q(z) \,]$$

$\Delta = \{a, b\}$
$P^{\mathcal{I}} = \{a\}$
$Q^{\mathcal{I}} = \{b\}$

Satisfiability and Validity

An interpretation $\mathcal{I}$ is a **model** of ϕ *under* α, if

$$\mathcal{I}, \alpha \models \phi.$$

Similarly as in propositional logic, a formula ϕ can be **satisfiable**, **unsatisfiable**, **falsifiable** or **valid**—the definition is in terms of the pair $(\mathcal{I}, \alpha)$.

A formula ϕ is

- **satisfiable**, if there is some $(\mathcal{I}, \alpha)$ that satisfies ϕ,
- **unsatisfiable**, if ϕ is not satisfiable,
- **falsifiable**, if there is some $(\mathcal{I}, \alpha)$ that does not satisfy ϕ,
- **valid** (i.e., a **tautology**), if every $(\mathcal{I}, \alpha)$ is a model of ϕ.

Analogously, two formulas are **logically equivalent** ($\phi \equiv \psi$), if for all $\mathcal{I}, \alpha$ we have:

$$\mathcal{I}, \alpha \models \phi \quad \text{iff} \quad \mathcal{I}, \alpha \models \psi$$

Note: $\mathtt{P}(\mathtt{x}) \not\equiv \mathtt{P}(\mathtt{y})$!

Free and Bound Variables

$$\forall \mathtt{x}.\ (\mathtt{R}(\boxed{\mathtt{y}},\boxed{\mathtt{z}})\ \wedge\ \exists\ \mathtt{y}.\ (\neg \mathtt{P}(\mathtt{y},\mathtt{x})\ \vee\ \mathtt{R}(\mathtt{y},\boxed{\mathtt{z}})))$$

Occurrences in boxes of y and z are **free**. All other occurrences of x, y, z are **bound**.

Free variables of a formula (inductively defined over the structure of expressions):

$$\begin{aligned}
\text{free}(x) &= \{x\} \\
\text{free}(f(t_1, \ldots, t_n)) &= \text{free}(t_1) \cup \ldots \cup \text{free}(t_n) \\
\text{free}(P(t_1, \ldots, t_n)) &= \text{free}(t_1) \cup \ldots \cup \text{free}(t_n) \\
\text{free}(\neg\phi) &= \text{free}(\phi) \\
\text{free}(\phi * \psi) &= \text{free}(\phi) \cup \text{free}(\psi),\ * = \vee, \wedge, \ldots \\
\text{free}(\forall x.\ \phi) &= \text{free}(\phi) - \{x\} \\
\text{free}(\exists x.\ \phi) &= \text{free}(\phi) - \{x\}
\end{aligned}$$

Open and Closed Formulas

- A formula is **closed** or a **sentence** if no free variables occurs in it. When formulating theories, we only use closed formulas.
- Note: For closed formulas, the properties *logical equivalence*, *satisfiability*, *entailment* etc. do not depend on variable assignments. If the property holds for one variable assignment then it holds for all of them.
- For closed formulas, the symbol α on the left hand side of the "$\models$" sign is omitted.

$$\mathcal{I} \models \phi$$

Entailment

Entailment is defined similarly as in propositional logic.

The formula ϕ is logically implied by a formula ψ, if ϕ is true in all models of ψ (symbolically, $\psi \models \phi$):

$$\psi \models \phi \quad \text{iff} \quad \mathcal{I} \models \phi \ \text{ for all models } \mathcal{I} \ \text{ of } \psi$$

More Exercises

- $\models \forall x.\ (P(x) \vee \neg P(x))$

- $\exists x.\ [\ P(x) \wedge (P(x) \rightarrow Q(x))\ ] \models \exists x.\ Q(x)$

- $\models \neg(\exists x.\ [\ \forall y.\ [\ P(x) \rightarrow Q(y)\ ]\ ])$

- $\exists y.\ [\ P(y) \wedge \neg Q(y)\ ] \wedge \forall z. [\ P(z) \vee Q(z)\ ]$ satisfiable

Equality

- Equality is a special predicate.
- $t_1 = t_2$ is true under a given interpretation $(\mathcal{I}, \alpha \models t_1 = t_2)$ if and only if t_1 and t_2 refer to the same object:
  ${t_1}^{\mathcal{I},\alpha} = {t_2}^{\mathcal{I},\alpha}$

E.g., $\forall x.\ (\times(sqrt(x), sqrt(x)) = x)$ is satisfiable

$2 = 2$ is valid

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\begin{aligned}
&\forall x, y.\, Sibling(x, y) \leftrightarrow \\
&\qquad (\neg(x = y) \wedge \\
&\qquad \exists m, f.\, \neg(m = f) \wedge \\
&\qquad\qquad Parent(m, x) \wedge Parent(f, x) \wedge \\
&\qquad\qquad Parent(m, y) \wedge Parent(f, y)\ )
\end{aligned}$$

Universal quantification

Everyone in England is smart: $\forall x.\ In(x, england) \rightarrow Smart(x)$

$(\forall x.\ \phi)$ is equivalent to the *conjunction* of all possible *instantiations* in x of ϕ:

$$\begin{array}{ll} & In(kingJohn, england) \rightarrow Smart(kingJohn) \\ \wedge & In(richard, england) \rightarrow Smart(richard) \\ \wedge & In(england, england) \rightarrow Smart(england) \\ \wedge & \ldots \end{array}$$

Typically, $\rightarrow$ is the main connective with $\forall$.
Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x.\ In(x, england) \wedge Smart(x)$$

means "Everyone is in England and everyone is smart"

Existential quantification

Someone in France is smart: $\exists x.\ In(x, france) \wedge Smart(x)$

$(\exists x.\ \phi)$ is equivalent to the *disjunction* of all possible *instantiations* in x of ϕ

$$
\begin{array}{ll}
 & In(kingJohn, france) \wedge Smart(kingJohn) \\
\vee & In(richard, france) \wedge Smart(richard) \\
\vee & In(france, france) \wedge Smart(france) \\
\vee & \ldots
\end{array}
$$

Typically, $\wedge$ is the main connective with $\exists$.

Common mistake: using $\rightarrow$ as the main connective with $\exists$:

$$\exists x.\ In(x, france) \rightarrow Smart(x)$$

is true if there is anyone who is not in France!

Properties of quantifiers

$(\forall x.\forall y.\phi)$ is the same as $(\forall y.\forall x.\phi)$ (Why?)

$(\exists x.\exists y.\phi)$ is the same as $(\exists y.\exists x.\phi)$ (Why?)

$(\exists x.\forall y.\phi)$ is **not** the same as $(\forall y.\exists x.\phi)$

$\exists x.\forall y.Loves(x, y)$
"There is a person who loves everyone in the world"

$\forall y.\exists x.Loves(x, y)$
"Everyone in the world is loved by at least one person" *(not necessarily the same)*

Quantifier duality: each can be expressed using the other:

$\forall x.Likes(x, iceCream) \qquad \neg\exists x.\neg Likes(x, iceCream)$

$\exists x.Likes(x, broccoli) \qquad \neg\forall x.\neg Likes(x, broccoli)$

Equivalences

$$
\begin{aligned}
(\forall x.\, \phi) \wedge \psi &\equiv \forall x.\, (\phi \wedge \psi) \text{ if } x \text{ not free in } \psi \\
(\forall x.\, \phi) \vee \psi &\equiv \forall x.\, (\phi \vee \psi) \text{ if } x \text{ not free in } \psi \\
(\exists x.\, \phi) \wedge \psi &\equiv \exists x.\, (\phi \wedge \psi) \text{ if } x \text{ not free in } \psi \\
(\exists x.\, \phi) \vee \psi &\equiv \exists x.\, (\phi \vee \psi) \text{ if } x \text{ not free in } \psi \\
\forall x.\, \phi \wedge \forall x.\, \psi &\equiv \forall x.\, (\phi \wedge \psi) \\
\exists x.\, \phi \vee \exists x.\, \psi &\equiv \exists x.\, (\phi \vee \psi)
\end{aligned}
$$

$$
\begin{aligned}
\neg \forall x.\, \phi &\equiv \exists x.\, \neg \phi \\
\neg \exists x.\, \phi &\equiv \forall x.\, \neg \phi
\end{aligned}
$$

& propositional equivalences

The Prenex Normal Form

Quantifier prefix + (quantifier free) matrix

$$\forall x_1 \forall x_2 \exists x_3 \ldots \forall x_n \phi$$

1. Elimination of $\rightarrow$ and $\leftrightarrow$
2. push $\neg$ inwards
3. pull quantifiers outwards

E.g. $\neg\forall \mathtt{x}.\ ((\forall \mathtt{x}.\ \mathtt{p}(\mathtt{x})) \rightarrow \mathtt{q}(\mathtt{x}))$

$\neg\forall \mathtt{x}.\ (\neg(\forall \mathtt{x}.\ \mathtt{p}(\mathtt{x})) \vee \mathtt{q}(\mathtt{x}))$

$\exists \mathtt{x}.\ ((\forall \mathtt{x}.\ \mathtt{p}(\mathtt{x})) \wedge \neg\mathtt{q}(\mathtt{x}))$

and now?

Notation: renaming of variables. Let $\phi[x/t]$ be the formula ϕ where all occurrences of x have been replaced by the term t.

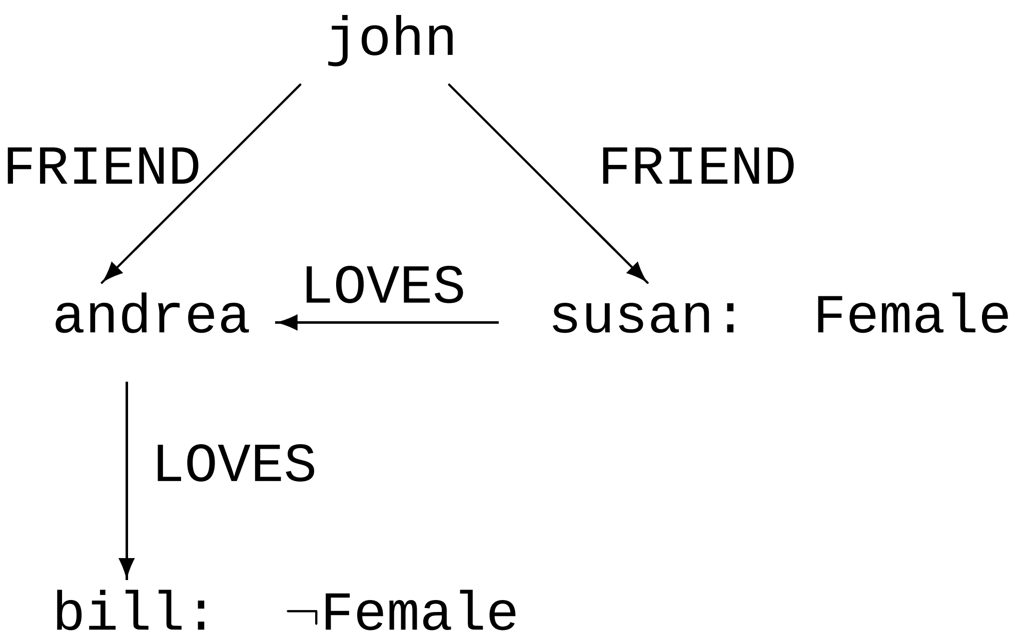
Lemma. Let y be a variable that does not occur in ϕ.
Then we have $\forall x \phi \equiv (\forall x \phi)[x/y]$ and $\exists x \phi \equiv (\exists x \phi)[x/y]$.

Theorem. There is an algorithm that computes for every formula its prenex normal form.

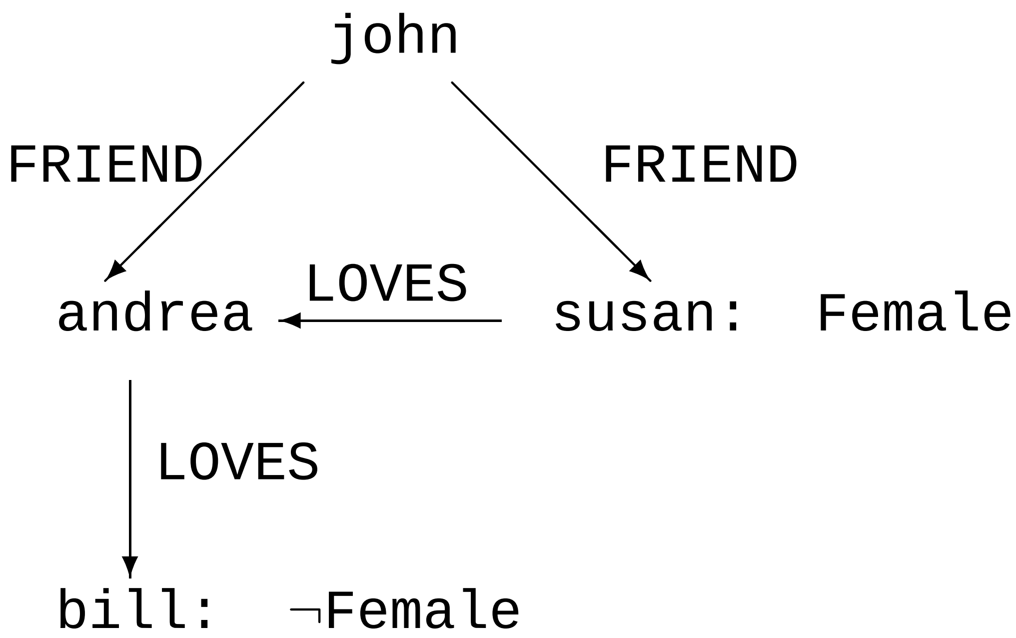
FOL at work: reasoning by cases

$\Gamma =$

FRIEND(john,susan) $\wedge$
FRIEND(john,andrea) $\wedge$
LOVES(susan,andrea) $\wedge$
LOVES(andrea,bill) $\wedge$
Female(susan) $\wedge$
$\neg$Female(bill)



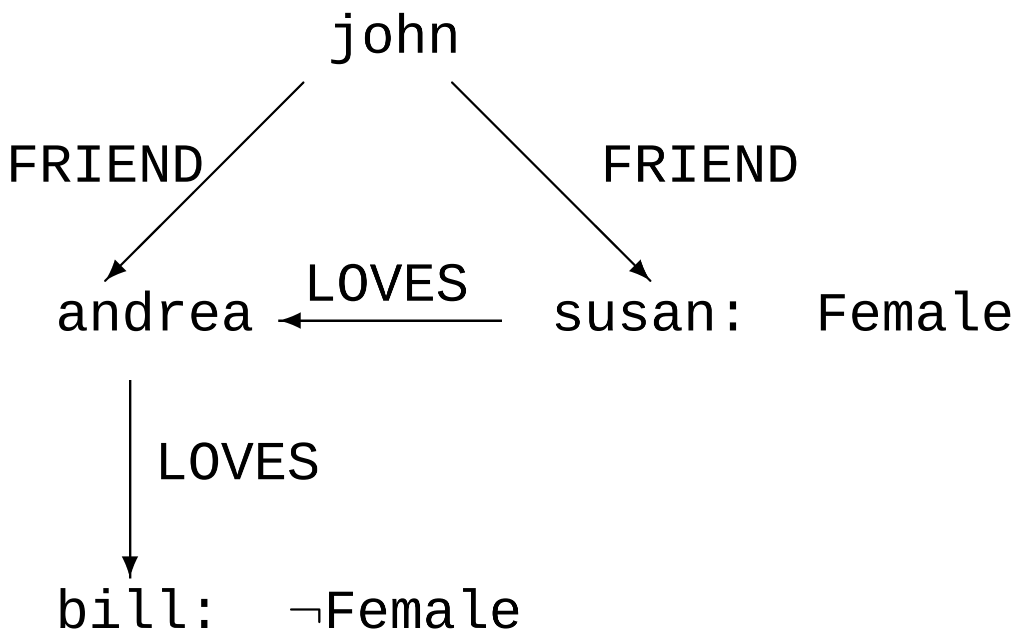
$\Gamma =$



Does John have a female friend loving a male (i.e. not female) person?

$$\Gamma \models \exists X, Y.\ \texttt{FRIEND}(\texttt{john}, X) \land \texttt{Female}(X) \land \texttt{LOVES}(X, Y) \land \neg\texttt{Female}(Y)$$

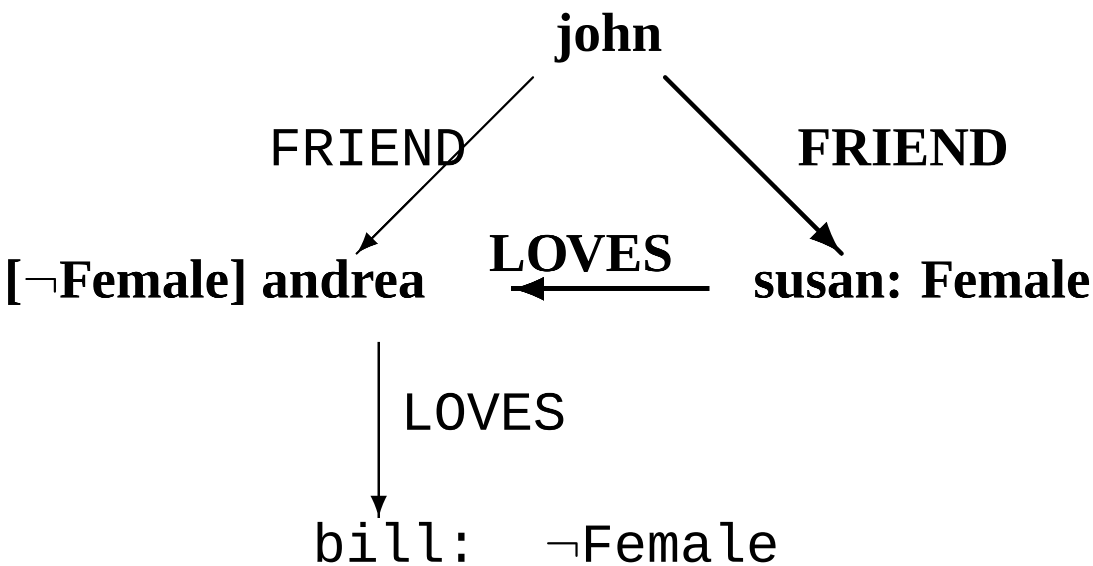
$\Gamma =$



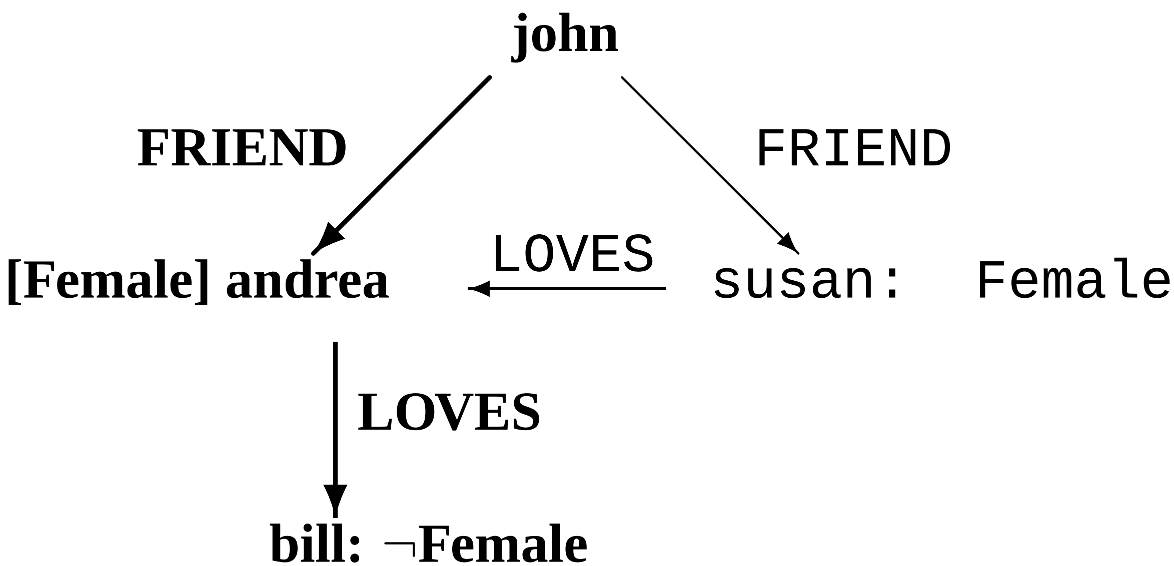
Does John have a female friend loving a male (i.e. not female) person?

YES!

$$\Gamma \models \exists X, Y.\ \texttt{FRIEND}(\texttt{john}, X) \wedge \texttt{Female}(X) \wedge \\ \texttt{LOVES}(X, Y) \wedge \neg\texttt{Female}(Y)$$



FRIEND(john,susan), Female(susan),
LOVES(susan,andrea), ¬ Female(andrea)



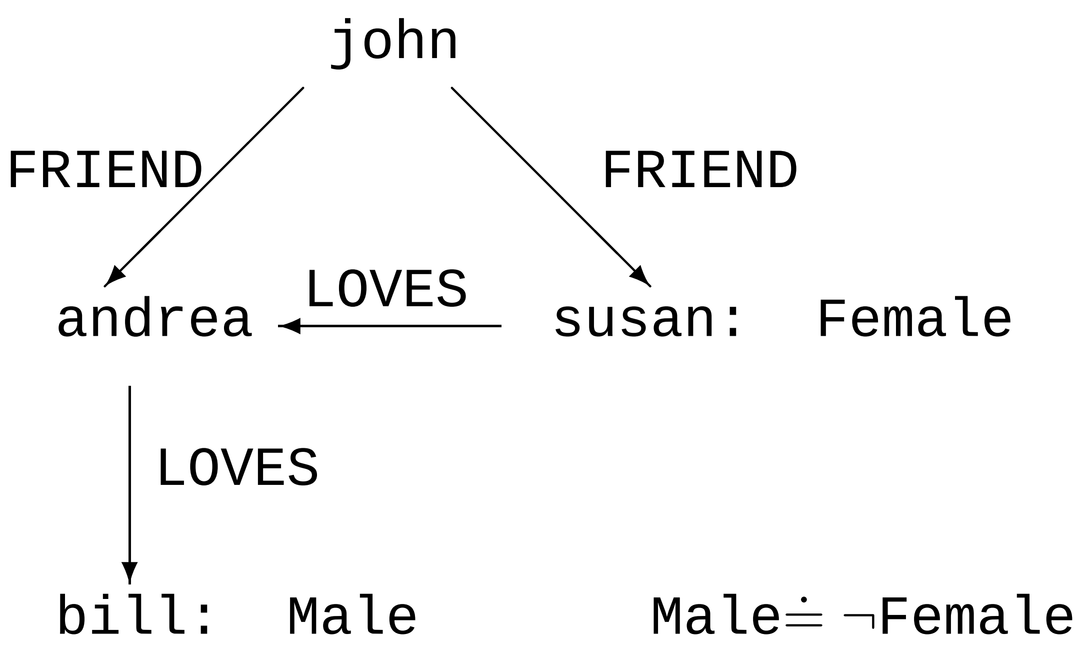
FRIEND(john,andrea), Female(andrea),
LOVES(andrea,bill), ¬ Female(bill)

Exercise

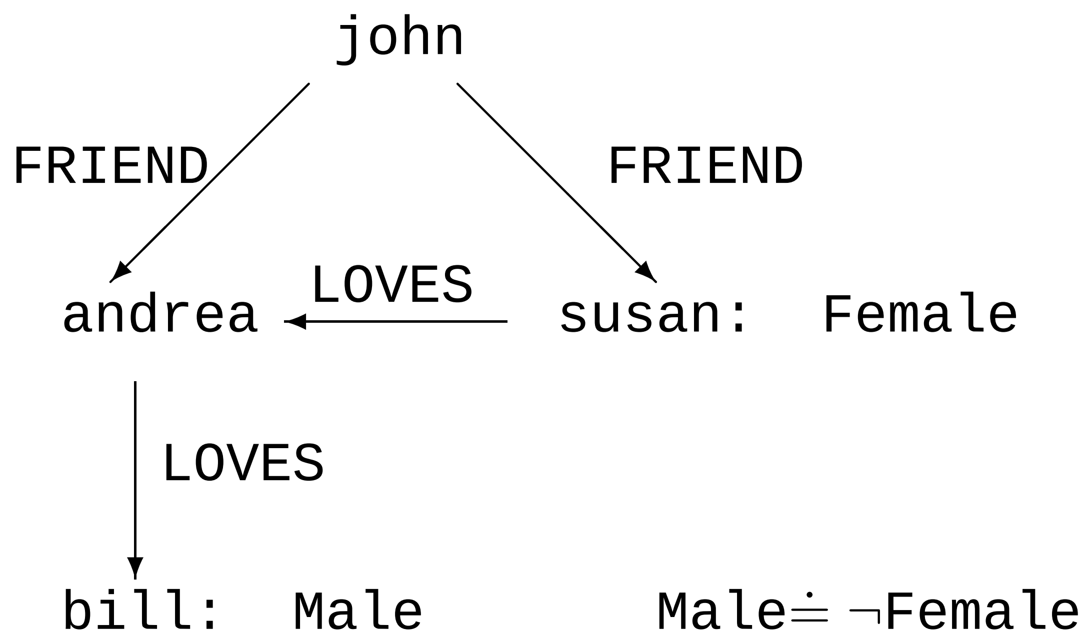
Write the possible models of the theory.

Theories and Models

$$\begin{aligned}\Gamma_1 = {} & \texttt{FRIEND}(\texttt{john}, \texttt{susan}) \wedge \\ & \texttt{FRIEND}(\texttt{john}, \texttt{andrea}) \wedge \\ & \texttt{LOVES}(\texttt{susan}, \texttt{andrea}) \wedge \\ & \texttt{LOVES}(\texttt{andrea}, \texttt{bill}) \wedge \\ & \texttt{Female}(\texttt{susan}) \wedge \\ & \texttt{Male}(\texttt{bill}) \wedge \\ & \forall X.\ \texttt{Male}(X) \leftrightarrow \neg\texttt{Female}(X)\end{aligned}$$



$\Gamma_1 =$



Does John have a female friend loving a male person?

$$\Gamma_1 \models \exists X, Y.\ \texttt{FRIEND}(\texttt{john}, X) \wedge \texttt{Female}(X) \wedge$$
$$\texttt{LOVES}(X, Y) \wedge \texttt{Male}(Y)$$

$$\begin{aligned}\Gamma = {} & \texttt{FRIEND(john,susan)} \wedge \\ & \texttt{FRIEND(john,andrea)} \wedge \\ & \texttt{LOVES(susan,andrea)} \wedge \\ & \texttt{LOVES(andrea,bill)} \wedge \\ & \texttt{Female(susan)} \wedge \\ & \neg\texttt{Female(bill)}\end{aligned}$$

$$\begin{aligned}\Gamma_1 = {} & \texttt{FRIEND}(\texttt{john}, \texttt{susan}) \wedge \\ & \texttt{FRIEND}(\texttt{john}, \texttt{andrea}) \wedge \\ & \texttt{LOVES}(\texttt{susan}, \texttt{andrea}) \wedge \\ & \texttt{LOVES}(\texttt{andrea}, \texttt{bill}) \wedge \\ & \texttt{Female}(\texttt{susan}) \wedge \\ & \texttt{Male}(\texttt{bill}) \wedge \\ & \forall X\texttt{.}\ \texttt{Male}(X) \leftrightarrow \neg\texttt{Female}(X)\end{aligned}$$

$$\Gamma = \texttt{FRIEND(john,susan)} \wedge$$
$$\texttt{FRIEND(john,andrea)} \wedge$$
$$\texttt{LOVES(susan,andrea)} \wedge$$
$$\texttt{LOVES(andrea,bill)} \wedge$$
$$\texttt{Female(susan)} \wedge$$
$$\neg\texttt{Female(bill)}$$

$$\Delta = \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\}$$
$$\texttt{Female}^{\mathcal{I}} = \{\texttt{susan}\}$$

$$\Gamma_1 = \texttt{FRIEND}(\texttt{john}, \texttt{susan}) \wedge$$
$$\texttt{FRIEND}(\texttt{john}, \texttt{andrea}) \wedge$$
$$\texttt{LOVES}(\texttt{susan}, \texttt{andrea}) \wedge$$
$$\texttt{LOVES}(\texttt{andrea}, \texttt{bill}) \wedge$$
$$\texttt{Female}(\texttt{susan}) \wedge$$
$$\texttt{Male}(\texttt{bill}) \wedge$$
$$\forall X.\ \texttt{Male}(X) \leftrightarrow \neg\texttt{Female}(X)$$

$$\Gamma = \texttt{FRIEND(john,susan)} \wedge \texttt{FRIEND(john,andrea)} \wedge \texttt{LOVES(susan,andrea)} \wedge \texttt{LOVES(andrea,bill)} \wedge \texttt{Female(susan)} \wedge \neg\texttt{Female(bill)}$$

$$\Delta = \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\}$$
$$\texttt{Female}^{\mathcal{I}} = \{\texttt{susan}\}$$

$$\Gamma_1 = \texttt{FRIEND}(\texttt{john}, \texttt{susan}) \wedge \texttt{FRIEND}(\texttt{john}, \texttt{andrea}) \wedge \texttt{LOVES}(\texttt{susan}, \texttt{andrea}) \wedge \texttt{LOVES}(\texttt{andrea}, \texttt{bill}) \wedge \texttt{Female}(\texttt{susan}) \wedge \texttt{Male}(\texttt{bill}) \wedge \forall X.\, \texttt{Male}(X) \leftrightarrow \neg\texttt{Female}(X)$$

$$\Delta^{\mathcal{I}_1} = \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\}$$
$$\texttt{Female}^{\mathcal{I}_1} = \{\texttt{susan}, \texttt{andrea}\}$$
$$\texttt{Male}^{\mathcal{I}_1} = \{\texttt{bill}, \texttt{john}\}$$

$$\Delta^{\mathcal{I}_2} = \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\}$$
$$\texttt{Female}^{\mathcal{I}_2} = \{\texttt{susan}\}$$
$$\texttt{Male}^{\mathcal{I}_2} = \{\texttt{bill}, \texttt{andrea}, \texttt{john}\}$$

$$\Delta^{\mathcal{I}_1} = \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\}$$
$$\texttt{Female}^{\mathcal{I}_1} = \{\texttt{susan}, \texttt{andrea}, \texttt{john}\}$$
$$\texttt{Male}^{\mathcal{I}_1} = \{\texttt{bill}\}$$

$$\Delta^{\mathcal{I}_2} = \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\}$$
$$\texttt{Female}^{\mathcal{I}_2} = \{\texttt{susan}, \texttt{john}\}$$
$$\texttt{Male}^{\mathcal{I}_2} = \{\texttt{bill}, \texttt{andrea}\}$$

$\Gamma \not\models \texttt{Female}(\texttt{andrea})$

$\Gamma \not\models \neg\texttt{Female}(\texttt{andrea})$

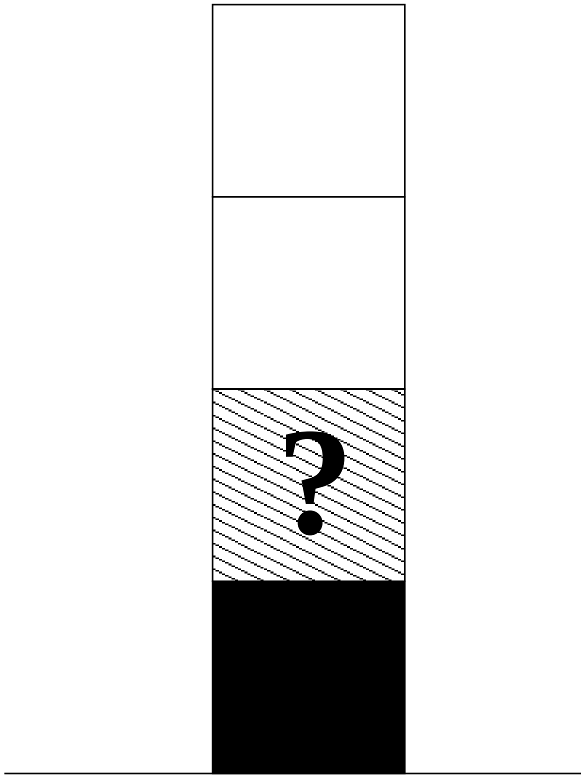
$\Gamma_1 \not\models \texttt{Female}(\texttt{andrea})$

$\Gamma_1 \not\models \neg\texttt{Female}(\texttt{andrea})$

$\Gamma_1 \not\models \texttt{Male}(\texttt{andrea})$

$\Gamma_1 \not\models \neg\texttt{Male}(\texttt{andrea})$

Exercise



Is it true that the top block is on a white block touching a black block?