Fixpoint Extensions of Temporal Description Logics

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Abstract

In this paper we introduce a decidable fixpoint extension of temporal Description Logics. We exploit the decidability results obtained for various monodic extensions of Description Logics to obtain decidability and tight complexity results for temporal fixpoint extensions of these Description Logics and more generally for the decidable monodic fragments of first order logic.

1 Introduction

Monodic temporal extensions of various (decidable) fragments of first-order logic have been studied employing the *quasi-model* approach of Wolter and Zacharyashev [Hodkinson *et al.*, 2000; 2001]. Their technique has been successfully applied to a variety of decidable fragments, e.g., to the $\mathcal{ALC}$ and $\mathcal{DLR}$ description logics, to the guarded fragment $\mathcal{GF}$, or to the two variables fragment. In addition, the complexity of the decision procedures for these fragments has been studied [Hodkinson *et al.*, 2003]. All these papers have focused on the standard first-order temporal logic that uses the $\mathcal{U}$ (until) and $\mathcal{S}$ (since) connectives, save [Gabbay *et al.*, 2003] that studies an extension of a multi-modal (but still first-order) logic.

This paper considers a different dimension of the problem: it proposes to enhance the *temporal part* of the language instead of varying the first-order fragment. The paper shows that the original quasimodel technique is amenable to using a much more expressive language over the temporal structure, while retaining decidability for many of the fragments studied in the $\mathcal{US}$ case.

Indeed, first-order temporal logics have been shown to lack certain expressiveness related, e.g., to expressing periodic events. This shortcoming has been identified by

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$$
\begin{array}{rcl}
\top^{\mathcal{I}(t)} & = & \Delta \\
\bot^{\mathcal{I}(t)} & = & \emptyset \\
(\neg C)^{\mathcal{I}(t)} & = & \Delta \setminus C^{\mathcal{I}(t)} \\
(C \sqcap D)^{\mathcal{I}(t)} & = & C^{\mathcal{I}(t)} \cap D^{\mathcal{I}(t)} \\
(C \sqcup D)^{\mathcal{I}(t)} & = & C^{\mathcal{I}(t)} \cup D^{\mathcal{I}(t)} \\
(\forall R \,.\, C)^{\mathcal{I}(t)} & = & \{i \in \Delta \mid \forall j \,.\, R^{\mathcal{I}(t)}(i,j) \Rightarrow C^{\mathcal{I}(t)}_{\alpha(t)}(j)\} \\
(\exists R \,.\, C)^{\mathcal{I}(t)} & = & \{i \in \Delta \mid \exists j \,.\, R^{\mathcal{I}(t)}_{\alpha(t)}(i,j) \wedge C^{\mathcal{I}(t)}_{\alpha(t)}(j)\} \\
(\bullet C)^{\mathcal{I}(t)} & = & C^{\mathcal{I}(t-1)} \\
(\circ C)^{\mathcal{I}(t)} & = & C^{\mathcal{I}(t+1)} \\
(\mu A \,.\, C)^{\mathcal{I}(t)} & = & \bigcup_{k \geq 0} (C^k[\bot/A])^{\mathcal{I}(t)}
\end{array}
$$

Figure 1: The semantics of $\mathcal{ALC}_{\mu\bullet\circ}$

Wolper [Wolper, 1983] and various extensions have been proposed, e.g., the extended temporal logic (ETL) [Wolper, 1983] or the temporal fixpoint calculus [Vardi, 1988].

In order to focus on the actual temporal dimension of the problem, the results this paper are formulated with respect to a very simple description logic, $\mathcal{ALC}$. However, the results can be easily extended to other decidable fragments of first-order logic, provided they satisfy the monodicity restriction. The results are as follows:

- We show a decision procedure for monodic $\mathcal{ALC}_{\mu\bullet\circ}$, a temporal description logic strictly more expressive than $\mathcal{ALC}_{\mathcal{US}}$, based on the temporal fixpoint calculus [Vardi, 1988]. We also show complexity bounds on the decision procedure that mirror those for $\mathcal{ALC}_{\mathcal{US}}$. Thus, from the complexity standpoint, the extension is *for free*.
- We show that the proposed extension is also applicable to more expressive dialects of description logics, e.g., $\mathcal{CIQ}$, $\mathcal{CIO}$, and $\mathcal{DLR}$, and to other decidable fragments of first order logic, e.g., $\mathcal{GF}$, that satisfy the monodic restriction.

The paper is organised as follows: Section 2 provides the necessary definitions. Section 3 presents decidability of the temporal fixpoint extension of a simple description logic $\mathcal{ALC}$. It also presents complexity bounds for the associated reasoning problems. However, due to a heavy reliance on a rather complex *quasimodel* machinery [Hodkinson *et al.*, 2000], the actual proofs are sketched only in an appendix of the paper. Section 4 discusses the applicability of the results to a wider range of description logics and other decidable fragments of first-order logic. Section 5 concludes with directions for future research.

2 Definitions

Definition 1 ($\mathcal{ALC}_{\mu\bullet\circ}$ syntax) *Concepts in the language $\mathcal{ALC}_{\mu\bullet\circ}$ is defined by the following abstract syntax.*

$$C, D ::= A \mid \top \mid \bot \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \forall R \,.\, C \mid \exists R \,.\, C \mid \bullet C \mid \circ C \mid \mu A \,.\, C$$

The $\circ$ and $\bullet$ are the usual next-time *and* previous-time *operators. Given a $\mu A.C$* fixpoint *concept expression—where A is a primitive concept description—we require that*

A occurs positively in the concept expression C, i.e., it appears within even numbers of negations ($\forall R . C$ counts as a negation). Informally, the concept $\mu A . C$ denotes the least fixpoint solution to the equation $A \equiv C(A)$, where here C is seen as a function of the concept variable A.

Formulæ of $\mathcal{ALC}_{\mu\bullet\circ}$ are defined by the abstract syntax

$$\varphi ::= C \sqsubseteq D \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, \bullet\varphi \,|\, \circ\varphi \,|\, \mu x.\varphi.$$

Similarly to concepts descriptions, the propositional variable x in $\mu x.\varphi$ can only appear in φ under an even number of negations.

We say that a fixpoint concept $\mu A . C$ is a *future concept* (*past concept*) if every occurrence of the concept variable A in C is in the scope of one or more $\circ$ ($\bullet$) operators but no $\bullet$ ($\circ$) operators, respectively.

Definition 2 (Monodic Fragment) *We define $\mathcal{ALC}^{\mathrm{mon}}_{\mu\bullet\circ}$ (the monodic fragment of $\mathcal{ALC}_{\mu\bullet\circ}$) to be $\mathcal{ALC}_{\mu\bullet\circ}$ restricted to concepts in which free fixpoint variables do not occur in the scope of role restrictions ($\forall R . C$ and $\exists R . C$).*

The monodic restriction essentially *separates* the handling of the *temporal* dimension using the fixpoint and next/previous time operators from the *data* dimension captured by $\mathcal{ALC}$ roles. The boolean structure is shared between these two parts. This restriction corresponds to our original intent of extending the *temporal part* of the language without affecting the first-order fragment. In the following, we consider only the monodic fragment. This restriction makes the expressiveness of the language substantially different from the standard extensions of non temporal description logics with fixpoints—e.g., [Calvanese *et al.*, 1999]—since the fixpoint operates only on the temporal part of the language.

Definition 3 ($\mathcal{ALC}_{\mu\bullet\circ}$ semantics) *An $\mathcal{ALC}_{\mu\bullet\circ}$ interpretation structure is a triple $\mathcal{I} = (\mathcal{T}, \Delta, (\cdot)^{\mathcal{I}(t)})$ where $\mathcal{T}$ is a flow of time (a discrete, linearly ordered set), Δ is a non-empty domain of* objects, *and $(\cdot)^{\mathcal{I}(t)}$ is an* interpretation function *that for every $t \in \mathcal{T}$ provides an interpretation for concepts and roles at time t (i.e., $C^{\mathcal{I}(t)} \subseteq \Delta$ and $R^{\mathcal{I}(t)} \subseteq \Delta \times \Delta$). The interpretation should satisfy the equations in Fig. 1; $C^k[\bot/A]$ stands for the concept description obtained by unfolding a fixpoint concept k times*[1]. *Note that we make the constant domain assumption, i.e., Δ does not change in time.*

Given a formula φ, an interpretation $\mathcal{I}$, and a time point $t \in \mathcal{T}$, the truth-relation $\mathcal{I}, t \models \varphi$ (φ holds in $\mathcal{I}$ at moment t) is defined inductively as follows:

$$\begin{array}{lll}
\mathcal{I}, t \models C \sqsubseteq D & \textit{iff} & C^{\mathcal{I}(t)} \subseteq D^{\mathcal{I}(t)} \\
\mathcal{I}, t \models \neg\varphi & \textit{iff} & \mathcal{I}, t \not\models \varphi \\
\mathcal{I}, t \models \varphi \wedge \psi & \textit{iff} & \mathcal{I}, t \models \varphi \textit{ and } \mathcal{I}, t \models \psi \\
\mathcal{I}, t \models \bullet\varphi & \textit{iff} & \mathcal{I}, t-1 \models \varphi \\
\mathcal{I}, t \models \circ\varphi & \textit{iff} & \mathcal{I}, t+1 \models \varphi \\
\mathcal{I}, t \models \mu x.\varphi & \textit{iff} & \mathcal{I}, t \models \varphi^k[\textit{false}/x] \textit{ for some } k \geq 0
\end{array}$$

[1] Due to the restrictions on the occurrence of A in C, this definition is equivalennt to the more common *intersection of models* definition [Calvanese *et al.*, 1999; Vardi, 1988].

A formula φ is satisfiable *if there is a temporal interpretation $\mathcal{I}$ such that $\mathcal{I}, t \models \varphi$, for some time point t; $\mathcal{I}$ is called a model for φ at t. A concept C is* satisfiable *if there is an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}(t)} \neq \emptyset$ for some time point t. We say that φ is* globally satisfiable *if there is an interpretation $\mathcal{I}$ such that $\mathcal{I}, t \models \varphi$ for every t ($\mathcal{I} \models \varphi$, in symbols). We say that φ (*globally*)* implies *ψ and write $\varphi \models \psi$ if we have $\mathcal{I} \models \psi$ whenever $\mathcal{I} \models \psi$.*

Note that a concept C is satisfiable iff $\neg(C \sqsubseteq \bot)$ is satisfiable, a formula φ is globally satisfiable iff $\Box\blacksquare\varphi$ is satisfiable, and $\varphi \models \psi$ iff $\Box\blacksquare\varphi \wedge \neg\psi$ is not satisfiable[2]. Thus, all reasoning tasks connected with the notions introduced above reduce to satisfiability of formulas.

3 Properties of $\mathcal{ALC}^{\mathrm{mon}}_{\mu\bullet\circ}$

The language we have introduced so far has clear advantages over the description logic $\mathcal{ALC}_{\mathcal{US}}$. The addition of the fixpoint allows to express the notion of evenness and periodicity, which is not expressible in $\mathcal{ALC}_{\mathcal{US}}$. On the other hand, the $\mathcal{U}$ (until), $\mathcal{S}$ (since), and the other standard temporal connectives of $\mathcal{ALC}_{\mathcal{US}}$ can be encoded in the logic with fixpoints $\mathcal{ALC}^{\mathrm{mon}}_{\mu\bullet\circ}$. So, for example:

$$
\begin{array}{rlrl}
\Diamond C & \equiv \mu A \,.\, (C \sqcup \circ A) & \blacklozenge C & \equiv \mu A \,.\, (C \sqcup \bullet A) \\
\Box C & \equiv \mu A \,.\, (C \sqcap \circ A) & \blacksquare C & \equiv \mu A \,.\, (C \sqcap \bullet A) \\
C \,\mathcal{U}\, D & \equiv \mu A \,.\, (D \sqcup (C \sqcap \circ A)) & C \,\mathcal{S}\, D & \equiv \mu A \,.\, (D \sqcup (C \sqcap \bullet A))
\end{array}
$$

Moreover, based on results on the expressive power of propositional linear time temporal logics [Wolper, 1983] we can prove the following:

Proposition 4 *$\mathcal{ALC}^{\mathrm{mon}}_{\mu\bullet\circ}$ ($\mathcal{ALC}^{\mathrm{mon}}_{\mu\circ}$) is more expressive than $\mathcal{ALC}_{\mathcal{US}}$ ($\mathcal{ALC}_{\mathcal{U}}$), respectively.*

A typical example of the additional expressive power of $\mathcal{ALC}^{\mathrm{mon}}_{\mu\bullet\circ}$ would be a property which should hold true every k time points, starting from the current one. For example, a catholic priest celebrates the Mass every seven days:

$$\texttt{Catholic}-\texttt{Priest} \sqsubseteq \mu\texttt{A} \,.\, (\forall\texttt{celebrate} \,.\, \texttt{Mass} \sqcap \circ\circ\circ\circ\circ\circ\circ\texttt{A})$$

Observe that, in the restricted monodic fragment introduced in the previous section, the descriptions within a single state are purely first-order; the fixpoint only affects the temporal part of the language and it does not change the first order nature of the pure description logic.

Note also that in the monodic fragment it is impossible to express temporalised roles. It is well known that these would lead to an undecidable satisfiability problem even in $\mathcal{ALC}_{\mathcal{U}}$ [Hodkinson *et al.*, 2000].

3.1 Complexity of Reasoning

We summarise the computational properties of $\mathcal{ALC}_{\mu\bullet\circ}$. We consider only the integer-like flow of time $\mathcal{T} = (Z, <)$.

[2]The $\Box$ and $\blacksquare$ operators are defined in Saction 3.

Theorem 5 *The formula satisfiability problem for* $\mathcal{ALC}^{\text{mon}}_{\mu\bullet\bigcirc}$ *is decidable.*

The restrictions on the occurrence of the fixpoint concept variable in the fixpoint constructor of $\mathcal{ALC}^{\text{mon}}_{\mu\bullet\bigcirc}$ guarantees that the fixpoint is only applied with respect to the temporal dimension of the underlying structures; within a single state, $\mathcal{ALC}^{\text{mon}}_{\mu\bullet\bigcirc}$ remains a fragment of first-order logic. Therefore, the quasi-model machinery can be applied in this setting. In addition, complexity results from the case of first-order temporal connectives transfer to our setting:

Theorem 6 *The formula satisfiability problem for* $\mathcal{ALC}^{\text{mon}}_{\mu\bullet\bigcirc}$ *is EXPSPACE-complete.*

Here, the upper bound relies on a temporal fixpoint extension of [Wolter and Zakharyaschev, 1999; Artale *et al.*, 2002; Gabbay *et al.*, 2003; Schild, 1993]; hardness holds even for $\mathcal{ALC}_{\Box}$.

Theorem 7 *The concept satisfiability problem for* $\mathcal{ALC}^{\text{mon}}_{\mu\bullet\bigcirc}$ *is PSPACE-complete.*

The upper bound follows from an extension of [Schild, 1993]; the PSPACE complexity of the propositional temporal logic [Sistla and Clarke, 1985] is the same as the temporal fixpoint calculus [Vardi, 1988]. The lower bound follows from PSPACE-hardness for $\mathcal{ALC}_{\mathcal{U}}$.

4 Temporal Fixpoints and Other Fragments

The first extension of $\mathcal{ALC}^{\text{mon}}_{\mu\bullet\bigcirc}$ is allowing ABox assertions of the form $a : C$ and aRb , for a and b names of individual objects in Δ, to be considered atomic formulas alongside $C \sqsubseteq D$. This extension does not change the computational properties of $\mathcal{ALC}^{\text{mon}}_{\mu\bullet\bigcirc}$:

Theorem 8 *The formula satisfiability problem for* $\mathcal{ALC}^{\text{mon}}_{\mu\bullet\bigcirc}$ *with ABox is decidable; concept satisfiability is PSPACE-complete, formula satisfiability is EXPSPACE-complete.*

The decidability result can also be extended to more powerful description logics and other decidable fragments of first-order logic, in particular to the following:

Theorem 9 *Satisfiability of the monodic fixpoint temporal extensions of*

- $\mathcal{CIQ}$, $\mathcal{CQO}$, *and* $\mathcal{CIO}$;
- $\mathcal{DLR}$;
- $\mathcal{GF}$

is decidable.

These results are based on *patching* the decidability proofs in, e.g., [Wolter and Zakharyaschev, 1999; Hodkinson, 2002; Artale *et al.*, 2002] using our technique. It also shows that the temporal fixpoint extension is orthogonal to the consideration of the first-order fragment (as long as integer-like flow of time is used).

5 Conclusion

The paper provides a *modular* extension of fragments of first-order temporal logics that have been shown decidable using the *quasimodel* technique to allowing fixpoints to be used in the temporal dimension. This extension enhances the expressive power of the languages (for example, evenness is now definable over the temporal dimension). The extension is modular in the sense that it can be applied to a large number of monodic decidable fragments of first-order temporal logic.

5.1 Open Problems

We are currently studying several extensions of the framework proposed in this paper, namely:

- Allowing fixpoint variables to occur in scopes of $\forall R.\cdot$ and $\exists R.\cdot$, while requiring the fixpoints to affect only the temporal dimension, e.g., by requiring all fixpoint variables to occur in the scope of the next time ($\bigcirc$) or previous time ($\bullet$) operators;
- Allowing full $\mathcal{ALC}_{\mu\bullet\bigcirc}$; in this case, the concept descriptions are no longer first-order in every state and the quasi-model technique cannot be applied directly.

Other extensions relate to studying temporal fixpoints for other flows of time, to allowing even more expressive temporal languages (e.g., S1S), and to interaction with queries [Artale *et al.*, 2002].

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Appendix. A quick introduction to Quasimodels

In this appendix we introduce the necessary background definitions related to the *quasimodel*-based technique used to show decidability of various monodic temporal extensions of decidable fragments of first-order logic [Hodkinson *et al.*, 2000; Wolter and Zakharyaschev, 1999]. We modify these to suit $\mathcal{ALC}^{\text{mon}}_{\mu\bigcirc}$ over the flow of time $(N, <)$ (for simplicity, we consider only the future fragment here, since treating the combined past/future fragment is essentially the same.) Consider a formula $\varphi \in \mathcal{ALC}^{\text{mon}}_{\mu\bigcirc}$. We define sets of concepts and formulas, $\mathsf{con}\,\varphi$ and $\mathsf{sub}\,\varphi$, to be the sets of all concepts and all subformulas of φ, and their negations, respectively, in which all the concepts D (formulas ϑ) of the form $\bigcirc C$ and $\mu A \,.\, C$ ($\bigcirc\psi$ and $\mu x.\psi$) are replaced by auxiliary primitive concepts A_D (propositions p_ϑ), respectively, that do not appear in φ.

Definition 10 (Quasiworlds for φ) *Given an $\mathcal{ALC}$ interpretation I (that interprets primitive symbols in φ and the above auxiliary symbols), we define a* quasiworld *w_I for φ to be the tuple*

$$\left\langle \left\{ \{C \in \mathsf{con}\,\varphi : a \in C^I\} : a \in \Delta \right\}, \{\psi \in \mathsf{sub}\,\varphi : I \models \psi\} \right\rangle$$

The important observations at this point are that, for a fixed formula φ, (1) there are only finitely many (distinct) quasiworlds, and (2) for decidable logics, they can be effectively constructed.

The interpretations I for the individual quasiworlds will serve as templates for *models of single states* in an overall model for φ. However, these models must be *coherent* along the temporal dimension. This, in particular, requires that the auxiliary concepts and formulas standing for temporal subconcepts/subformulas must behave according to the definitions of the temporal connectives and fixpoints.

Definition 11 (Runs and Quasimodel for φ) *Let $W = \langle w_i : i \in N \rangle$ be a sequence of quasiworlds of the form $\langle T_i, \Psi_i \rangle$, indexed by natural numbers. We say that a sequence $r = \langle t_i : i \in N \rangle$, where $t_i \subseteq \mathsf{con}\,\varphi$, is a* run *if*

1. $t_i \in T_i$,
2. $A_{\bigcirc C} \in t_i$ *iff* $C \in t_{i+1}$; *and*
3. $A_{\mu A.C} \in t_i$ *iff* $C \in t_i$ *and that there is no infinitely regenerating sequence of* $A_{\mu A.C}$ *in the run* r.

We say that W is a quasimodel *for φ if*

1. *for every* $t \in T_i$ *and* $i \in N$ *there is a run* r *such that* $t \in r$;
2. $p_{\bigcirc\psi} \in \Psi_i$ *iff* $\psi \in \Psi_{i+1}$; *and*
3. $p_{\mu x.\psi} \in \Psi_i$ *iff* $\psi \in \Psi_i$ *and that there is no infinitely regenerating sequence of* $p_{\mu x.\psi} \in \Psi_i$ *in the* W.

We say that W satisfies *φ iff $\varphi \in \Psi_0$.*

Runs in quasimodels *relate* domain elements from the domains of different quasiworlds yielding a coherent model for φ (here, the ability to *copy* domain elements in the individual states sufficiently many times is essential to have sufficiently many runs). The remaining conditions then ensure, in a similar way, that the formulas holding in the individual quasiworlds are also coherent with the temporal structure $(N, <)$. The conditions associated with the auxiliary concepts and propositions related to fixpoints are analogous to those introduced by Vardi in the case of the temporal fixpoint calculus [Vardi, 1988] to guarantee the *minimality* of the interpretations of the fixpoints.

Proposition 12 *An $\mathcal{ALC}^{\mathrm{mon}}_{\mu\bigcirc}$ formula φ is satisfiable if and only if it is satisfiable in a quasimodel for φ.*

Thus, it suffices to check whether φ has a quasimodel. This can be done, e.g., by embedding the quasimodel conditions into S1S, similarly to [Hodkinson *et al.*, 2000]. In particular, S1S is more than sufficient to enforce the temporal fixpoint conditions. To achieve tight complexity bounds, we can employ the fact that, for satisfiable $\mathcal{ALC}^{\mathrm{mon}}_{\mu\bigcirc}$ formulas, periodic quasimodels exist, similarly to the $\mathcal{US}$ case [Hodkinson *et al.*, 2000]. This yields the required complexity bounds.