Description Logics

Foundations of First Order Logic

Enrico Franconi

franconi@cs.man.ac.uk
http://www.cs.man.ac.uk/~franconi

Department of Computer Science, University of Manchester
Motivation

- We can already do a lot with propositional logic.
- But it is unpleasant that we cannot access the \textit{structure} of atomic sentences.
- Atomic formulas of propositional logic are \textit{too atomic} – they are just statements which may be true or false but which have no internal structure.
- In \textit{First Order Logic} (FOL) the atomic formulas are interpreted as statements about \textit{relationships between objects}. 
Predicates and Constants

Let’s consider the statements:

- *Mary is female*
  
  *John is male*
  
  *Mary and John are siblings*

In propositional logic the above statements are atomic propositions:

- *Mary-is-female*
  
  *John-is-male*
  
  *Mary-and-John-are-siblings*

In FOL atomic statements use predicates, with constants as argument:

- *Female(mary)*
  
  *Male(john)*
  
  *siblings(mary, john)*
Variables and Quantifiers

Let’s consider the statements:

- *Everybody is male or female*
- *A male is not a female*

In FOL predicates may have variables as arguments, whose value is bounded by quantifiers:

- $\forall x. \text{Male}(x) \lor \text{Female}(x)$
- $\forall x. \text{Male}(x) \rightarrow \neg \text{Female}(x)$

Deduction (why?):

- *Mary is not male*
- $\neg \text{Male}(\text{mary})$
Functions

Let's consider the statement:

- *The father of a person is male*

In FOL objects of the domain may be denoted by functions applied to (other) objects:

- \( \forall x. \text{Male}(\text{father}(x)) \)
Syntax of FOL: atomic sentences

Countably infinite **supply of symbols** *(signature)*:

- variable symbols: $x, y, z, \ldots$
- $n$-ary function symbols: $f, g, h, \ldots$
- individual constants: $a, b, c, \ldots$
- $n$-are predicate symbols: $P, Q, R, \ldots$

**Terms:** $t \rightarrow x$ variable

- $a$ constant
- $f(t_1, \ldots, t_n)$ function application

**Ground terms:** terms that do not contain variables

**Formulas:** $\phi \rightarrow P(t_1, \ldots, t_n)$ atomic formulas

E.g., $\text{Brother}(\text{kingJohn}, \text{richardTheLionheart})$

$> (\text{length}(\text{leftLegOf}(\text{richard})), \text{length}(\text{leftLegOf}(\text{kingJohn})))$
Syntax of FOL: propositional sentences

Formulas: \( \phi, \psi \rightarrow P(t_1, \ldots, t_n) \)  atomic formulas

- \( \bot \) false
- \( \top \) true
- \( \neg \phi \) negation
- \( \phi \land \psi \) conjunction
- \( \phi \lor \psi \) disjunction
- \( \phi \rightarrow \psi \) implication
- \( \phi \leftrightarrow \psi \) equivalence

• (Ground) **atoms** and (ground) **literals**.

E.g. \( \text{Sibling}(\text{kingJohn}, \text{richard}) \rightarrow \text{Sibling}(\text{richard}, \text{kingJohn}) \)

\( \geq (1, 2) \lor \leq (1, 2) \)

\( \geq (1, 2) \land \neg \geq (1, 2) \)
Syntax of full FOL

Formulas: \( \phi, \psi \rightarrow P(t_1, \ldots, t_n) \) atomic formulas

- \( \bot \) false
- \( \top \) true
- \( \neg \phi \) negation
- \( \phi \land \psi \) conjunction
- \( \phi \lor \psi \) disjunction
- \( \phi \rightarrow \psi \) implication
- \( \phi \leftrightarrow \psi \) equivalence
- \( \forall x. \phi \) universal quantification
- \( \exists x. \phi \) existential quantification

E.g. Everyone in England is smart: \( \forall x. \text{In}(x, \text{england}) \rightarrow \text{Smart}(x) \)

Someone in France is smart: \( \exists x. \text{In}(x, \text{france}) \land \text{Smart}(x) \)
Summary of Syntax of FOL

• Terms
  • variables
  • constants
  • functions

• Literals
  • atomic formula
    • relation (predicate)
  • negation

• Well formed formulas
  • truth-functional connectives
  • existential and universal quantifiers
Semantics of FOL: intuition

- Just like in propositional logic, a (complex) FOL formula may be true (or false) with respect to a given interpretation.

- An interpretation specifies referents for:
  - *constant symbols* $\rightarrow$ *objects*
  - *predicate symbols* $\rightarrow$ *relations*
  - *function symbols* $\rightarrow$ *functional relations*

- An atomic sentence $P(t_1, \ldots, t_n)$ is true in a given interpretation iff the *objects* referred to by $t_1, \ldots, t_n$ are in the *relation* referred to by the predicate $P$.

- An interpretation in which a formula is true is called a *model* for the formula.
Models for FOL: Example

objects

relations: sets of tuples of objects

functional relations: all tuples of objects + "value" object
Semantic of FOL: Interpretations

**Interpretation:** $\mathcal{I} = \langle \Delta, \cdot^\mathcal{I} \rangle$ where $\Delta$ is an arbitrary non-empty set and $\mathcal{I}$ is a function that maps

- *n*-ary function symbols to functions over $\Delta$:
  \[ f^\mathcal{I} \in [\Delta^n \rightarrow \Delta] \]

- individual constants to elements of $\Delta$:
  \[ a^\mathcal{I} \in \Delta \]

- *n*-ary predicate symbols to relation over $\Delta$:
  \[ P^\mathcal{I} \subseteq \Delta^n \]
Semantic of FOL: Satisfaction

**Interpretation** of ground terms:

\[(f(t_1, \ldots, t_n))^I = f^I(t_1^I, \ldots, t_n^I) \ (\in \Delta)\]

**Satisfaction** of ground atoms \(P(t_1, \ldots, t_n)\):

\[\mathcal{I} \models P(t_1, \ldots, t_n) \text{ iff } \langle t_1^I, \ldots, t_n^I \rangle \in P^I\]
Examples

\[ \Delta = \{d_1, \ldots, d_n, n > 1\} \]
\[ a^I = d_1 \]
\[ b^I = d_2 \]
\[ \text{Block}^I = \{d_1\} \]
\[ \text{Red}^I = \Delta \]

\[ \Delta = \{1, 2, 3, \ldots\} \]
\[ 1^I = 1 \]
\[ 2^I = 2 \]
\[ \vdots \]
\[ \text{Even}^I = \{2, 4, 6, \ldots\} \]
\[ \text{succ}^I = \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \]
Examples

\[ \Delta = \{d_1, \ldots, d_n, n > 1\} \]
\[ a_I = d_1 \]
\[ b_I = d_2 \]
\[ \text{Block}_I = \{d_1\} \]
\[ \text{Red}_I = \Delta \]
\[ \mathcal{I} \models \text{Red}(b) \]
\[ \mathcal{I} \not\models \text{Block}(b) \]
\[ \Delta = \{1, 2, 3, \ldots\} \]
\[ 1^I = 1 \]
\[ 2^I = 2 \]
\[ \vdots \]
\[ \text{Even}_I = \{2, 4, 6, \ldots\} \]
\[ \text{succ}_I = \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \]
Examples

\[ \Delta = \{d_1, \ldots, d_n, n > 1\} \]
\[ a^I = d_1 \]
\[ b^I = d_2 \]
\[ \text{Block}^I = \{d_1\} \]
\[ \text{Red}^I = \Delta \]
\[ I \models \text{Red}(b) \]
\[ I \not\models \text{Block}(b) \]

\[ \Delta = \{1, 2, 3, \ldots\} \]
\[ 1^I = 1 \]
\[ 2^I = 2 \]
\[ \vdots \]
\[ \text{Even}^I = \{2, 4, 6, \ldots\} \]
\[ \text{succ}^I = \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \]
\[ I \not\models \text{Even}(3) \]
\[ I \models \text{Even(succ}(3)) \]
Semantics of FOL: Variable Assignments

$V$ set of all variables. Function $\alpha: V \rightarrow \Delta$.

**Notation:** $\alpha[x/d]$ is identical to $\alpha$ except for the variable $x$.

Interpretation of terms *under* $\mathcal{I}$, $\alpha$:

$$
\begin{align*}
x_{\mathcal{I},\alpha} & = \alpha(x) \\
a_{\mathcal{I},\alpha} & = a_{\mathcal{I}} \\
(f(t_1, \ldots, t_n))_{\mathcal{I},\alpha} & = f_{\mathcal{I}}(t_1_{\mathcal{I},\alpha}, \ldots, t_n_{\mathcal{I},\alpha})
\end{align*}
$$

Satisfiability of atomic formulas:

$$
\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) \text{ iff } \langle t_1_{\mathcal{I},\alpha}, \ldots, t_n_{\mathcal{I},\alpha} \rangle \in P_{\mathcal{I}}
$$
Variable Assignment example

\[ \alpha = \{ (x \mapsto d_1), (y \mapsto d_2) \} \]

\[ \mathcal{I}, \alpha \models \text{Red}(x) \]

\[ \mathcal{I}, \alpha[y/d_1] \models \text{Block}(y) \]
Semantics of FOL: Satisfiability of formulas

A formula $\phi$ is satisfied by (is true in) an interpretation $\mathcal{I}$ under a variable assignment $\alpha$,

$\mathcal{I}, \alpha \models \phi$:

- $\mathcal{I}, \alpha \models P(t_1, \ldots, t_n)$ if $\langle t_1^{\mathcal{I},\alpha}, \ldots, t_n^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}}$
- $\mathcal{I}, \alpha \models \neg \phi$ if $\mathcal{I}, \alpha \not\models \phi$
- $\mathcal{I}, \alpha \models \phi \land \psi$ if $\mathcal{I}, \alpha \models \phi$ and $\mathcal{I}, \alpha \models \psi$
- $\mathcal{I}, \alpha \models \phi \lor \psi$ if $\mathcal{I}, \alpha \models \phi$ or $\mathcal{I}, \alpha \models \psi$
- $\mathcal{I}, \alpha \models \forall x. \phi$ if for all $d \in \Delta$:
  \[ \mathcal{I}, \alpha[x/d] \models \phi \]
- $\mathcal{I}, \alpha \models \exists x. \phi$ if there exists a $d \in \Delta$:
  \[ \mathcal{I}, \alpha[x/d] \models \phi \]
Examples

\[ \Delta = \{d_1, \ldots, d_n, \} \quad n > 1 \]

\[ a^\mathcal{I} = d_1 \]

\[ b^\mathcal{I} = d_1 \]

\[ \text{Block}^\mathcal{I} = \{d_1\} \]

\[ \text{Red}^\mathcal{I} = \Delta \]

\[ \alpha = \{(x \mapsto d_1), (y \mapsto d_2)\} \]

1. \( \mathcal{I}, \alpha \models \text{Block}(c) \lor \neg \text{Block}(c) ? \)

2. \( \mathcal{I}, \alpha \models \text{Block}(x) \rightarrow \text{Block}(x) \lor \text{Block}(y) ? \)

3. \( \mathcal{I}, \alpha \models \forall x \: \text{Block}(x) \rightarrow \text{Red}(x) ? \)

4. \( \Theta = \left\{ \begin{array}{c}
\text{Block}(a), \: \text{Block}(b) \\
\forall x \: (\text{Block}(x) \rightarrow \text{Red}(x))
\end{array} \right\} \)

\( \mathcal{I}, \alpha \models \Theta ? \)
Example

Find a model of the formula:

$$\exists y. \left[ P(y) \land \neg Q(y) \right] \land \forall z. \left[ P(z) \lor Q(z) \right]$$
Example

Find a model of the formula:

$$\exists y. \ [ P(y) \land \neg Q(y) ] \land \forall z. \ [ P(z) \lor Q(z) ]$$

$$\Delta = \{ a, b \}$$

$$P^\mathcal{I} = \{ a \}$$

$$Q^\mathcal{I} = \{ b \}$$
Satisfiability and Validity

An interpretation $\mathcal{I}$ is a model of $\phi$ under $\alpha$, if

$$\mathcal{I}, \alpha \models \phi.$$  

Similarly as in propositional logic, a formula $\phi$ can be satisfiable, unsatisfiable, falsifiable or valid—the definition is in terms of the pair $(\mathcal{I}, \alpha)$.

A formula $\phi$ is

- **satisfiable**, if there is some $(\mathcal{I}, \alpha)$ that satisfies $\phi$,
- **unsatisfiable**, if $\phi$ is not satisfiable,
- **falsifiable**, if there is some $(\mathcal{I}, \alpha)$ that does not satisfy $\phi$,
- **valid** (i.e., a tautology), if every $(\mathcal{I}, \alpha)$ is a model of $\phi$. 
Equivalence

Analogously, two formulas are **logically equivalent** ($\phi \equiv \psi$), if for all $\mathcal{I}$, $\alpha$ we have:

\[
\mathcal{I}, \alpha \models \phi \iff \mathcal{I}, \alpha \models \psi
\]

**Note:** $P(x) \neq P(y)$!
Free and Bound Variables

$$\forall x. \ (R(y, z) \land \exists y. \ (\neg P(y, x) \lor R(y, z)))$$

Variables in boxes are free; other variables are bound.

Free variables of a formula (inductively defined over the structure of expressions):

$$\text{free}(x) = \{x\}$$
$$\text{free}(a) = \emptyset$$
$$\text{free}(f(t_1, \ldots, t_n)) = \text{free}(t_1) \cup \ldots \cup \text{free}(t_n)$$
$$\text{free}(P(t_1, \ldots, t_n)) = \text{free}(t_1) \cup \ldots \cup \text{free}(t_n)$$
$$\text{free}(\neg \phi) = \text{free}(\phi)$$
$$\text{free}(\phi \ast \psi) = \text{free}(\phi) \cup \text{free}(\psi), \ \ast = \lor, \land, \ldots$$
$$\text{free}(\forall x. \ \phi) = \text{free}(\phi) - \{x\}$$
$$\text{free}(\exists x. \ \phi) = \text{free}(\phi) - \{x\}$$
Open and Closed Formulas

- A formula is **closed** or a **sentence** if no free variables occurs in it. When formulating theories, we only use closed formulas.

- Note: For closed formulas, the properties *logical equivalence*, *satisfiability*, *entailment* etc. do not depend on variable assignments. If the property holds for one variable assignment then it holds for all of them.

- For closed formulas, the symbol $\alpha$ on the left hand side of the “$\models$” sign is omitted.

$$\mathcal{I} \models \phi$$
Entailment

Entailment is defined similarly as in propositional logic.

The formula $\phi$ is logically implied by a formula $\psi$, if $\phi$ is true in all models of $\psi$ (symbolically, $\psi \models \phi$):

$$\psi \models \phi \iff \mathcal{I} \models \phi \text{ for all models } \mathcal{I} \text{ of } \psi$$
More Exercises

- $\models \forall x. (P(x) \lor \neg P(x))$
- $\exists x. [ P(x) \land (P(x) \rightarrow Q(x)) ] \models \exists x. Q(x)$
- $\models \neg(\exists x. [ \forall y. [ P(x) \rightarrow Q(y) ] ])$
- $\exists y. [ P(y) \land \neg Q(y) ] \land \forall z. [ P(z) \lor Q(z) ]$ satisfiable
Equality

• Equality is a special predicate.

• $t_1 = t_2$ is true under a given interpretation $(\mathcal{I}, \alpha \models t_1 = t_2)$ if and only if $t_1$ and $t_2$ refer to the same object:

$$t_1^{\mathcal{I},\alpha} = t_2^{\mathcal{I},\alpha}$$

E.g., $\forall x. (\times (\sqrt{x}, \sqrt{x})) = x$ is satisfiable

$2 = 2$ is valid

E.g., definition of (full) $\text{Sibling}$ in terms of $\text{Parent}$:

$$\forall x, y. \quad Sibling(x, y) \leftrightarrow$$

$$(\neg(x = y) \land$$

$$\exists m, f. \neg(m = f) \land \text{Parent}(m, x) \land \text{Parent}(f, x) \land$$

$$\text{Parent}(m, y) \land \text{Parent}(f, y))$$
Universal quantification

Everyone in England is smart: \( \forall x. \ In(x, \text{england}) \rightarrow Smart(x) \)

\((\forall x. \ \phi)\) is equivalent to the \textit{conjunction} of all possible \textit{instantiations} in \(x\) of \(\phi\):

\[
\begin{align*}
& In(\text{kingJohn, england}) \rightarrow Smart(\text{kingJohn}) \\
& \land \quad In(\text{richard, england}) \rightarrow Smart(\text{richard}) \\
& \land \quad In(\text{england, england}) \rightarrow Smart(\text{england}) \\
& \land \quad \ldots
\end{align*}
\]

Typically, \(\rightarrow\) is the main connective with \(\forall\).

Common mistake: using \(\land\) as the main connective with \(\forall\):

\[ \forall x. \ In(x, \text{england}) \land Smart(x) \]

means “Everyone is in England and everyone is smart”
Existential quantification

Someone in France is smart: $\exists x. \, \text{In}(x, \text{france}) \land \text{Smart}(x)$

$(\exists x. \, \phi)$ is equivalent to the disjunction of all possible instantiations in $x$ of $\phi$

$$\text{In}(\text{kingJohn}, \text{france}) \land \text{Smart}(\text{kingJohn})$$
$$\lor \quad \text{In}(\text{richard}, \text{france}) \land \text{Smart}(\text{richard})$$
$$\lor \quad \text{In}(\text{france}, \text{france}) \land \text{Smart}(\text{france})$$
$$\lor \quad \ldots$$

Typically, $\land$ is the main connective with $\exists$.

Common mistake: using $\rightarrow$ as the main connective with $\exists$:

$$\exists x. \, \text{In}(x, \text{france}) \rightarrow \text{Smart}(x)$$

is true if there is anyone who is not in France!
Properties of quantifiers

(∀x.∀y. φ) is the same as (∀y.∀x. φ) (Why?)

(∃x.∃y. φ) is the same as (∃y.∃x. φ) (Why?)

(∃x.∀y. φ) is not the same as (∀y.∃x. φ)

∃x.∀y. Loves(x, y)
“There is a person who loves everyone in the world”

∀y.∃x. Loves(x, y)
“Everyone in the world is loved by at least one person” (not necessarily the same)

**Quantifier duality**: each can be expressed using the other:

∀x. Likes(x, iceCream)       ¬∃x.¬Likes(x, iceCream)

∃x. Likes(x, broccoli)       ¬∀x.¬Likes(x, broccoli)
Equivalences

\((\forall x. \phi) \land \psi \equiv \forall x. (\phi \land \psi)\) if \(x\) not free in \(\psi\)

\((\forall x. \phi) \lor \psi \equiv \forall x. (\phi \lor \psi)\) if \(x\) not free in \(\psi\)

\((\exists x. \phi) \land \psi \equiv \exists x. (\phi \land \psi)\) if \(x\) not free in \(\psi\)

\((\exists x. \phi) \lor \psi \equiv \exists x. (\phi \lor \psi)\) if \(x\) not free in \(\psi\)

\(\forall x. \phi \land \forall x. \psi \equiv \forall x. (\phi \land \psi)\)

\(\exists x. \phi \lor \exists x. \psi \equiv \exists x. (\phi \lor \psi)\)

\(\neg \forall x. \phi \equiv \exists x. \neg \phi\)

\(\neg \exists x. \phi \equiv \forall x. \neg \phi\)

& propositional equivalences
The Prenex Normal Form

Quantifier prefix + (quantifier free) matrix

$$\forall x_1 \forall x_2 \exists x_3 \ldots \forall x_n \phi$$

1. Elimination of → and ↔

2. push ¬ inwards

3. pull quantifiers outwards

E.g. $$\neg \forall x. ((\forall x. p(x)) \rightarrow q(x))$$

$$\neg \forall x. (\neg (\forall x. p(x)) \lor q(x))$$

$$\exists x. ((\forall x. p(x)) \land \neg q(x))$$

and now?

Notation: renaming of variables. Let $$\phi[x/t]$$ be the formula $$\phi$$ where all occurrences of $$x$$ have been replaced by the term $$t$$. 
The Prenex Normal Form: theorems

**Lemma.** Let $y$ be a variable that does not occur in $\phi$. Then we have $\forall x \phi \equiv (\forall x \phi)[x/y]$ and $\exists x \phi \equiv (\exists x \phi)[x/y]$.

**Theorem.** There is an algorithm that computes for every formula its prenex normal form.
FOL at work: reasoning by cases

\[ \Gamma = \text{FRIEND}(\text{john}, \text{susan}) \land \]
\[ \text{FRIEND}(\text{john}, \text{andrea}) \land \]
\[ \text{LOVES}(\text{susan}, \text{andrea}) \land \]
\[ \text{LOVES}(\text{andrea}, \text{bill}) \land \]
\[ \text{Female}(\text{susan}) \land \]
\[ \neg \text{Female}(\text{bill}) \]

\[
\text{john} \\
\text{FRIEND} \quad \text{FRIEND} \\
\text{andrea} \quad \text{LOVES} \quad \text{susan: Female} \\
\text{LOVES} \\
\text{bill: } \neg \text{Female}
\]
Does John have a female friend loving a male (i.e. not female) person?

\[ \Gamma = \text{bill: } \neg \text{Female} \]

\[ \Gamma \models \exists X, Y. \text{FRIEND}(\text{john}, X) \land \text{Female}(X) \land \text{LOVES}(X, Y) \land \neg \text{Female}(Y) \]
Does John have a female friend loving a male (i.e. not female) person?

YES!

\[ \Gamma = \text{bill: } \neg \text{Female} \]

\[ \Gamma \models \exists X, Y. \text{FRIEND}(\text{john}, X) \land \text{Female}(X) \land \text{LOVES}(X, Y) \land \neg \text{Female}(Y) \]
FRIEND(john, susan), Female(susan),
LOVES(susan, andrea), \neg Female(andrea)
FRIEND(john, andrea), Female(andrea),
LOVES(andrea, bill), ¬ Female(bill)
Theories and Models

\[
\Gamma_1 = \text{FRIEND}(\text{john}, \text{susan}) \land \\
\text{FRIEND}(\text{john}, \text{andrea}) \land \\
\text{LOVES}(\text{susan}, \text{andrea}) \land \\
\text{LOVES}(\text{andrea}, \text{bill}) \land \\
\text{Female}(\text{susan}) \land \\
\text{Male}(\text{bill}) \land \\
\forall X. \text{Male}(X) \leftrightarrow \neg \text{Female}(X)
\]

\[
\text{joh}n \\
\text{FRIEND} \quad \text{FRIEND} \\
\text{a}ndr\text{ea} \quad \text{LOVES} \quad \text{su}\text{s}an: \quad \text{Female} \\
\text{LOVES} \\
\text{bi}ll: \quad \text{Male} \quad \text{Male} \equiv \neg \text{Female}
\]
\[ \Gamma_1 = \text{bill: Male} \] 
\[ \text{Male} \models \neg \text{Female} \]

Does John have a female friend loving a male person?

\[ \Gamma_1 \models \exists X, Y. \ \text{FRIEND}(\text{john}, X) \land \text{Female}(X) \land \text{LOVES}(X, Y) \land \text{Male}(Y) \]
\[ \Gamma = \text{FRIEND}(\text{john}, \text{susan}) \land \\
\quad \text{FRIEND}(\text{john}, \text{andrea}) \land \\
\quad \text{LOVES}(\text{susan}, \text{andrea}) \land \\
\quad \text{LOVES}(\text{andrea}, \text{bill}) \land \\
\quad \text{Female}(\text{susan}) \land \\
\quad \neg \text{Female}(\text{bill}) \]

\[ \Gamma_1 = \text{FRIEND}(\text{john}, \text{susan}) \land \\
\quad \text{FRIEND}(\text{john}, \text{andrea}) \land \\
\quad \text{LOVES}(\text{susan}, \text{andrea}) \land \\
\quad \text{LOVES}(\text{andrea}, \text{bill}) \land \\
\quad \text{Female}(\text{susan}) \land \\
\quad \text{Male}(\text{bill}) \land \\
\quad \forall X. \text{Male}(X) \leftrightarrow \neg \text{Female}(X) \]
Γ = FRIEND(john, susan) ∧
    FRIEND(john, andrea) ∧
    LOVES(susan, andrea) ∧
    LOVES(andrea, bill) ∧
    Female(susan) ∧
    ¬Female(bill)

Γ₁ = FRIEND(john, susan) ∧
    FRIEND(john, andrea) ∧
    LOVES(susan, andrea) ∧
    LOVES(andrea, bill) ∧
    Female(susan) ∧
    Male(bill) ∧
    ∀X. Male(X) ↔ ¬Female(X)

Δ = {john, susan, andrea, bill}
Femaleᵢ = {susan}
\[\Gamma = \text{FRIEND}(\text{john}, \text{susan}) \land \text{FRIEND}(\text{john}, \text{andrea}) \land \text{LOVES}(\text{susan}, \text{andrea}) \land \text{LOVES}(\text{andrea}, \text{bill}) \land \text{Female}(\text{susan}) \land \neg \text{Female}(\text{bill})\]

\[\Gamma_1 = \text{FRIEND}(\text{john}, \text{susan}) \land \text{FRIEND}(\text{john}, \text{andrea}) \land \text{LOVES}(\text{susan}, \text{andrea}) \land \text{LOVES}(\text{andrea}, \text{bill}) \land \text{Female}(\text{susan}) \land \text{Male}(\text{bill}) \land \forall X. \text{Male}(X) \leftrightarrow \neg \text{Female}(X)\]

\[\Delta = \{\text{john, susan, andrea, bill}\} \quad \text{Female}^I = \{\text{susan}\}\]

\[\Delta^I_1 = \{\text{john, susan, andrea, bill}\} \quad \text{Female}^I_1 = \{\text{susan, andrea}\} \quad \text{Male}^I_1 = \{\text{bill, john}\}\]

\[\Delta^I_2 = \{\text{john, susan, andrea, bill}\} \quad \text{Female}^I_2 = \{\text{susan}\} \quad \text{Male}^I_2 = \{\text{bill, andrea, john}\}\]
\( \Gamma \not\models \text{Female}(\text{andrea}) \)
\( \Gamma \not\models \neg \text{Female}(\text{andrea}) \)

\( \Gamma_1 \not\models \text{Female}(\text{andrea}) \)
\( \Gamma_1 \not\models \neg \text{Female}(\text{andrea}) \)
\( \Gamma_1 \not\models \text{Male}(\text{andrea}) \)
\( \Gamma_1 \not\models \neg \text{Male}(\text{andrea}) \)
Is it true that the top block is on a white block touching a black block?