- syntax
- semantics
- normal forms
- calculi
- other properties
- variants of FOL
- automated theorem provers

- all calculi presented for propositional logic can be extended to FOL
- the extension is performed by extending the language (straightforward), and the addition of new rules of inferences and axioms
- we will only present the case of natural deduction, resolution and tableaux calculi

Natural Deduction Additional Inference Rules for FOL

universal elimination rules

$$\frac{(\forall X)F}{F[X/t]}$$

$$\frac{\neg(\forall X)F}{\neg F[X/a]}$$

existential elimination rules

$$\frac{(\exists X)F}{F[X/a]}$$

$$\frac{\neg(\exists X)F}{\neg F[X/t]}$$

universal introduction rules

$$\frac{F}{(\forall X)F}$$

$$\frac{\neg F[X/t]}{\neg(\forall X)F}$$

existential introduction rules

$$\frac{\neg F}{\neg(\exists X)F}$$

$$\frac{F[X/t]}{(\exists X)F}$$

- in the previous rules, t is any ground term, and a is a constant not occurring previously in the deduction

Example of proof for $(\exists X)(\forall Y)p(X,Y) \Rightarrow (\forall Y)(\exists X)p(X,Y)$

$$(\exists X)(\forall Y)p(X,Y)$$
$$\neg(\forall Y)(\exists X)p(X,Y)$$
$$\neg(\exists X)p(X,b)$$
$$\neg p(a,b)$$
$$(\forall Y)p(a,Y)$$
$$p(a,b)$$
$$\emptyset$$
$$(\forall Y)(\exists X)p(X,Y)$$
$$(\exists X)(\forall Y)p(X,Y) \Rightarrow (\forall Y)(\exists X)p(X,Y)$$

- we can observe that the constants a, b introduced in rules 3,4 are Skolem constants
- in these steps we had no choice but to introduce new constants
- on the other hand, in steps 5,6 we were guessing the ground term to substitute
- **Exercise**: prove with natural deduction that $(\exists X)(F \Rightarrow G) \Rightarrow ((\forall X)F \Rightarrow G)$ is valid for all formulas F, G such that G does not contain a free occurrence of X
- **Theorem 20.** *The first order calculus of natural deduction is sound and complete**

- as in propositional logic, resolution is introduced as a negative analyzing calculus
- the language is the language of FOL in clausal form
- the only axiom is the empty clause $\{\}$
- we have two rules: the resolution rule, and the factoring rule

- Resolution Rule
  if $C_1 = \{L, L_1, \ldots, L_n\}$ and $C_2 = \{\neg L', L'_1, L'_2, \ldots, L'_m\}$ are two clauses without common variables
  such that L and L' are unifiable with mgu θ
  then the clause
  $$\{L_1\theta, L_2\theta, \ldots, L_n\theta, L'_1\theta, L'_2\theta, \ldots, L'_m\theta\}$$
  is the resolvent of C_1 and C_2
- the variables appearing in the clause are universally quantified, so the condition of no common variables can be easily obtained by variable renaming
- Factoring Rule
  let $C = \{L_1, L_2, \ldots, L_n\}$ be a clause such that L_1 and L_2 are unifiable with mgu θ
  then the clause
  $$\{L_2\theta, \ldots, L_n\theta\}$$
  is a factor of C

- let $\mathcal{F}$ be a set of clauses, a deduction of C from $\mathcal{F}$ is a finite sequence $C_1, C_2, \ldots, C_n$ such that $C_n = C$
  and $C_i, 1 \leq i \leq n$ can be either elements of $\mathcal{F}$, or obtained from previous clauses in the sequence by resolution or factoring
- a deduction of the empty clause is called a refutation
- **Theorem 21.** *The first order resolution calculus is sound and complete**
- **Theorem 22.** *$\mathcal{F}$ is unsatisfiable iff there is a refutation for $\mathcal{F}$*

Example: Russel antinomy the barber shaves a person iff that person does not shave himself

$$(\forall X)(shaves(barber, X) \Leftrightarrow \neg shaves(X, X))$$
$$\{\neg shaves(b, X), \neg shaves(X, X)\}$$
$$\{shaves(b, Y), shaves(Y, Y)\}$$

Example: $(\exists X)(\forall Y)p(X,Y) \Rightarrow (\forall Y)(\exists X)p(X,Y)$ is valid

$$\begin{array}{c}\neg((\exists X)(\forall Y)p(X,Y) \Rightarrow (\forall Y)(\exists X)p(X,Y)) \\ \{p(a,Y)\} \\ \{\neg p(W,b)\}\end{array}$$

Example: Show the following logical consequence relation:

$$\begin{array}{ll}(\forall X,Z)((A(X,Y) \wedge A(Y,Z)) \Rightarrow \neg B(X,Z)) & \\ \wedge\ (\forall X,Y)(A(X,Y) \Leftrightarrow (B(Y,X) \vee C(x,Y))) & \\ \wedge\ (\forall X)(\exists Y)A(X,Y) & \models (\forall X)A(X,X)\end{array}$$

Example:

If one number is less than or equal to a second number, and the second number is less than or equal to a third, then the first number is not greater than the first.

A number is less than or equal to a second number if and only if the second number is greater than the first or is equal to the second.

Given any number, there is another number that is less than or equal to.

Therefore, every number is less than or equal to itself.

Example:

Some students attend logic lectures diligently.
No students attend boring logic lectures diligently.
John's logic lectures are attended diligently by all students.
Therefore, none of John's lectures are boring.

- resolution defines a search procedure, in each step there are several possible paths in which proceed
- SLD resolution is a refinement of resolution which reduces the number of possibilities
- a fair strategy for these search procedures is one such that each branch of the search has to be considered after a finite time
- fair strategies are needed in order to guarantee completeness for SLD resolution

- **Exercise**: Use resolution to find the validity of the following arguments
  - Some students are anxious. Some students study. If a student is anxious he will not pass his examination unless he studies. Therefore, no student will pass the examinations.
  - Some students are anxious. Some students study. If a student is anxious he will not pass his examination unless he studies. Therefore, some students will pass the examinations.
  - Some students are anxious. All students study. If a student is anxious he will not pass his examination unless he studies. Therefore, all students will pass the examinations.
  - All students are anxious. Some students study. If a student is anxious he will not pass his examination unless he studies. Therefore, some students will pass the examinations.

- semantic tableaux in FOL are basically the same as in propositional logic
- a sequence of formulas is constructed according to certain rules, and usually depicted in the form of a tree
- the rules that were used in propositional logic are still valid in FOL
- in addition, we have to introduce new rules for handling quantifiers

- rule for elimination of universal quantifier:
  if a branch contains $(\forall X)F$, then expand the branch with a new node $F[X \backslash t]$, where t is a term
- rule for elimination of existential quantifier:
  if a branch contains $(\exists X)F$, then expand the branch with a new node $F[X \backslash t]$, where t is a term that has not been used in the derivation so far.
- a standard new constant a is in general used in this case

- rule for elimination of the negation of a universal quantifier:
  if a branch contains $\neg(\forall X)F$, then expand the branch with a new node $\neg F[X \backslash t]$, where t is a term that has not been used in the derivation so far
- rule for elimination of the negation of a existential quantifier:
  if a branch contains $\neg(\exists X)F$, then expand the branch with a new node $\neg F[X \backslash t]$, where t is a term

- the rule for closing branches is the same as in propositional logic
- if all branches of the tableau are closed the the set of formulas from which the tableau was constructed is mutually inconsistent
- Example: $(\forall X, Y) R(X, Y) \Rightarrow R(a, a)$ is valid
- Example: $(\forall X) A(X) \Rightarrow (\exists Y) A(Y)$ is valid

- when we use the rule for existential quantification, we cannot use the same term more than once
- Example: $(\exists X, Y)R(X, Y) \Rightarrow R(a, a)$ is not valid but...
- the restriction does not hold for universal quantification
- Example: $(\forall X)A(X) \wedge B(X) \Rightarrow (\forall X)A(X)$
- this gives us a useful hint for developing tableaux in FOL: always apply when possible existential elimination before universal elimination

- **Theorem 23.** *The first order tableaux calculus is sound and complete**
- Example: $(\forall X)(A(X) \vee (\forall X)B(X)) \Rightarrow (\forall X)(A(X) \vee B(X))$
- Example: let's see if $(\forall X)(A(X) \vee B(X)) \Rightarrow ((\forall X)A(X) \vee (\forall X)B(X))$ is valid
- suppose a pre-interpretation with domain the natural numbers, $A(X)$ are the even numbers, and $B(X)$ are the odd numbers

- Example: all men are mortal, and Socrates is a man. Therefore, Socrates is mortal.
- Example: all men are mortal, and Socrates is mortal. Therefore, Socrates is a man.
- **Exercise**: solve using tableaux
  1. $(\exists X)A(X) \Rightarrow (\forall X)A(X)$
  2. $(\exists X)A(X) \Rightarrow (\exists Y)A(Y)$
  3. $(\forall X)(A(X) \wedge B(X)) \Rightarrow (\exists X)B(X)$
  4. $(\forall X)(A(X) \vee B(X)) \Rightarrow A(X)$
  5. $(\forall X)(A(X) \Rightarrow (\neg A(X) \Rightarrow B(X)))$
  6. $(\forall X)(A(X) \Rightarrow (B(X) \Rightarrow C(X)) \Rightarrow ((A(X) \Rightarrow B(X)) \Rightarrow C(X)))$

- **Exercise**: determine if the argument is valid using tableaux
  1. All fruit is tasty if it is not cooked. This apple is cooked. Therefore, it is not tasty.
  2. All lecturers are determined. Anyone who is determined and intelligent will give satisfactory service. Clare is an intelligent lecturer. Therefore, Clare will give satisfactory service.
  3. All those who honour both parents are blessed. If anyone dislikes any of his siblings he does not honour his parents. Jack likes his sister Jill. Therefore, Jack is blessed.
  4. Some lecturers are imaginative but poor communicators. Only good students are lecturers. Good students are not imaginative. Every artist is imaginative. Therefore not every good student is an artist.
  5. Dilly loves all and only those who love Milly. Milly loves all and only those who do not love Dilly. Dilly loves herself. Therefore, Milly loves herself.

- each sound and complete calculus for FOL defines a derivability relation $\vdash$ between a set of formulas $\mathcal{F}$ and a formula G such that $\mathcal{F} \vdash G$ iff $\mathcal{F} \models G$
- **Theorem 24.** *$\mathcal{F} \vdash G$ iff there is a finite subset $\mathcal{F}'$ such that $\mathcal{F}' \vdash G$*
- compactness property for the entailment relation

  Theorem 25. *$\mathcal{F} \models G$ iff there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ such that $\mathcal{F}' \models G$*
- **Theorem 26.** *If all finite subsets of $\mathcal{F}$ are satisfiable, then $\mathcal{F}$ is satisfiable*

- **Theorem 27.** ***Löwenheim-Skolem*** *if $\mathcal{F}$ is satisfiable, then $\mathcal{F}$ is satisfiable in a model with countable domain*
  (or equivalently if $\mathcal{F}$ has finite models of arbitrary cardinality, then $\mathcal{F}$ has an infinite model)
- this theorem has remarkable consequences about the inexpressibility of FOL
  for example, the REACHABILITY graph problem cannot be expressed in FOL

- another important property about FOL
- **Theorem 28.** ***Gödel's incompleteness*** *there is no recursively enumerable calculus (set of axioms and inference rules) for FOL that produces exactly all true properties about natural numbers*
- this means that any sound axiomatic system for the natural numbers in FOL must necessarily be incomplete, ie there must exist a property that is not provable in it