- syntax
- semantics
- normal forms
- calculi
- other properties
- variants of FOL
- automated theorem provers

- as in propositional logic, FOL has a two-vauled semantics
- however, before assigning meaning to formulas, the meaning of terms have to be defined
- a pre-interpretation J for a fol language $\mathcal{L}$ consists of a non-empty set D, called the domain of J, and an assignment mapping $f^J : D^n \longrightarrow D$ to each n-ary function symbol in the alphabet of $\mathcal{L}$
- the meaning of a ground term t under J is then defined as

$$t^J = \begin{cases} t^J & \text{if } t \text{ is a constant} \\ f^J({t_1}^J, \ldots, {t_n}^J) & \text{if } t \text{ is of the form } f(t_1, \ldots, t_n) \end{cases}$$

- example: consider the term $f(h(c), g(a, b))$
- let J be defined with $D = \mathbb{N}$ with $+, -, \surd$ and

$$\begin{array}{lll} a^J = 5 & b^J = 3 & c^J = 16 \\ f^J = + & g^J = - & h^J = \surd \end{array}$$

  then $f(h(c), g(a, b))^J = +(\surd(16), -(5, 3)) = 6$
- let J be defined with $D =$nodes and arcs in a directed graphs, a, b some nodes, c an arc, $-$ the constructor function for directed arcs, $\surd$ the consulting function for giving an arc returning its starting element, and $+$ the constructor function that given a node and an arc, builds a new arc with the given node and the ending node of the given arc
- these examples show that the same term can have different meanings

- to assign meaning to non-ground terms we introduce the notion of state
- a state is a mapping that assigns to each variable an element in the domain of a pre-interpretation
- given a pre-interpretation J and a state σ, the meaning of a term t is

$$t^{J,\sigma} = \begin{cases} \sigma(t) & \text{if } t \text{ is a variable} \\ t^{J} & \text{if } t \text{ is a constant} \\ f^{J}({t_1}^{J,\sigma}, \ldots, {t_n}^{J,\sigma}) & \text{if } t \text{ is of the form } f(t_1, \ldots, t_n) \end{cases}$$

- hence, pre-interpretations with a state maps all terms onto elements of D

- let σ be a state, X a variable and $e \in D$, then $\sigma X \mapsto e$ is a new state that only varies from σ in that the assignment to X is e
- an interpretation I to a fol language $\mathcal{L}$ consists of a pre-interpretation J with domain D, and an assignment $p^I \subseteq D^n$ to each n-ary relation symbol in $\mathcal{L}$
- p^I is called the extension of p under I
- I is said to be based on J, and so f^J is also noted f^I
- D is also called the domain of I

- example: suppose in $\mathcal{L}$ there are two binary predicated p and q
- an interpretation I to $\mathcal{L}$ can set $D = \mathbb{N}$ and

$$p^I = \{(n,m) | n < m, n, m \in \mathbb{N}\}$$
$$q^I = \{(n,m) | n > m, n, m \in \mathbb{N}\}$$

- given an interpretation I, and a state σ for a language $\mathcal{L}$
- we define for any formula F the relation $I, \sigma \models F$ (I, σ satisfies F) as follows:

$I, \sigma \models p(t_1, \ldots, t_n)$	iff $({t_1}^{I,\sigma}, \ldots, {t_n}^{I,\sigma}) \in p^I$
$I, \sigma \models \neg F$	iff $I, \sigma \not\models F$
$I, \sigma \models F \wedge G$	iff $I, \sigma \models F$ and $I, \sigma \models G$
$I, \sigma \models F \vee G$	iff $I, \sigma \models F$ or $I, \sigma \models G$
$I, \sigma \models F \Rightarrow G$	iff $I, \sigma \not\models F$ or $I, \sigma \models G$
$I, \sigma \models F \Leftrightarrow G$	iff $I, \sigma \models F \Rightarrow G$ and $I, \sigma \models G \Rightarrow F$
$I, \sigma \models (\exists X)F$	iff there exists $e \in D$ such that $I, \sigma\{X \mapsto e\} \models F$
$I, \sigma \models (\forall X)F$	iff for all $e \in D$ it holds that $I, \sigma\{X \mapsto e\} \models F$

- an interpretation I is said to be a model for F, written $I \models F$, iff for all states σ we have that $I, \sigma \models F$
- then, if F is a sentence $I \models F$ iff $I, \sigma \models F$
- if $\mathcal{F}$ is a set of formulas, an interpretation I is a model of $\mathcal{F}$ iff it is a model for each $F \in \mathcal{F}$

- the logical entailment $\models$ relation between sets of formulas and formulas can also be extended
- likewise, the concept of validity, satisfiability, falsifiability and unsatisfiability
- theorems 3 and 4 can also be applied to universally closed FOL formulas
- in contrast to propositional logic, the question of whether a FOL formula is a logical consequence of a set of formulas is undecidable
- this can be formally shown by reducing some known undecidable problem (like the halting problem for Turing machines) to logical entailment in FOL
- however, logical entailment in FOL is semidecidable

- Herbrand interpretations will help in the process of computing logical entailment
- instead of searching models in all interpretations of a set of formulas, we will show that searching only on Herbrand interpretations is enough
- given an alphabet $\mathcal{A}$ that contains at least one constant, let $\mathcal{L}$ be the language based on it
- the Herbrand universe $U_H^{\mathcal{A}}$ is the set of ground terms from $\mathcal{A}$
- the Herbrand base $B_H^{\mathcal{A}}$ is the set of ground atoms from $\mathcal{A}$
- example: let $\mathcal{A}$ contain function symbols a^0, f^1, and the relation symbol p^2

$$U_H^{\mathcal{A}} = \{a, f(a), f(f(a)), \ldots\}$$
$$B_H^{\mathcal{A}} = \{p(a, a), p(a, f(a)), p(f(a), a), p(f(a), f(a)), \ldots\}$$

- the Herbrand pre-interpretation for $\mathcal{L}$ consists of the domain $U_H^{\mathcal{A}}$, and the assignment to each f^n the function in $U_H^{\mathcal{A}^n} \longrightarrow U_H^{\mathcal{A}}$ that maps $(t_1, \ldots, t_n)$ to $f(t_1, \ldots, t_n)$
- for each alphabet, there is only a unique Herbrand pre-interpretation
- a Herbrand interpretation consists of the Herbrand pre-interpretation, and an assignment of a set of n-tuples to each relation symbol in the alphabet
- a Herbrand model for a set of formulas $\mathcal{F}$ is a Herbrand interpration that is a model of $\mathcal{F}$

- there is a one-to-one corrispondence between each Herbrand interpretation I, and a particular subset of the $B_H^{\mathcal{A}}$
  defined by

$$\{p(t_1, \ldots, t_n) | p \in \mathcal{A} \text{ is a } n\text{-ary relation symbol and } (t_1, \ldots, t_n) \in p^I\}$$

- so we can identify Herbrand interpretation with the set of those ground atoms that are true under it, and viceversa
- example:

$$\mathcal{F} = \{p(a, a), (\forall X, Y)(p(X, Y) \Rightarrow p(X, f(Y)))\}$$

  then

$$\{p(a, a), p(a, f(a)), p(a, f(f(a))), \ldots\}$$

  is a Herbrand model for $\mathcal{F}$
- we will see the importance of Herbrand interpretation after introducing normal forms for FOL

- the notion of semantic equivalence, and equivalences between formulas can be extended to FOL
- some additional equivalences for FOL

$$F \equiv (\forall X)F$$

$$\neg(\forall X)F \equiv (\exists X)\neg F \text{ nq universal law}$$

$$\neg(\exists X)F \equiv (\forall X)\neg F \text{ nq existential law}$$

$$(\forall X)F \wedge (\forall X)G \equiv (\forall X)(F \wedge G)$$

$$(\exists X)F \vee (\exists X)G \equiv (\exists X)(F \vee G)$$

$$(\forall X)(\forall Y)F \equiv (\forall Y)(\forall X)F$$

$$(\exists X)(\exists Y)F \equiv (\exists Y)(\exists X)F$$

$$(\forall X)F \wedge G \equiv (\forall X)(F \wedge G) \text{ gq law}$$

$$(\forall X)F \vee G \equiv (\forall X)(F \vee G) \text{ gq law}^*$$

$$(\exists X)F \wedge G \equiv (\exists X)(F \wedge G) \text{ gq law}^*$$

$$(\exists X)F \vee G \equiv (\exists X)(F \vee G) \text{ gq law}^*$$

- where * only applies if X dows not occur free in G
- the replacement theorem for proposition logic can be extended to FOL, ie if G and H are semantically equivalent, so are $F[G]$ and $F[G/H]$

- besides semantic equivalences (model preserving transformations), there are additional transformations
- these only preserve the validity or the satisfiability of formula (validity preserving or satisfiability preserving transformations)
- a formula $(\exists X_1, \ldots, X_n)(\forall Y)F$ is valid iff

$$(\exists X_1, \ldots, X_n)F[Y/f(X_1, \ldots, X_n)]$$

  is valid, with f is a function symbol not appearing in F
- this transformation is called Skolemization, and f a Skolem function

- analogously, a formula $(\forall X_1, \ldots, X_n)(\exists Y)F$ is satisfiable iff

$$(\forall X_1, \ldots, X_n)F[Y/f(X_1, \ldots, X_n)]$$

  is satisfiable, with f is a function symbol not appearing in F
- we can now extend the algorithm to transform a FOL formula F into a conjunctive normal form G such that F is unsatisfiable iff G is unsatisfiable
- after the skolemization step, the formula only contains universal quantifiers, which can be dropped
- as in the propositional case, the clausal form denotes the set of sets notations for the formula in CNF

Problem: transform a fol formula into CNF

Input a fol formula F

Output a fol formula G in CNF such that F unsatisfiable iff G unsatisfiable

1: Universally close the formula
2: Eliminate all equivalence connectives using the equivalence law
3: Eliminate all implication connectives using the implication law
4: Eliminate all negation connectives (except those in front of propositions) using the de Morgan laws, the double negation law and the nq laws
5: Apply the gq laws to obtain a sequence of quantifiers applied to a quantifier-free formula
6: Eliminate existential quantifiers by Skolemization (drop remaining universal quantifiers)
7: Distribute all disjunctions over conjunctions using the distributivity, conmutativity and associativity laws.

- the formula obtained after step 5 is called in prenex normal form, consisting of a sequence of quantifiers applied to a quantifiers-free formula called matrix
- **Theorem 17.** *If F' is a CNF formula obtained from a FOL formula F under this algorithm, then F' is unsatisfiable iff F is unsatisfiable*

- example:

$$(\forall V)(q(V,a) \Rightarrow (\exists W)(q(W,a) \wedge (\forall X,Y)(p(h(f(X,Y)),W) \Rightarrow p(h(f(g(X),g(Y))),V))$$

- interpretation I with domain $\mathcal{R}$

$$\begin{aligned} a^I &= 0 \in \mathbb{R} \\ f^I &= - : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \\ h^I &= |\cdot| : \mathbb{R} \rightarrow \mathbb{R} \\ q^I &= \{(n,m) : n,m \in \mathbb{R} \wedge n > m\} \\ p^I &= \{(n,m) : n,m \in \mathbb{R} \wedge n < m\} \end{aligned}$$

$$(\forall \epsilon)(\epsilon > 0 \Rightarrow (\exists \delta)(\delta > 0 \wedge (\forall X,Y)(|X - Y| < \delta \Rightarrow |g(X) - g(Y)| < \epsilon)))$$

- as demonstrated by Herbrand in 1930

  Theorem 18. *For each interpretation J over some domain D for a formula F in CNF, there exists a Herbrand interpretation I for F such that $J \models F$ iff $I \models F$*

- **Theorem 19.** *A FOL formula F in CNF is unsatisfiable iff F is false under all Herbrand interpretations*

- this last result tell us how to show logical consequence in FOL
  1. following theorem 3 logical consequence can be mapped to validity
  2. following theorem 4 validity is equivalent to unsatisfiability of the complement
  3. so the complement should be transformed in CNF (theorem 19 shows this step is satisfiability preserving)
  4. and then we check for unsatisfiability under all Herbrand interpretation
- the last step can be performed by one of the negative calculi that will be presented
- but this is not a decision procedure, since the calculus may not terminate when the answer is negative