

- syntax
- semantics
- normal forms
- calculi
- other properties
- variants of FOL
- automated theorem provers

- propositional logic has many applications but its expressive power is restricted
- for example

all men are mortal  
each child must have parents

cannot be directly expressed

- Gottlob Frege developed what is now **First Order Logic** (FOL) in 1879
- extending propositional logic with **functions**, **variables** and **quantifiers**

- $\mathcal{L}$  is a language of first order logic
- the **alphabet** of  $\mathcal{L}$  is composed of
  - a countably infinite set of **variables**:  $X, Y, Z, \dots$
  - a set of **function** symbols:  $f, g, \dots$
  - a set of **predicates** or **relation** symbols:  $P, Q, R, \dots$
  - the following **connectives**:  $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
  - the following **quantifiers**:  $\exists, \forall$
  - the **punctuation symbols**:  $(, )$
- each functions and relation symbol has a fixed **arity** which denotes the number of arguments associated to it
- we will denote arities when necessary as a superscript. For example  $P^2$  is a binary predicate
- nullary functions are called **constants**
- nullary predicates are **propositions**

- **Terms** denote all the objects  $\mathcal{L}$  is able to speak about
- they are inductively defined as:
  - a variable  $X$  is a term
  - if  $f^n$  is a function symbol, and  $t_1, \dots, t_n$  are terms, then  $f(t_1, \dots, t_n)$  is a term
- generic terms are denoted by  $s, t, \dots$
- a term  $c()$  is simplified as  $c$

- FOL formulas are inductively defined as:
  - if  $P^n$  is a relation symbol, and  $t_1, \dots, t_n$  are terms, then  $P(t_1, \dots, t_n)$  is a formula called **atom**
  - if  $F$  and  $G$  are formulas, and  $X$  is a variable, then

$$\begin{aligned} & \neg F \\ & (F \vee G) \\ & (F \wedge G) \\ & (F \Rightarrow G) \\ & (F \Leftrightarrow G) \\ & (\exists X)F \\ & (\forall X)F \end{aligned}$$

are formulas

- **literals** are atoms (positive literals), or negated atoms (negative literals)
- parenthesis are omitted whenever possible with the following precedence order:

$$\exists, \forall > \neg > \wedge > \vee > \Rightarrow > \Leftrightarrow$$

- also for convenience,  $(\exists X_1) \dots (\exists X_n) F$  and  $(\forall X_1) \dots (\forall X_n) F$  are abbreviated as  $(\exists X_1, \dots, X_n) F$  and  $(\forall X_1, \dots, X_n) F$
- no meaning is attached to the symbols of the alphabet (this is the role of the "semantics")

- for example, *all humans are mortal and each child must have parents* can be written as

$$(\forall X)h(X) \Rightarrow m(X)$$

$$(\forall X)c(X) \Rightarrow (\exists Y, Z)p(X, Y, Z)$$

- **Exercise:** express the transitivity and symmetry properties in FOL

- an **expression** is either a term, or a formula. Just a string, no meaning assigned yet
- **Exercise:** express the principles of structural induction and structural recursion in terms of FOL
- a **subterm** of an expression  $E$  is a substring of  $E$  which is also a term in the same language
- if  $t$  is a subterm of  $E$ , then it is said that  $t$  **occurs** in  $E$
- every term is a subterm of itself; so a subterm  $t$  of  $s$  is a **proper subterm** if  $t \neq s$

- $VAR(E)$  denotes the set of variables occurring in  $E$
- if  $VAR(E) = \emptyset$  then  $E$  is said to be **ground**
- if  $(QX)F$  is a formula, and  $Q$  a quantifier, then  $F$  is said to be the **scope** of  $Q$ , and  $Q$  is said to be **applied** to  $F$
- an occurrence of a variable in a formula is **bound** iff the occurrence is within the scope of a quantifier employing the variable
- otherwise it is called **free**
- example:  $(\forall X)((\forall Y)p(X, Y) \wedge q(X, Y))$
- a formula is **closed** iff it does not contain free occurrences of variables
- closed formulas are also called **sentences**
- the **universal closure** of a formula  $F$  is obtained by adding as a prefix a universal quantifier for each free variable occurring in  $F$

- a **substitution**  $\theta$  is a mapping from a set of variables into a set of terms, which is equal to the identity mapping almost everywhere except for a finite set of variables
- this means that  $|\{X \mid \theta(X) \neq X\}| < \aleph_0$
- $\theta$  is represented with this **finite set of pairs**  $\{X_1 \setminus t_1, \dots, X_n \setminus t_n\}$  where all  $X_i, 1 \leq i \leq n$  are different variables and  $X_i \neq t_i$
- substitutions are denoted by  $\theta, \rho, \dots$
- the identity mapping  $\{\}$ , or  $\varepsilon$  is called the **empty substitution**
- a pair  $X \setminus t$  is called a **binding**
- if a substitution  $\theta$  is a one-to-one, onto mapping different from the empty substitution then  $\theta$  is called a **renaming**

- let  $\theta = \{X_1 \setminus t_1, \dots, X_n \setminus t_n\}$  be a substitution, and  $V$  a set of variables then
  - $DOM(\theta)$  denotes the set  $\{X_1, \dots, X_n\}$
  - $RANGE(\theta)$  denotes the set  $\{t_1, \dots, t_n\}$
  - $VRANGE(\theta)$  denotes the set of variables occurring in  $RANGE(\theta)$
  - $VAR(\theta) = DOM(\theta) \cup VRANGE(\theta)$
  - $\theta|_V = \{X \setminus t \in \theta \mid X \in V\}$  is the **restriction** of  $\theta$  to  $V$
- example: consider  $\theta = \{X \setminus f(Y, Z), Y \setminus g(h(W), a)\}$  and  $\sigma = \{X \setminus Y, Y \setminus Z, Z \setminus X\}$

- the result of **applying** a substitution  $\theta$  to an expression  $E$ , denoted  $E\theta$  is the expression obtained by simultaneously replacing each occurrence of a variable from  $DOM(\theta)$  in  $E$  by the corresponding term in  $RANGE(\theta)$
- $E\theta$  is called an **instance** of  $E$
- $\theta$  is said to be a **grounding** substitution for  $E$  if  $E\theta$  is ground
- if  $\theta$  is a renaming substitution, then  $E\theta$  is called a **variant** of  $E$
- example: let  $s = g(X, g(Y, Z))$ , and substitutions  $\theta = \{X \setminus f(Y, Z), Y \setminus g(h(W), a)\}$  and  $\sigma = \{X \setminus Y, Y \setminus Z, Z \setminus X\}$

- if  $\theta$  and  $\sigma$  are two substitutions, then the **composition**  $\theta\sigma$  is the substitution

$$\{X \setminus t\sigma \mid X \setminus t \in \theta \text{ and } X \neq t\sigma\} \cup \{Y \setminus t \in \sigma \mid Y \notin \text{DOM}(\theta)\}$$

- the composition of substitutions is associative, and has  $\varepsilon$  as left and right identity, but it is not commutative
- **Exercise:** prove this
- example:  $\theta = \{X \setminus f(Y, Z), Y \setminus g(h(W), a)\}$  and  $\sigma = \{X \setminus Y, Y \setminus Z, Z \setminus X\}$  and  $s = g(X, g(Y, Z))$
- substitution  $\theta$  is said to be **more general** than  $\sigma$  if for some substitution  $\tau$  we have that  $\sigma = \theta\tau$
- example:  $\theta = \{X \setminus f(Y, Z)\}$  and  $\sigma = \{X \setminus f(a, b), Y \setminus a, Z \setminus b\}$
- example:  $\theta = \{X \setminus f(Y, Z)\}$  and  $\sigma' = \{X \setminus f(a, b)\}$

- a **unification problem** consists of two terms  $t$  and  $s$ , and is the question whether there exists a substitution  $\theta$  such that  $s\theta = t\theta$
- if such substitution  $\theta$  exists, then it is called a **unifier** for  $s$  and  $t$
- $\theta$  is called the **most general unifier**, or **mgu**, for  $s$  and  $t$  if it is a unifier which is more general than all unifiers for  $s$  and  $t$
- example:  $f(A, b, C, D)$ ,  $f(X, Y, Z, e)$  are unifiable

$$\theta_1 = \{A \setminus a, X \setminus a, Y \setminus b, C \setminus c, Z \setminus c, D \setminus e\}$$

$$\theta_2 = \{X \setminus A, Y \setminus b, C \setminus A, Z \setminus A, D \setminus e\}$$

$$\theta_3 = \{X \setminus A, Y \setminus b, Z \setminus C, D \setminus e, W \setminus a\}$$

$$\theta_4 = \{X \setminus A, Y \setminus b, Z \setminus C, D \setminus e\}$$

$$\theta_5 = \{A \setminus X, Y \setminus b, C \setminus Z, D \setminus e\}$$

$$\theta_6 = \{A \setminus V, X \setminus V, Y \setminus b, C \setminus W, Z \setminus W, D \setminus e\}$$

- a mgu  $\theta$  of  $s$  and  $t$  is called **strong** if for all unifiers  $\sigma$  of  $s$  and  $t$  we have that  $\sigma = \theta\sigma$
- the unification problem is decidable
- if  $s$  and  $t$  are unifiable, then there exists a strong mgu of  $s$  and  $t$  which is unique modulo variable renaming
- several algorithms are known for computing the strong mgu of two unifiable terms [Robinson65] [MartelliMontanari82]

Robinson's unification algorithm

Problem: find mgu of two terms, if it exists

Input two terms  $t_1, t_2$

Output a strong mgu  $\theta$  of  $t_1$  and  $t_2$  if it exists, **no** otherwise

- 1:  $\theta := \emptyset; E := \{t_1 = t_2\}$
- 2: **while**  $E$  is not empty **do**
- 3:     Delete an equation  $T = S$  from  $E$
- 4:     **if**  $T$  is a variable **then**
- 5:         (occur check) if  $T$  occurs in  $S$  then halt with **no**
- 6:          $\theta := \theta\{T \setminus S\} \cup \{T \setminus S\}$
- 7:          $E := E\{T \setminus S\}$
- 8:     **end if**
- 9:     **if**  $S$  is a variable **then**
- 10:         (occur check) if  $S$  occurs in  $T$  then halt with **no**

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11:    $\theta := \theta\{S \setminus T\} \cup \{S \setminus T\}$ 
12:    $E := E\{S \setminus T\}$ 
13: end if
14: if  $T = f(T_1, \dots, T_n)$  and  $S = g(S_1, \dots, S_m)$  are both non-variable terms then
15:   if  $f \neq g$  or  $n \neq m$  then
16:     halt with no
17:   else
18:     add  $S_1 = T_1, \dots, S_n = T_n$  to  $E$ 
19:   end if
20: end if
21: end while
22: return  $\theta$ 

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- **Theorem 16.** *Robinson's algorithm is correct and terminating.*
- Robinson's algorithm is quadratic on the length of the terms.
- Martelli-Montanari algorithm is in  $O(n \log n)$ , and linear algorithm are also known (Paterson-Wegman)

- example:  $f(X, g(Y), Y)$  and  $f(h(Z), g(a), h(b))$
- example:  $f(X)$  and  $f(g(X))$
- example:  $f(X, X)$  and  $f(a, b)$