- introduction
- default logic
- circumscription
- autoepistemic logic
- predicate completion
- answer sets
- well-founded models

- propositional logic, FOL and most of other logics (traditional modal logics, intuitionistic, etc) all exhibit the monotonicity property

$$\text{if } \mathcal{F} \models G \text{ then for all } F,\ \mathcal{F} \cup F \models G$$

- this kind of exact, deductive reasoning is very different from commonsense
- example: let

$$\begin{aligned} \mathcal{F} = \{&bird(tweety), (\forall X)(penguin(X) \Rightarrow bird(X)), \\ &(\forall X)(penguin(X) \Rightarrow \neg fly(X)), \\ &(\forall X)(bird(X) \Rightarrow fly(X))\} \end{aligned}$$

  then $\mathcal{F} \models fly(tweety)$, and $\mathcal{F} \cup \{penguin(tweety)\}$ is inconsistent.
- we would expect $\mathcal{F} \cup \{penguin(tweety)\} \models \neg fly(tweety)$, making $fly(tweety)$ a defeasible consequence.

- this problem is also known as qualification problem in knowledge representation, or the frame problem in planning
- it is not a probabilistic reasoning
- for example, in legal reasoning the principle of presumption of innocence, or also defeasible reasoning is nonmonotonic
- the negation-as-failure operator in Prolog (early 70s) was the first nonmonotonic constructor in a logic-based system
- SLDNF calculus was introduced extending SLD resolution with negation-as-failure

- but SLDNF is an operational semantics dependent on the underlying search strategy in SLD, without declarative counterpart
- there is also the problem of floundering on resolving non-ground negative literals
- so negation-as-failure is not an appropriate semantics for nonmonotonic reasoning
- other logical systems were needed to formalize this reasoning

- introduced by R. Reiter in 1980. Default Logic = classical logic + default rules
- a default rule is a defeasible inference rule of the form

$$\frac{F : G}{H}$$

where F, G, H are sentences in the language called the pre-requisite, the justification, and the conclusion of the default rule

- the interpretation of the rule is that if F is known, and there is no evidence that G is false, then H can be inferred
- a normal default rule is a default rule of the form

$$\frac{F : G}{G}$$

- default reasoning

$$\frac{quacker(X) : pacifist(X)}{pacifist(X)}$$

- rules with exceptions

$$\frac{bird(X) : \neg abnormal(X)}{fly(X)}$$

- frame axioms

$$\frac{on(X, Y, T : \neg moving(X, T)}{on(X, T + 1)}$$

- but the application of a default rule requires a consistency condition to be satisfied
- what makes this condition complicate is that this consistency depends on the application or not of all default rules in the theory
- also, rules can interact in complex ways
- in order to provide a precise semantics for this logic, Reiter introduced the notion of an extension in place of models in classical logic
- a default theory is a pair $(\mathcal{F}, \Gamma)$ being $\mathcal{F}$ a set of FOL sentences and Γ a set of default rules
- $\mathcal{F}$ represents the strict or background information, and Γ represent the defeasible information

- Let D_1 be the default theory $\mathcal{F} = \{a\}$, $\Gamma = \{\frac{\neg\neg a:}{c}, \frac{\neg\neg c:a}{d}\}$
- Let D_2 be the default theory

$$\mathcal{F} = \{quacker(nixon), republican(nixon)\}$$
$$\Gamma = \{\frac{quacker(X):pacifist(X)}{pacifist(X)}, \frac{republican(X):\neg pacifist(X)}{\neg pacifist(X)}\}$$

- a pair of sets of sentences $(\mathcal{F}_1, \mathcal{F}_2)$ triggers a default rule $\frac{F:G}{H}$iff

$$\mathcal{F}_1 \models F \text{ and } \mathcal{F}_2 \not\models \neg G$$

- an extension for a default theory $(\mathcal{F}, \Gamma)$ is a set of sentences E such that

$$E = E_0 \cup E_1 \cup E_2 \cup \ldots \cup E_n \cup \ldots$$

where

$$E_0 = \mathbf{Cn}(\mathcal{F})$$

$$E_{i+1} = \mathbf{Cn}(E_i \cup \{H : \text{ there is } \frac{F : G}{H} \in \Gamma \text{ which is triggered by } (E_i, E)\})$$

- the above definition is not recursive; it is a truly circular characterization of E

- $E = \mathbf{Cn}(\{a, d\})$ is an extension of D_1
- $E = \mathbf{Cn}(\{quacker(nixon), republican(nixon), pacifist(nixon)\})$ is an extension of D_2
- but also $E' = \mathbf{Cn}(\{quacker(nixon), republican(nixon), \neg pacifist(nixon)\})$ is an extension of D_2
- existence of extension is not guaranteed: $D_3 = (\{\}, \{\frac{:p}{\neg p}\})$
- extensions are minimal, if E is an extension then there is no extension E' such that $E \subseteq E'$

- example: let $\mathcal{F} = \{F\}$ and $\Gamma = \{\frac{F:G}{G}\}$. This theory has $\mathbf{Cn}(\{F, G\})$ as its only extension.
- **Theorem 30.** *Let $(\mathcal{F}, \Gamma)$ be a default theory. If Γ has only normal default rules, then the theory has at least an extension.*
- alternative characterization of extensions: A set of sentences E is an extension of default theory $(\mathcal{F}, \Gamma)$ iff
  1. $\mathcal{F} \subseteq E$
  2. E is closed under $\mathbf{Cn}()$ the classical logical consequences operator
  3. E is closed under all default rules $\frac{F:G}{H}$ such that $G \notin E$
  4. every formula in E is justified, ie "derivable" from $\mathcal{F}$ and the default rules

- example: consider the default theory $(\mathcal{F}, \Gamma)$:

$$\begin{aligned}\mathcal{F} =& \{bird(tweety), \\ & (\forall X)(penguin(X) \Rightarrow bird(X)), \\ & (\forall X)(penguin(X) \Rightarrow \neg fly(X)\} \\ \Gamma =& \{\frac{bird(X) : fly(X)}{fly(X)}\}\end{aligned}$$

Then $fly(tweety)$ is contained in its only extension, but it is not in the extension of

$$(\mathcal{F} \cup \{penguin(tweety)\}, \Gamma)$$

- these examples are enough to show that there is by no way an "iterative" process in order to construct an extension
- one has to guess the set of sentences E, and then verify that it satisfies the definition
- from extensions we can define both
  - credulous semantics: the consequences of the default theory are the sentences in one chosen extension of the theory
  - skeptical semantics: the consequences of the default theory are the sentences belonging to all the extensions of the theory

- default logic seems to be very hard to handle computationally because of
  - the extensions are infinite sets of formulas
  - the definition of extension is non-constructive
- in fact, first order default reasonin is not even semi-decidable
- propositional skeptical reasoning is Π_2^P-complete
- propositional credulous reasonin and extension existence are Σ_2^P-complete
- this means it is highly unlikely that default reasoning can be implemented on top of a classical theorem prover with polynomial overhead.

- introduced by McCarthy in 1980
- circumscription select from a FOL theory those models that minimally satisfy a given predicate
- example: normally a block is on the table, a, b are different blocks, a is not on the table

$$block(X) \wedge \neg abnormal(X) \Rightarrow on(X, table)$$
$$block(a), block(b), a \neq b, \neg on(a, table)$$

- classical logic does not conclude $on(b, table)$ because there is a model with too many abnormal objects
- so circumscription is based on the idea to consider only logical consequences of minimal models

- let F, G be two FOL formulas with the same free variables $X_1, \ldots, X_n$, then we write $F \leq G$ for

$$(\forall X_1, \ldots, X_n)(F \Rightarrow G)$$

and $F < G$ for

$$(\forall X_1, \ldots, X_n)(F \Rightarrow G) \wedge (\forall X_1, \ldots, X_n)(\neg(F \Leftrightarrow G))$$

- this notation has an intuitive reading within the semantics of FOL
- let F be a FOL sentence containing a predicate $p(X_1, \ldots, X_n)$, and Φ a predicate variable with the same arity thant p
- then the circumscription of p in F is the second order sentence

$$(F[p] \wedge \neg(\exists \Phi)(F[\Phi] \wedge (\Phi < p))$$

We write $\mathrm{CIRC}[F; p]$

- example $\mathrm{CIRC}[p(a);p] \equiv (\forall X)(p(X) \Leftrightarrow X = a)$
- example $\mathrm{CIRC}[\neg p(a);p] \equiv (\forall X)(\neg p(X))$
- example $\mathrm{CIRC}[p(a) \wedge p(b);p] \equiv (\forall X)(p(X) \equiv (X = a \vee X = b))$
- example $\mathrm{CIRC}[p(a) \vee p(b);p] \equiv (\forall X)(p(X) \Leftrightarrow X = a) \vee (\forall X)(p(X) \Leftrightarrow X = a)$
- example $\mathrm{CIRC}[p(a) \Rightarrow p(b);p] \equiv (\forall X)(\neg p(X))$
- example $\mathrm{CIRC}[p(a) \vee (p(b) \wedge p(c));p] \equiv$

$$(\forall X)(p(X) \Leftrightarrow X = a) \vee ((\forall X)(p(X) \Leftrightarrow X = b \vee X = c) \wedge a \neq b \wedge a \neq c)$$

- example $\mathrm{CIRC}[(\forall X)(q(X) \Rightarrow p(X));p] \equiv (\forall X)(q(X) \Leftrightarrow p(X))$

- this can be regarded as asserting that the only tuples that satisfy p are those that have to, as long as F is true
- for the semantics of circumscription we need the notion of minimal model
- let M and N be two models of sentence F. We say that M is a submodel of N in p, $M \leq_p N$ iff M and N have the same domain, all other predicate and function symbols in A besides p have the same extensions, but the extension of p in M is included in its extension in N
- a model M is minimal in p if $M' \leq_p M$ only if $M' = M$
- minimal models don't always exists, so the circumscriptive theory may be inconsistent

- **Theorem 31.** $\mathrm{CIRC}[F;p] \models G$ *iff* G *is true in all minimal models of* F *in* p
- **Exercise**: prove this result
- example: let $F = p(a) \wedge p(b) \wedge p(c)$, then

$$\mathrm{CIRC}[F;p] \models X = a \vee X = b \vee X = c$$

- example: let $F = p(a) \vee p(b)$, then

$$\mathrm{CIRC}[F;p] \models (\forall X)(p(X) \Rightarrow X = a) \vee (\forall X)(p(X) \Rightarrow X = b)$$

- in most application, the basic form of circumscription is too special. It doesn't allow to formalize for example the blocks world theories.
- this is because minimality is understood as the impossibility of making the extend of the circumscribed predicate smaller, but without changing anything else in the world
- for example let $F = p(a) \vee p(b)$, then

$$\mathrm{CIRC}[F; p] \not\models (\forall X)(p(X) \Rightarrow X = a) \wedge a = b$$

- or $\mathrm{CIRC}[(\forall X)(q(X) \Rightarrow p(X)); p] \not\equiv (\forall X)(\neg p(X))$
- we need to be able to specify that some objects, function or predicate symbols occurring in the theory, are able to vary along the process of minimizing p.

- so let t F be a FOL sentence containing a predicate p and function/constants $z_1, \ldots, z_n$ (with possible other functions and constants).
- and Φ a predicate variable with the same arity thant p
- then the circumscription of p in F with varied $z_1, \ldots, z_n$ is the second order sentence

$$(F[p, z_1, \ldots, z_n] \wedge \neg(\exists \Phi, \Psi_1, \ldots, \Psi_n)(F[\Phi, \Psi_1, \ldots, \Psi_n] \wedge (\Phi < p))$$

We write $\mathrm{CIRC}[F; p; z_1, \ldots, z_n]$

- example: let $F = p(a) \vee p(b)$, then

$$\mathrm{CIRC}[F; p; a, b] \models (\forall X)(p(X) \Rightarrow X = a) \wedge a = b$$

- example: $\mathrm{CIRC}[(\forall X)(q(X) \Rightarrow p(X)); p; q] \equiv (\forall X)(\neg p(X))$

- example: let F be

$$bird(tweety) \wedge (\forall X)(bird(X) \wedge \neg abnormal(X) \Rightarrow fly(X))$$
$$\wedge (\forall X)(penguin(X) \Rightarrow bird(X))$$
$$\wedge (\forall X)(penguin(X) \Rightarrow \neg fly(X))$$

then $\mathrm{CIRC}[F; abnormal; fly] \models fly(tweety)$
but $\mathrm{CIRC}[F \cup \{penguin(tweety)\}; abnormal; fly] \models \neg fly(tweety)$

- example: let F be

$$\neg abnormal0(X) \Rightarrow \neg ontable(X)$$
$$block(X) \wedge \neg abnormal1(X) \Rightarrow ontable(X)$$
$$heavyBlock(X) \wedge \neg abnormal2(X) \Rightarrow \neg ontable(X)$$
$$heavyBlock(b1), block(b2), b1 \neq b2, b1 \neq b3, b2 \neq b3$$
$$\textbf{circ } abnormal0 \textbf{ var } ontable$$
$$\textbf{circ } abnormal1 \textbf{ var } ontable, abnormal0$$
$$\textbf{circ } abnormal2 \textbf{ var } ontable, abnormal0, abnormal1, abnormal2$$

  then this theory has as logical consequences $\neg ontable(b1)$, $ontable(b2)$, $\neg ontable(b3)$. We give preferences to the minimization policy.

- several other different variants of circumscription exist: domain circumscription, pointwise circumscription, parallel circumscription, etc.

- introduced by R. Moore in 1985
- it is one of several possible nonmonotonic modal logics studied by McDermott & Doyle
- let $\mathcal{L}$ be a propositional language, we define $\mathcal{L}_{\mathbf{B}}$ the smallest set such that
  - $\mathcal{L} \subseteq \mathcal{L}_{\mathbf{B}}$
  - if $F, G \in \mathcal{L}_{\mathbf{B}}$ then so are $\neg F$, $F \wedge G$, $F \vee G$, $F \Rightarrow G$, $F \Leftrightarrow G$
  - if $F \in \mathcal{L}_{\mathbf{B}}$ then $\mathbf{B}F \in \mathcal{L}_{\mathbf{B}}$
- an autoepistemic theory is any set of formulas in $\mathcal{L}_B$

- an autoepistemic theory T is said to be stable iff satisfies all
  - $T = \{F : T \models F\}$
  - if $F \in T$ then $\mathbf{B}F \in F$ (necessitation)
  - if $F \notin T$ then $\neg\mathbf{B}F \in T$
- **Theorem 32.** *if T is stable then*
  - *if $\mathbf{B}F \in T$ then $F \in T$*
  - *if T is consistent and $\neg\mathbf{B}F \in T$ then $F \notin T$*
- **Exercise**: prove this result
- then if T is stable and consistent, $F \in T$ iff $\mathbf{B}F \in T$, and $F \notin T$ iff $\neg\mathbf{B}F \in T$

- intuitively, an agent set of beliefs is a stable and consistent set of formulas which include some predefined facts T
- for a semantic characterization we will introduce the notion of stable expansions
- a set of formulas E is said to be an stable expansion of T iff

$$E = \{F \ : \ T \cup \{\mathbf{B}F : F \in E\} \cup \{\neg\mathbf{B}F : F \notin E\} \models F\}$$

- this is a circular definition like of expansions in default logic
- a stable expansion not always exist: $\{p \Rightarrow \mathbf{B}\neg p\}$
- some theories have more than one stable expansion: $\{\mathbf{B}p \Rightarrow p\}$

- the kernel E_0 of a stable theory $E \subseteq \mathcal{L}_B$ is defined as the propositional subset of E
- **Lemma 1.** *if E is a stable set, then E is an expansion of E_0*
- **Lemma 2.** *if E and F are stable sets such that $E_0 = F_0$ then $E = F$*
- this means that two stable expansions with the same set of propositional knowledge must necessarily be the same expansions
- **Exercise**: prove the previous two lemmas

- **Theorem 33.** *if* $T \subseteq \mathcal{L}_B$ *is stable, then* T *is closed under* $S5(K+T+4+5)$ *consequences*
- remind the modal axioms
  - **K**: $\mathbf{B}(F \Rightarrow G) \Rightarrow (\mathbf{B}F \Rightarrow \mathbf{B}G)$
  - **T**: $\mathbf{B}F \Rightarrow F$
  - **4**: $\mathbf{B}F \Rightarrow \mathbf{BB}F$
  - **5**: $\neg\mathbf{B}F \Rightarrow \mathbf{B}\neg\mathbf{B}F$

- but not every theory closed under $S5$ is stable: $\{\mathbf{B}p\}$ has only one expansion $\{\}$ but under $S5$ it will produce $\{p\}$
- so the strongest modal logic S that can be used when defining expansions is $K45$ (weak $S5$)
- example: $\{\mathbf{B}p \Rightarrow p\}$
- example: $\{\mathbf{B}p \Rightarrow q, \mathbf{B}q \Rightarrow p\}$
- example: $\{\neg Bp \Rightarrow q, \mathbf{B}p \Rightarrow p\}$

- example: consider the theory T

$$bird(tweety)$$
$$(\forall X)(bird(X) \wedge \neg \mathbf{B} fly(X) \Rightarrow fly(X))$$
$$(\forall X)(penguin(X) \Rightarrow bird(X))$$
$$(\forall X)(penguin(X) \Rightarrow \neg fly(X))$$

then it has an expansion based on kernel

$$T \cup \{fly(tweety)\}$$

Also the theory $T \cup \{penguin(tweety)\}$ has only one expansion based on kernel

$$T \cup \{penguin(tweety), \neg fly(tweety)\}$$