5. Reasoning in Description Logics

Exercise 5.1 Let $\mathcal{T}$ be a TBox consisting of concept inclusions of the form $A_1 \sqsubseteq A_2$ and concept disjointness assertion of the form $A_1 \sqsubseteq \neg A_2$, for atomic concepts $A_1$ and $A_2$.

Describe an algorithm for checking concept satisfiability with respect to $\mathcal{T}$, i.e., whether for some concept $A$ it holds that $A$ is satisfiable with respect to $\mathcal{T}$.

Exercise 5.2 Consider TBoxes $\mathcal{T}$ consisting of axioms of the form $B_1 \sqsubseteq B_2$, where

\[ B_1, B_2 ::= A \mid \exists R \mid \exists R^-, \]

$A$ denotes an atomic concept, and $R$ an atomic role.

1. Describe an algorithm for checking subsumption with respect to a given $\mathcal{T}$, i.e., whether for two concepts $B_1$ and $B_2$ it holds that $\mathcal{T} \models B_1 \sqsubseteq B_2$.

2. Let $\mathcal{A} = \{ A_0(a) \}$ and $\mathcal{T}$ a (n arbitrary) TBox of the above form. Can we determine whether $(\mathcal{T}, \mathcal{A})$ is satisfiable?

Exercise 5.3 Show that concept satisfiability in $\mathcal{ALC}$ is NP-hard.

Hint: show the claim by reduction from SAT.

Exercise 5.4 Let $q_n$, for $n \geq 2$, be a Boolean conjunctive query with $n$ existential variables of the form $\exists x_1, \ldots, x_n. P(x_1, x_2) \land \cdots \land P(x_{n-1}, x_n)$. Given $n \geq 2$:

1. construct an $\mathcal{ALC}$ KB $\mathcal{K}_n$ such that $\mathcal{K}_n \models q_n$.

2. construct an $\mathcal{ALC}$ KB $\mathcal{K}'_{2n}$ of size polynomial in $n$ such that $\mathcal{K}'_{2n} \models q_{2n}$ and $\mathcal{K}'_{2n} \not\models q_{2n+1}$.

Hint: $\mathcal{K}'_{2n}$ “implements” a binary counter by means of $n$ atomic concepts representing the bits of the counter, and such that the models of $\mathcal{K}'_{2n}$ contain a $P$-chain of objects of length $2^n$. 