Circumscribing *DL-Lite*

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Abstract. Classical logics (and hence Description Logics) are monotonic: the set of conclusions increases monotonically with the set of premises. Instead, common-sense reasoning is characterized as non-monotonic: new information can invalidate some of the previously made conclusions. Circumscription is one of the main non-monotonic formalisms whose idea is to minimize (circumscribe) the extension of given predicates. In this paper we study circumscribed *DL-Lite* knowledge bases and show how to compute circumscription of a single predicate (either a concept or a role) in a *DL-Lite*$^{\mathcal{H}}_{bool}$ knowledge base. Unlike other works on circumscribed Description Logics KBs, we are interested not only in checking entailment, but actually in computing circumscription itself. We show that circumscription of a role in *DL-Lite*$^{\mathcal{H}}_{bool}$ requires the language of $\mathcal{ALCHOIQ}$ extended with union or roles, thus is first-order expressible.

1 Introduction

Description Logics (DLs) [2] are acknowledged as computationally well-behaved fragments of first-order logic, and widely used in areas such as Knowledge Representation, Semantic Web and Ontology-Based Data Access for automated reasoning. There has been a continuous interest in non-monotonic extensions of DLs, and a considerable amount of work in that field includes extensions of DLs with default logic [28, 3, 30], with preference relation [19, 13, 7, 10], with circumscription [23, 6, 15, 5], with defeasible logic [25, 14, 32, 16] and with logic of Minimal Knowledge and Negation as Failure [22, 12, 17, 24].

Motivation for non-monotonic reasoning comes from the need to handle real life scenarios when the knowledge about the world is incomplete or changing. One of the motivations for non-monotonic DLs stems from the biomedical domain [26] where DLs are used as a tool for the formalization of ontologies such as SNOMED [11] and GALEN [27]. Another motivation comes from policy languages based on DLs [31, 34], which require non-monotonic reasoning. Prototypical properties and defeasible inheritance in DLs can also be added to the wish list.

In our work we have chosen circumscription as the underlying non-monotonic formalism for two main reasons. First, the semantics of circumscription is sufficiently simple, so circumscribed DLs can be defined in a straightforward way. Second, the existing works on circumscribed DLs [6, 5] show that they are interesting objects to be investigated and one could get nice results if one used a low complexity DL. More precisely, the current approaches to circumscribed DL knowledge bases (KBs) can be divided in two according to the DL used: expressive DLs such as $\mathcal{ALC}$, $\mathcal{ALCIO}$ and $\mathcal{ALCQO}$ [6], and tractable DLs such as $\mathcal{EL}$ and $\mathcal{EL}^{++}$ [4, 5]. The former showed that reasoning in circumscribed $\mathcal{ALC}$ KBs is NExpTime$^{\text{NP}}$-hard, while some forms of reasoning in circumscribed $\mathcal{EL}$ KBs are tractable.

In this paper we investigate circumscription in *DL-Lite*$^{\mathcal{H}}_{bool}$, which is a sub-logic of the expressive DL $\mathcal{ALCHI}$ (essentially $\mathcal{ALC}$ with role hierarchy and inverse roles), and a super-logic of *DL-Lite*$_{\mathcal{R}}$, the basic *DL-Lite* logic. *DL-Lite*$^{\mathcal{H}}_{bool}$ is a member of the extended *DL-Lite* family [1], popular for its low complexity of reasoning, notably AC^0 data complexity of answering (atomic) queries. In contrast with the previous works on circumscribed DLs we not only want to check entailment, but also to compute circumscription of *DL-Lite*$^{\mathcal{H}}_{bool}$ KBs. We show that the circumscription of a single predicate (concept or role) in *DL-Lite*$^{\mathcal{H}}_{bool}$ can be expressed in $\mathcal{ALCHOIQ}$ with union of roles.

The paper is organized as follows: In Section 2 we introduce the logic *DL-Lite*$^{\mathcal{H}}_{bool}$ and the notion of circumscription. Section 3 presents circumscribed *DL-Lite* and includes a motivating example. In Section 4, we show how to compute circumscription of a single predicate in *DL-Lite*$^{\mathcal{H}}_{bool}$, and in Section 5, we show how to check entailment in the circumscribed KB. Finally, in Section 6, we draw some conclusions and outline issues for future work.

2 Preliminaries

We introduce the DLs that we adopt in this paper, and then recall the notions about circumscription.

2.1 Description Logics

Here, we present the DL *DL-Lite*$^{\mathcal{H}}_{bool}$, a member of the extended *DL-Lite* family of DLs known for their nice computational properties [8, 1]. Good computational behavior of *DL-Lite*$^{\mathcal{H}}_{bool}$, which is a sub-logic of $\mathcal{ALCHI}$, is achieved by prohibiting concepts of the form $\exists R.C$ and $\forall R.C$. Satisfiability checking in *DL-Lite*$^{\mathcal{H}}_{bool}$ can be done in NP in combined complexity and in AC^0 in data complexity [1].

Let N_C, N_R, and N_a be countably infinite sets of concept, role and individual names, respectively. The language of *DL-Lite*$^{\mathcal{H}}_{bool}$ contains individual names $a, b \in N_a$, atomic concepts $A \in N_C$, and atomic roles $P \in N_R$. Complex roles Q and concepts C of this language are defined as follows:

$$
\begin{array}{rcllll}
R & ::= & P & \mid & P^- & \\
Q & ::= & R & \mid & \neg R & \\
B & ::= & \bot & \mid & A & \mid \exists R \\
C & ::= & B & \mid & \neg C & \mid C_1 \sqcap C_2
\end{array}
$$

The concepts of the form B are called *basic* concepts and roles of the form R are called *basic* roles. Moreover, for a role R, we use R^- to denote P^- when $R = P$, and P when $R = P^-$. A predicate in DLs is either an atomic concept or an atomic role.

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A *DL-Lite$^{\mathcal{H}}_{bool}$ TBox*, $\mathcal{T}$, is a finite set of concept and role inclusion axioms (or simply concept and role *inclusions*) of the form:

$$C_1 \sqsubseteq C_2 \qquad \text{and} \qquad R \sqsubseteq Q,$$

and an *ABox*, $\mathcal{A}$, is a finite set of membership assertions:

$$A(a), \quad \neg A(a), \quad P(a,b), \quad \text{and} \quad \neg P(a,b).$$

A *DL-Lite$^{\mathcal{H}}_{bool}$* KB $\mathcal{K}$ is a pair $\langle \mathcal{T}, \mathcal{A} \rangle$.

The semantics of *DL-Lite$^{\mathcal{H}}_{bool}$* is defined as usual in DLs. An *interpretation* $\mathcal{I}$ is a pair $\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ with non-empty domain $\Delta^{\mathcal{I}}$ and interpretation function $\cdot^{\mathcal{I}}$ that assigns *(i)* to every concept name A a subset $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ of the domain; *(ii)* to every role name P a binary relation $P^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ over the domain; *(iii)* to every individual a an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.

Concept and role constructs are interpreted as follows

$$\begin{array}{rclrcl}
(P^-)^{\mathcal{I}} &=& \{(y,x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (x,y) \in P^{\mathcal{I}}\} & & & \\
(\neg R)^{\mathcal{I}} &=& \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \setminus R^{\mathcal{I}} & \bot^{\mathcal{I}} &=& \emptyset \\
(\exists R)^{\mathcal{I}} &=& \{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}}, (x,y) \in R^{\mathcal{I}}\} & & & \\
(\neg C)^{\mathcal{I}} &=& \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} & (C_1 \sqcap C_2)^{\mathcal{I}} &=& {C_1}^{\mathcal{I}} \cap {C_2}^{\mathcal{I}}
\end{array}$$

We will use standard abbreviations such as $C_1 \sqcup C_2$ for $\neg(\neg C_1 \sqcap \neg C_2)$, and $\top$ for $\neg\bot$.

The satisfaction relation is defined as follows:

$$\begin{array}{llll}
\mathcal{I} \models C_1 \sqsubseteq C_2 & \text{iff } {C_1}^{\mathcal{I}} \subseteq {C_2}^{\mathcal{I}} & \mathcal{I} \models R \sqsubseteq Q & \text{iff } R^{\mathcal{I}} \subseteq Q^{\mathcal{I}} \\
\mathcal{I} \models A(a) & \text{iff } a^{\mathcal{I}} \in A^{\mathcal{I}} & \mathcal{I} \models P(a,b) & \text{iff } (a^{\mathcal{I}}, b^{\mathcal{I}}) \in P^{\mathcal{I}} \\
\mathcal{I} \models \neg A(a) & \text{iff } a^{\mathcal{I}} \notin A^{\mathcal{I}} & \mathcal{I} \models \neg P(a,b) & \text{iff } (a^{\mathcal{I}}, b^{\mathcal{I}}) \notin P^{\mathcal{I}}
\end{array}$$

We say that $\mathcal{I}$ is a *model* of a TBox if (resp., ABox) it satisfies all its axioms (resp., assertions). $\mathcal{I}$ is a *model* of a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ if it is a model of both $\mathcal{T}$ and $\mathcal{A}$. $\mathcal{K}$ is said to be *satisfiable* (or *consistent*) if it has a model.

DL-Lite$^{\mathcal{H}}_{bool}$ is a super-logic of other three *DL-Lite* DLs that differ in the form of allowed TBox inclusions [1]. Here we mention only *DL-Lite$^{\mathcal{H}}_{core}$*, also known as *DL-Lite$_{\mathcal{R}}$* (in the original paper [8]). A TBox $\mathcal{T}$ is a *DL-Lite$^{\mathcal{H}}_{core}$* TBox if its concept inclusions are of the form

$$B_1 \sqsubseteq B_2 \quad \text{or} \quad B_1 \sqsubseteq \neg B_2.$$

A *signature* Σ is a set of concept and role names, that is, $\Sigma \subseteq N_C \cup N_R$. Given a KB $\mathcal{K}$, the *signature* $\Sigma(\mathcal{K})$ *of* $\mathcal{K}$ is the alphabet of concept and role names occurring in $\mathcal{K}$ (and likewise for a TBox $\mathcal{T}$, an ABox $\mathcal{A}$, a concept C, and a role R).

We are going to express circumscription in the language of $\mathcal{ALCHOIQ}$, which is $\mathcal{ALCHI}$ extended with nominals ($\mathcal{O}$) and qualified number restrictions ($\mathcal{Q}$). Here we present the missing constructs.

Let R be a basic role, as defined in the previous section. Then complex concepts C in $\mathcal{ALCHOIQ}$ are built according to the following syntax:

$$C ::= A \mid \exists R.C \mid \neg C \mid C_1 \sqcap C_2 \mid \{a_1, \ldots, a_n\} \mid \geq k\, R.C$$

where k is a non-negative integer and n is a positive integer. Here we have three new constructs: qualified existential restriction $\exists R.C$, nominals $\{a_1, \ldots, a_n\}$, and qualified number restriction $\geq k\, R.C$. Note that the construct $\exists R$ can be seen an abbreviation for $\exists R.\top$.

The new constructs are interpreted as follows:

$$\begin{array}{rcl}
(\exists R.C)^{\mathcal{I}} &=& \{x \in \Delta^{\mathcal{I}} \mid \exists y \in C^{\mathcal{I}}, (x,y) \in R^{\mathcal{I}}\} \\
\{a_1, \ldots, a_n\}^{\mathcal{I}} &=& \{{a_1}^{\mathcal{I}}, \ldots, {a_n}^{\mathcal{I}}\} \\
(\geq k\, R.C)^{\mathcal{I}} &=& \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \in C^{\mathcal{I}} \mid (x,y) \in R^{\mathcal{I}}\} \geq k\}
\end{array}$$

In the following, we will use $\mathsf{Funct}(P)$ to abbreviate $\geq 2\, P \sqsubseteq \bot$.

2.2 Circumscription

Circumscription was introduced by McCarthy [23], and has been well studied and explored by Lifschitz [20, 21] and others [18, 33]. It is an important formalism of common-sense reasoning that offers non-monotonic reasoning abilities by circumscribing (minimizing) the extension of specific predicates. Below we briefly present the notion of circumscription and circumscribed theories.

First, for any predicate symbols P, Q of the same arity, $P = Q$ stands for $\forall x(P(x) \equiv Q(x))$, $P \leq Q$ stands for $\forall x(P(x) \rightarrow Q(x))$, and $P < Q$ stands for $(P \leq Q) \land \neg(P = Q)$.

Let $\Phi(P)$ be a first-order sentence containing a predicate constant P. Then, by definition, the *circumscription* of P in $\Phi(P)$, denoted $\mathsf{Circ}(\Phi; P)$, is the second-order formula

$$\Phi(P) \land \forall p \neg(\Phi(p) \land p < P),$$

where p is a predicate variable of the same arity as P.

More generally, we can simultaneously minimize several predicates, which gives *parallel* and *prioritized* circumscription (here we introduce only parallel circumscription). Moreover, we may allow the extension of some predicates to vary in order to make the extension of the minimized predicates smaller. Let P be a tuple of predicate constants, and Z a tuple of function and/or predicate constants disjoint with P, and $\Phi(P, Z)$ a sentence. Then the circumscription of P in $\Phi(P, Z)$ with variable Z, denoted $\mathsf{Circ}(\Phi; P; Z)$, is the sentence

$$\Phi(P, Z) \land \forall pz \neg(\Phi(p, z) \land p < P),$$

where the notation $P \sim Q$, with $\sim$ being one of $=, \leq, <$, is generalized to tuples of predicates: $P \leq Q$ stands for $P_1 \leq Q_1 \land \cdots \land P_n \leq Q_n$, similar for $P = Q$. Finally, $P < Q$ stands again for $(P \leq Q) \land \neg(P = Q)$.

The models of $\mathsf{Circ}(\Phi; P; Z)$ are the models of Φ such that the extension of P cannot be made smaller without losing the property Φ, even at the price of changing the interpretations of Z. In order to define a model formally, we need to define an order on interpretations.

Let $\mathcal{I}$ and $\mathcal{J}$ be two classical interpretations of Φ. Then we write $\mathcal{I} \leq^{P;Z} \mathcal{J}$ if

- $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$,
- $X^{\mathcal{I}} = X^{\mathcal{J}}$ for every X that does not belong to P, nor to Z,
- $X^{\mathcal{I}} \subseteq X^{\mathcal{J}}$ for every $X \in P$.

We write $\mathcal{I} <^{P;Z} \mathcal{J}$ if $\mathcal{I} \leq^{P;Z} \mathcal{J}$ but not $\mathcal{J} \leq^{P;Z} \mathcal{I}$.

An interpretation $\mathcal{I}$ is a *model* of $\mathsf{Circ}(\Phi; P; Z)$ if it is a model of Φ and it is minimal relative to $\leq^{P;Z}$, i.e., there is no other model $\mathcal{J}$ of Φ such that $\mathcal{J} <^{P;Z} \mathcal{I}$.

To ensure the existence of a model of $\mathsf{Circ}(\Phi; P; Z)$ we need Φ to be well-founded w.r.t. $(P; Z)$. Φ is said to be *well-founded* w.r.t. $(P; Z)$ if for every model $\mathcal{J}$ of Φ there exists a model $\mathcal{I}$ of Φ minimal relative to $\leq^{P;Z}$ and such that $\mathcal{I} \leq^{P;Z} \mathcal{J}$.

A lot of effort has been made to understand in which cases circumscription is first-order expressible, and what its computational properties are [21, 18, 33]. A simple case when circumscription is not first-order expressible is circumscribing a transitive binary predicate. Then circumscription of that predicate is equivalent to the transitive closure, and it cannot be reduced to a first-order sentence.

Below we present some results that help to compute circumscription:

- [21] if Ψ does not contain P, Z, then

$$\mathsf{Circ}(\Phi(P, Z) \land \Psi; P; Z) \equiv \mathsf{Circ}(\Phi(P, Z); P; Z) \land \Psi$$

- [21] if $\Psi(P)$ contains only negative occurrences of P, then
$$\mathsf{Circ}(\Phi(P) \wedge \Psi(P); P) \equiv \mathsf{Circ}(\Phi(P); P) \wedge \Psi(P),$$
where an occurrence of P in $\Psi(P)$ is said to be *negative* if P appears negated in the negation normal form (NNF) of $\Psi(P)$.
- [21] if Ψ does not contain P, then
$$\mathsf{Circ}(\forall x(\Psi(x) \to P(x)); P) = \forall x(\Psi(x) \equiv P(x)),$$
and it is called *predicate completion*.
- [21] if a sentence Φ is satisfiable and well-founded w.r.t. $(P; Z)$, then $\mathsf{Circ}(\Phi; P; Z)$ is satisfiable.
- [33] the finite model property of a first-order fragment implies its decidability under the circumscriptive semantics.

3 Circumscribed *DL-Lite*

Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be a *DL-Lite*$^{\mathcal{H}}_{bool}$ KB. Let M and V be sets of predicates from the signature of $\mathcal{K}$, such that $M \cap V = \emptyset$. Then $\mathcal{K}$ *circumscribed w.r.t. minimized predicates* M *and varied predicates* V is an expression:
$$\mathsf{Circ}(\mathcal{K}; M; V).$$
If V is empty, we write $\mathsf{Circ}(\mathcal{K}; M)$. The rest of the predicates are assumed to be fixed and predicates from M are assumed to be minimized in parallel. If the ABox is empty, then we circumscribe only $\mathcal{T}$ and write $\mathsf{Circ}(\mathcal{T}; M; V)$.

We rely on the notion of a model as defined for the classical circumscription. An interpretation $\mathcal{I}$ is a *model* of $\mathsf{Circ}(\mathcal{K}; M; V)$ if it is a model of $\mathsf{Circ}(\Phi_{\mathcal{K}}; M; V)$, where $\Phi_{\mathcal{K}}$ is the standard translation of $\mathcal{K}$ to first-order logic.

Theorem 1. *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be a DL-Lite$^{\mathcal{H}}_{bool}$ KB, M, V sets of predicates from the signature of $\mathcal{K}$, such that $M \cap V = \emptyset$. Then $\mathcal{K}$ is well-founded w.r.t. $(M; V)$, and if $\mathcal{K}$ is satisfiable, then $\mathsf{Circ}(\mathcal{K}; M; V)$ is satisfiable.*

Proof. Well-foundedness follows from the finite model property of *DL-Lite*$^{\mathcal{H}}_{bool}$ and the last claim follows from Proposition 11 in [21]. □

The typical example for non-monotonic reasoning is the Tweety example. It can be encoded in circumscribed *DL-Lite*$^{\mathcal{H}}_{bool}$ as follows:

Example 1. Assume an ontology about birds. We want to express the following commonsense facts: typically birds fly, penguins are birds, and they cannot fly. Let *Bird*, *Abnormal*, *Penguin*, *Flier* be concept names and $\mathcal{T}$ the following TBox
$$\begin{aligned} \mathit{Bird} \sqcap \neg \mathit{Abnormal} &\sqsubseteq \mathit{Flier} \\ \mathit{Penguin} &\sqsubseteq \mathit{Bird} \\ \mathit{Penguin} &\sqsubseteq \mathit{Abnormal} \\ \mathit{Abnormal} &\sqsubseteq \neg \mathit{Flier} \end{aligned}$$
Moreover, we assume that birds are considered normal if there is no evidence to the contrary. Therefore, we minimize the set of abnormal birds.

Suppose we know that Tweety is a bird, which is encoded in the ABox $A = \{\mathit{Bird}(\mathsf{tweety})\}$. Then, we obtain that
$$\mathsf{Circ}(\langle \mathcal{T}, \mathcal{A} \rangle; \mathit{Abnormal}) \models \mathit{Flier}(\mathsf{tweety}).$$
Now, assume we learn that Tweety is not just a bird, but a penguin, $\mathcal{A}' = \mathcal{A} \cup \{\mathit{Penguin}(\mathsf{tweety})\}$. Then
$$\mathsf{Circ}(\langle \mathcal{T}, \mathcal{A}' \rangle; \mathit{Abnormal}) \models \neg \mathit{Flier}(\mathsf{tweety}).$$
Thus, we have invalidated the previous conclusion that Tweety flies.

4 Computing Circumscription in *DL-Lite*$^{\mathcal{H}}_{bool}$

In this section we show how to compute circumscription of a *DL-Lite*$^{\mathcal{H}}_{bool}$ KB with respect to a single predicate, that is, for $M = \{X\}$.

It is easy to compute circumscription of an atomic concept A in a *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox $\mathcal{T}$. We start by observing that we can assume w.l.o.g. that each concept inclusion axiom in $\mathcal{T}$ has the 'normal' form $\top \sqsubseteq L_1 \sqcup \cdots \sqcup L_n$, where each L_i is either a basic concept or a negated basic concept, and no basic concept appears both positively and negatively in the same axiom. Indeed, the transformation of an arbitrary set of concept inclusions into this form is analogous to the conversion of a propositional formula into an equivalent set of clauses.[2] Hence, for a concept inclusion α of the above form, we say that α is *positive* (resp., *negative*) w.r.t. A if A appears positively (resp., negatively) in α. Let $\mathsf{Pos}_{\mathcal{T}}(A)$ be the set of all inclusions in $\mathcal{T}$ positive w.r.t. A, and $\mathsf{Neg}_{\mathcal{T}}(A)$ the set of all inclusions in $\mathcal{T}$ negative w.r.t. A. Moreover, again w.l.o.g., we may consider that each axiom in $\mathsf{Pos}_{\mathcal{T}}(A)$ has the form $C \sqsubseteq A$, for some concept C not containing A.

Proposition 2. *Let $\mathcal{T}$ be a DL-Lite$^{\mathcal{H}}_{bool}$ TBox and A an atomic concept of $\mathcal{T}$. Then*
$$\mathsf{Circ}(\mathcal{T}; \{A\}) = \mathcal{T} \cup \{C_1 \sqcup \cdots \sqcup C_n \equiv A\},$$
where $\mathsf{Pos}_{\mathcal{T}}(A) = \{C_i \sqsubseteq A\}_{i=1}^{n}$.

Proof. Since $\mathcal{T}$ is a *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox, we have that
$$\mathcal{T} = \mathsf{Pos}_{\mathcal{T}}(A) \cup \mathsf{Neg}_{\mathcal{T}}(A) \cup \mathcal{T}',$$
where $\mathcal{T}'$ is the set of inclusions in $\mathcal{T}$ that do not contain A. Therefore, A does not appear in the concept $C_1 \sqcup \cdots \sqcup C_n$, and the result follows directly from the properties of circumscription. □

Notice that in *DL-Lite*$^{\mathcal{H}}_{bool}$, circumscribing an atomic concept corresponds to predicate completion. Also notice, that if $\mathcal{T}$ is a *DL-Lite*$^{\mathcal{H}}_{core}$ TBox, $\mathsf{Circ}(\mathcal{T}; \{A\})$ is a *DL-Lite*$^{\mathcal{H}}_{bool}$ KB.

Computing circumscription is not so trivial when X is an atomic role P. In the following we compute circumscription of P in *DL-Lite*$^{\mathcal{H}}_{core}$ and *DL-Lite*$^{\mathcal{H}}_{bool}$ TBoxes.

4.1 Circumscribing a *DL-Lite*$^{\mathcal{H}}_{core}$ TBox

We start by circumscribing *DL-Lite*$^{\mathcal{H}}_{core}$ TBoxes. In *DL-Lite*$^{\mathcal{H}}_{core}$ a role P can appear positively in the assertions of the form:
$$R \sqsubseteq P, \qquad B \sqsubseteq \exists P, \qquad B \sqsubseteq \exists P^-,$$
where R is a basic role and B is an atomic concept. First, we compute circumscription of P for several easy cases.

Let P be a role name, R a role, and C_1, C_2 concepts such that $P \notin \Sigma(\{R, C_1, C_2\})$. For an interpretation $\mathcal{I}$ and a tuple of domain elements (a, b), we denote by $\mathcal{I} \setminus P(a, b)$ the interpretation $\mathcal{I}'$ that agrees with $\mathcal{I}$ on all predicates except P and $P^{\mathcal{I}'} = P^{\mathcal{I}} \setminus \{(a, b)\}$).

1. $\mathsf{Circ}(\{R \sqsubseteq P\}; P) \equiv \{R \equiv P\}$
Proof. Follows from predicate completion. □

2. $\mathsf{Circ}(\{C_1 \sqsubseteq \exists P\}; P) \equiv \{C_1 \equiv \exists P, \mathsf{Funct}(P)\}$
Proof. Let $\mathcal{I}$ be a model of $\mathsf{Circ}(\{C_1 \sqsubseteq \exists P\}; P)$. Then $\mathcal{I} \models C_1 \sqsubseteq \exists P$ and $\mathcal{I}$ is minimal relative to P. Assume that $\mathcal{I} \not\models \exists P \sqsubseteq C_1$, hence there exists a tuple $(a, b) \in P^{\mathcal{I}}$ s.t. $a \notin C_1{}^{\mathcal{I}}$. Then $\mathcal{I}$ can be

[2] Note that such transformation might be exponential.

improved by removing (a, b) from $P^{\mathcal{I}}$: let $\mathcal{I}' = \mathcal{I} \setminus P(a, b)$. Then $\mathcal{I}' \models C_1 \sqsubseteq \exists P$ and $\mathcal{I}' <^P \mathcal{I}$, which contradicts with $\mathcal{I}$ being a model of $\mathsf{Circ}(\{C_1 \sqsubseteq \exists P\}; P)$. Hence, $\mathcal{I} \models \exists P \sqsubseteq C_1$. Now, assume $\mathcal{I} \not\models \mathsf{Funct}(P)$, that is, there exist two tuples $(a, b) \in P^{\mathcal{I}}$, $(a, b') \in P^{\mathcal{I}}$, $b \neq b'$. Again, $\mathcal{I}$ can be improved by removing one of these tuples from $P^{\mathcal{I}}$, which contradicts with $\mathcal{I}$ being a model of $\mathsf{Circ}(\{C_1 \sqsubseteq \exists P\}; P)$. Thus, $\mathcal{I} \models \mathsf{Funct}(P)$.

Let $\mathcal{I}$ be a model of $\{C_1 \equiv \exists P, \mathsf{Funct}(P)\}$. Then it is a model of $C_1 \sqsubseteq \exists P$. Let us show it is minimal relative to P: no tuple can be removed from $P^{\mathcal{I}}$ without violating the axiom $C_1 \sqsubseteq \exists P$. By contradiction, assume that $(a, b) \in P^{\mathcal{I}}$ can be removed while still satisfying the axiom $C_1 \sqsubseteq \exists P$. Then, there must exist another tuple $(a, b') \in P^{\mathcal{I}}$ such that $b \neq b'$, which contradicts that $\mathcal{I} \models \mathsf{Funct}(P)$. Hence, $\mathcal{I}$ is minimal relative to P. □

3. $\mathsf{Circ}(\{C_2 \sqsubseteq \exists P^-\}; P) \equiv \{C_2 \equiv \exists P^-, \mathsf{Funct}(P^-)\}$
Proof. Similar to 1. □

Conversely, if we combine case 1 and case 2, circumscription does not entail equivalences for the domain and the range of P:

$$\mathsf{Circ}(\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}; P) \not\models C_1 \equiv \exists P$$
$$\mathsf{Circ}(\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}; P) \not\models C_2 \equiv \exists P^-$$

as an interpretation $\mathcal{I}$ that for each element $c_1 \in {C_1}^{\mathcal{I}}$ contains a tuple $(c_1, f(c_1)) \in P^{\mathcal{I}}$ and for each element $c_2 \in {C_2}^{\mathcal{I}}$ contains a tuple $(f(c_2), c_2) \in P^{\mathcal{I}}$, where f is a bijection and $P^{\mathcal{I}}$ contains nothing else, is a model of $\mathsf{Circ}(\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}; P)$.

However, we can entail a weaker statement. Below we actually compute circumscription of P in the TBox $\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}$.

Proposition 3. *Let P be a role name, C_1, C_2 arbitrary DL concepts (not necessarily DL-Lite$^{\mathcal{H}}_{core}$) such that $P \notin \Sigma(\{C_1, C_2\})$.*
Then $\mathsf{Circ}(\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}; P)$ is equivalent to the following TBox Π:

$$\begin{array}{ll} C_1 \sqsubseteq \exists P & \exists P.\neg C_2 \sqsubseteq C_1 \;\; (DRC) \\ C_2 \sqsubseteq \exists P^- & \geq 2\, P.\neg C_2 \sqsubseteq \bot \;\; (F1a) \\ & \geq 2\, P^-.\neg C_1 \sqsubseteq \bot \;\; (F1b) \end{array}$$
$$\begin{array}{rl} \exists P.C_2 \sqcap \exists P.\neg C_2 & \sqsubseteq \bot \;\; (F2a) \\ \exists P^-.C_1 \sqcap \exists P^-.\neg C_1 & \sqsubseteq \bot \;\; (F2b) \\ \geq 2\, P \sqcap \exists P.(\geq 2\, P^-) & \sqsubseteq \bot \;\; (NZa) \end{array}$$

Before proving the above result, we provide an intuitive explanation of the axioms in Π (cf. Figure 1). Axioms (DRC), (F1a-b), (F2a-b), and (NZa) encode minimality of P. Intuitively, axiom (DRC) closes the domain and the range of P by saying that P cannot connect an object lying outside C_1 with an object lying outside C_2. Axiom (F1a) asserts local functionality of P: an object cannot have two successors that are not in C_2. Axiom (F1b) says the same about the inverse P^- and C_1. Axioms (F2a) and (F2b) can also be seen as a sort of functionality restrictions: axiom (F2a) states that if an object has a P-successor in C_2, then it cannot have a second P-successor not in C_2; axiom (F2b) states the same about P^- and C_1. Finally, axiom (NZa) assures that P does not form a zigzag: it says that there cannot exist an object that has at least two P successors, and one of its successors has at least two P-predecessors.

Interpretations forbidden by axioms (DRC), (F1a), (F2a), and (NZa) are depicted in Figure 1. Dots denote objects, edges denote P connections and ovals denote the extensions of classes C_1 and C_2.

Proof. (⇒) Let $\mathcal{I}$ be a model of $\mathsf{Circ}(\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}; P)$. It means that $\mathcal{I}$ is a model of $\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}$ and it is minimal relative to P. We show that $\mathcal{I}$ is a model of Π, i.e., satisfies axioms (DRC), (F1a-b), (F2a-b), and (NZa).

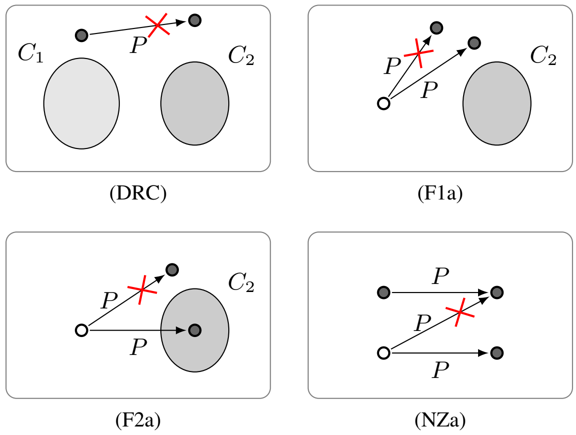

Figure 1. Interpretations forbidden by axioms (DRC), (F1a), (F2a) and (NZa). White objects denote elements whose existence is ruled out by the axioms. Crossed out edges can be deleted to improve the interpretations.

First, assume by contradiction that $\mathcal{I}$ does not satisfy axiom (DRC): $\mathcal{I} \not\models \exists P.\neg C_2 \sqsubseteq C_1$. Then, there should exist a tuple $(a, b) \in P^{\mathcal{I}}$ such that $b \in (\neg C_2)^{\mathcal{I}}$ and $a \notin (C_1)^{\mathcal{I}}$. Hence, $b \notin {C_2}^{\mathcal{I}}$ and $\mathcal{I}$ can be improved: $\mathcal{I}' = \mathcal{I} \setminus P(a, b)$ is a model of $\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}$ and $\mathcal{I}' <^P \mathcal{I}$. Contradiction with $\mathcal{I}$ being minimal relative to P.

Next, assume that $\mathcal{I}$ does not satisfy axiom (F1a), i.e., $\mathcal{I} \not\models {\geq 2\, P.\neg C_2} \sqsubseteq \bot$. That means there exist elements a, b, and b' such that $b \neq b'$, $(a, b) \in P^{\mathcal{I}}$, $(a, b') \in P^{\mathcal{I}}$, and $b \in (\neg C_2)^{\mathcal{I}}$, $b' \in (\neg C_2)^{\mathcal{I}}$. Again, we can improve $\mathcal{I}$ by removing one of the tuples (a, b) or (a, b') from $P^{\mathcal{I}}$, which contradicts that $\mathcal{I}$ is minimal relative to P. Hence, $\mathcal{I}$ is a model of axiom (F1a). It can be shown similarly that $\mathcal{I}$ is a model of axiom (F1b).

Now, we prove that $\mathcal{I}$ satisfies axiom (F2a), i.e., $\mathcal{I} \models \exists P.C_2 \sqcap \exists P.\neg C_2 \sqsubseteq \bot$. Assume the contrary, i.e., for some elements a, b, and b', $(a, b) \in P^{\mathcal{I}}$, $(a, b') \in P^{\mathcal{I}}$, $b \in {C_2}^{\mathcal{I}}$, and $b' \notin {C_2}^{\mathcal{I}}$. Then it is easy to see that $\mathcal{I}$ is not minimal relative to P: $\mathcal{I}' = \mathcal{I} \setminus P(a, b')$ is a model of $\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}$ and $\mathcal{I}' <^P \mathcal{I}$. Contradiction, therefore $\mathcal{I}$ is a model of axiom (F2a). Satisfaction of axiom (F2b) can be proved analogously.

Finally, we show that $\mathcal{I}$ satisfies axiom (NZa), that is $\mathcal{I} \models {\geq 2\, P} \sqcap \exists P.(\geq 2\, P^-) \sqsubseteq \bot$. Assume the contrary, that is for some elements a, a', b, and b', $b \neq b'$, $(a, b) \in P^{\mathcal{I}}$, $(a, b') \in P^{\mathcal{I}}$ ($a \in (\geq 2\, P)^{\mathcal{I}}$), and $a \neq a'$, $(a', b) \in P^{\mathcal{I}}$ ($b \in (\geq 2\, P^-)^{\mathcal{I}}$). Obviously, $\mathcal{I}$ is not minimal relative to P: $\mathcal{I}' = \mathcal{I} \setminus P(a, b)$ is a model of $\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}$ and $\mathcal{I}' <^P \mathcal{I}$. Contradiction with $\mathcal{I}$ being minimal relative to P. Therefore, axiom (NZa) is satisfied by $\mathcal{I}$.

(⇐) Let $\mathcal{I}$ be a model of Π. Then $\mathcal{I}$ is a model of $\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}$. Hence, to prove that $\mathcal{I}$ is a model of $\mathsf{Circ}(\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}; P)$ it remains to show that it is minimal relative to P.

By contradiction, assume that $\mathcal{I}$ is not minimal, that is, there exists a tuple $(a, b) \in P^{\mathcal{I}}$ such that the interpretation $\mathcal{I}' = \mathcal{I} \setminus P(a, b)$ is a model of $\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^-\}$ and $\mathcal{I}' <^P \mathcal{I}$. There are four cases:

1. $a \notin {C_1}^{\mathcal{I}}, b \notin {C_2}^{\mathcal{I}}$. Contradiction with axiom (DRC), $\exists P.\neg C_2 \sqsubseteq C_1$.
2. $a \notin {C_1}^{\mathcal{I}}, b \in {C_2}^{\mathcal{I}}$. By the assumption that (a, b) can be removed from the interpretation of P while satisfying $C_2 \sqsubseteq \exists P^-$, there

must exist a tuple $(a', b) \in P^{\mathcal{I}}$ with $a' \neq a$. Now, if $a' \notin C_1{}^{\mathcal{I}}$, then it contradicts $\mathcal{I} \models \geq 2\, P^{-}.\neg C_1 \sqsubseteq \bot$, and if $a' \in C_1{}^{\mathcal{I}}$, it contradicts $\mathcal{I} \models \exists P^{-}.C_1 \sqcap \exists P^{-}.\neg C_1 \sqsubseteq \bot$.

3. $a \in C_1{}^{\mathcal{I}}$, $b \notin C_2{}^{\mathcal{I}}$. Symmetric to the previous case.
4. $a \in C_1{}^{\mathcal{I}}$, $b \in C_2{}^{\mathcal{I}}$. By the assumption, (a, b) can be removed from $P^{\mathcal{I}}$. To satisfy $C_1 \sqsubseteq \exists P$, $C_2 \sqsubseteq \exists P^{-}$, there must exist two tuples $(a', b) \in P^{\mathcal{I}}$ and $(a, b') \in P^{\mathcal{I}}$ with $a \neq a'$ and $b \neq b'$. Then $a \in (\geq 2\, P)^{\mathcal{I}}$ and $b \in (\geq 2\, P^{-})^{\mathcal{I}}$. Contradiction with $\mathcal{I} \models \geq 2\, P \sqcap \exists P.(\geq 2\, P^{-}) \sqsubseteq \bot$.

In every case we derive a contradiction. Therefore, $\mathcal{I}$ is minimal, and hence, is a model of $\mathsf{Circ}(\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^{-}\}; P)$. ☐

Notice that the resulting TBox Π is no longer a *DL-Lite*$^{\mathcal{H}}_{core}$ TBox. The minimal language required is that of $\mathcal{ALCIQ}$.

Let us denote by $\mathsf{min}_{core}(P, C_1, C_2)$ the set formed by axioms (DRC), (F1a-b), (F2a-b), and (NZa) as a function of role P and concepts C_1 and C_2. Now, we can add to the TBox a role inclusion $R \sqsubseteq P$ and compute circumscription of P in a similar fashion. To address the additional role inclusion we make sure that the part of P disjoint from R is minimal. Note also that R does not have to satisfy axioms (DRC), (F1a-b), (F2a-b), and (NZa).

Proposition 4. *Let P be a role name, C_1, C_2 arbitrary DL concepts (not necessarily DL-Lite$^{\mathcal{H}}_{core}$) and R an arbitrary DL role such that $P \notin \Sigma(\{C_1, C_2, R\})$.*

Then $\mathsf{Circ}(\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^{-}, R \sqsubseteq P\}; P)$ is equivalent to the following TBox Π:

$$\begin{array}{ll} C_1 \sqsubseteq \exists P & \mathsf{min}_{core}(P', C_1 \sqcap \neg\exists R, C_2 \sqcap \neg\exists R^{-}) \\ C_2 \sqsubseteq \exists P^{-} & P \equiv P' \sqcup R \end{array}$$

where P' is a fresh role name and the Boolean constructors on roles are defined similarly to the Boolean constructors on concepts.

Note that though the axiom $P' \sqsubseteq \neg R$ is not explicitly asserted in Π, it is implied by Π. So, in fact it is not necessary to use a new name P' and we can replace each occurrence of P' by $P \sqcap \neg R$. Note also that in this case Π is an $\mathcal{ALCHIQ}$ plus union of roles TBox.

For the general case, it remains to consider inclusions of the form $\exists P^{-} \sqsubseteq \exists P$, $\exists P \sqsubseteq \exists P^{-}$, and $P^{-} \sqsubseteq P$. Interestingly, the former two inclusions act as inclusions positive w.r.t. P, i.e., inclusions where P occurs positively as $\exists P$, $\exists P^{-}$, P, or P^{-} (recall the normal form of concept inclusion axioms), whereas the latter inclusion acts as an inclusion negative w.r.t. P, i.e., inclusions where P occurs negatively as $\neg\exists P$, $\neg\exists P^{-}$, $\neg P$, or $\neg P^{-}$. Therefore, for a *DL-Lite*$^{\mathcal{H}}_{core}$ TBox $\mathcal{T}$, define $\mathsf{Pos}^{*}_{\mathcal{T}}(P)$ to be the set of all inclusions implied by $\mathcal{T}$ and positive w.r.t. P, or inclusions in $\mathcal{T}$ of the form $\exists P^{-} \sqsubseteq \exists P$, $\exists P \sqsubseteq \exists P^{-}$ if $\mathcal{T} \not\models P^{-} \sqsubseteq P$, and $\mathsf{Neg}^{*}_{\mathcal{T}}(P)$ to be the set of inclusions in $\mathcal{T}$ negative w.r.t. P, or inclusion $P^{-} \sqsubseteq P$ if $\mathcal{T} \models P^{-} \sqsubseteq P$. Finally, circumscription of an atomic role P in an arbitrary *DL-Lite*$^{\mathcal{H}}_{core}$ TBox can be computed as follows.

Theorem 5. *Let $\mathcal{T}$ be a DL-Lite$^{\mathcal{H}}_{core}$ TBox and P an atomic role. Further, let $\mathsf{Pos}^{*}_{\mathcal{T}}(P)$ be the set of the form*

$$\{R_i \sqsubseteq P\}_{i=0}^{m} \cup \{B_i \sqsubseteq \exists P\}_{i=0}^{n} \cup \{B'_i \sqsubseteq \exists P^{-}\}_{i=0}^{l},$$

(without loss of generality we can assume that P^{-} does not appear on the right-hand side of role inclusions in $\mathsf{Pos}^{*}_{\mathcal{T}}(P)$ and it does not contain inclusions of the form $X \sqsubseteq X$, where X is the domain or the range of P, or P itself). Then $\mathsf{Circ}(\mathcal{T}; P)$ can be computed as the union of $\mathcal{T}$ and the TBox Π:

$$\begin{array}{l} C_1 \equiv (B_1 \sqcup \cdots \sqcup B_n) \sqcap \neg(\exists R_1 \sqcup \cdots \sqcup \exists R_m) \\ C_2 \equiv (B'_1 \sqcup \cdots \sqcup B'_l) \sqcap \neg(\exists R_1^{-} \sqcup \cdots \sqcup \exists R_m^{-}) \\ \mathsf{min}_{core}(P', C_1, C_2) \\ P \equiv P' \sqcup R_1 \sqcup \cdots \sqcup R_m \end{array}$$

with P' a fresh atomic role, and C_1 and C_2 fresh atomic concepts. Note that here the empty union of concepts is equivalent to the bottom concept $\bot$.

Proof. By the properties of circumscription it holds that $\mathsf{Circ}(\mathcal{T}; P) = \mathsf{Circ}(\mathcal{T}_P; P) \wedge \mathcal{T}'$, where $\mathcal{T}'$ is the set of inclusions in $\mathcal{T}^{*}$ that do not contain P and $\mathcal{T}_P = \mathcal{T} \setminus \mathcal{T}'$.

Let $\mathcal{T}^{*}_{P}$ be the deductive closure of $\mathcal{T}_P$. Clearly, $\mathsf{Circ}(\mathcal{T}_P; P) \equiv \mathsf{Circ}(\mathcal{T}^{*}_{P}; P)$. Next, $\mathcal{T}^{*}_{P}$ can be partitioned in the following way:

$$\mathcal{T}^{*}_{P} = \mathsf{Pos}^{*}_{\mathcal{T}_P}(P) \cup \mathsf{Neg}^{*}_{\mathcal{T}_P}(P),$$

and similarly to Propositions 3 and 4 it can be shown that $\mathsf{Neg}^{*}_{\mathcal{T}_P}(P) \cup \Pi$ is equivalent to $\mathsf{Circ}(\mathcal{T}^{*}_{P}; P)$. It follows that $\mathcal{T} \cup \Pi$ is equivalent to $\mathsf{Circ}(\mathcal{T}; P)$. ☐

4.2 Circumscribing a *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox

In *DL-Lite*$^{\mathcal{H}}_{bool}$ inclusions positive w.r.t. a role P have the form:

$$\begin{array}{ll} R \sqsubseteq P, & C \sqsubseteq \exists P, \\ C \sqsubseteq \exists P \sqcup \exists P^{-}, & C \sqsubseteq \exists P^{-}, \end{array}$$

where R is a basic role and C is a complex concept.

In order to be able to compute circumscription of a *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox it remains to address positive occurrences of P in inclusions of the form $C \sqsubseteq \exists P \sqcup \exists P^{-}$. It turns out that circumscription of P in the TBox $\{C \sqsubseteq \exists P \sqcup \exists P^{-}\}$ is very similar to that in the TBox $\{C_1 \sqsubseteq \exists P, C_2 \sqsubseteq \exists P^{-}\}$ (see Proposition 3), with the difference that variations of axioms (F1a-b), (F2a-b), and (NZa) need to be added. More precisely, it is equivalent to the TBox:

$$\begin{array}{lrcll} & \exists P.\neg C_2 & \sqcap\ \exists P^{-}.\neg C_1 & \sqsubseteq \bot & \textit{(F1c)} \\ & \exists P.C_2 & \sqcap\ \exists P^{-}.\neg C_1 & \sqsubseteq \bot & \textit{(F2c)} \\ C \sqsubseteq \exists P \sqcup \exists P^{-} & \exists P.\neg C_2 & \sqcap\ \exists P^{-}.C_1 & \sqsubseteq \bot & \textit{(F2d)} \\ \mathsf{min}_{core}(P, C, C) & \exists P^{-} & \sqcap\ \exists P.(\geq 2\, P^{-}) & \sqsubseteq \bot & \textit{(NZb)} \\ & \geq 2\, P & \sqcap\ \exists P.(\exists P) & \sqsubseteq \bot & \textit{(NZc)} \\ & \exists P^{-} & \sqcap\ \exists P.(\exists P) & \sqsubseteq \bot & \textit{(NZd)} \end{array}$$

where C_1 and C_2 denote C. Let us denote by $\mathsf{min}_{bool}(P, C_1, C_2)$ the set formed by axioms (F1c), (F2c-d), and (NZb-d) as a function of role P and concepts C_1 and C_2. It will become clear later why we need to distinguish between C_1 and C_2 here.

Interpretations forbidden by the new axioms (F1c), (F2c), and (NZb-d) are depicted in Figure 2.

Now, when circumscribing an arbitrary *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox, some of these new axioms, e.g. (NZd), can contradict other TBox axioms, such as $\exists P^{-} \sqsubseteq \exists P$, therefore we cannot simply augment the theory with the new axioms to compute circumscription of a *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox. To this purpose, we first transform the given TBox into an equivalent TBox, and then provide an algorithm to compute circumscription in the new TBox. This transformation exploits the fact that the following two TBoxes are equivalent to each other: $\{C \sqsubseteq \exists P \sqcup \exists P^{-}, \exists P^{-} \sqsubseteq \exists P\}$ and $\{C \sqsubseteq \exists P, \exists P^{-} \sqsubseteq \exists P\}$. More precisely, for a *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox $\mathcal{T}$ and a role P, denote by $\mathcal{T}^{P,\sqcup}$ the TBox equivalent to $\mathcal{T}$ constructed as follows: if $\mathcal{T}$ implies

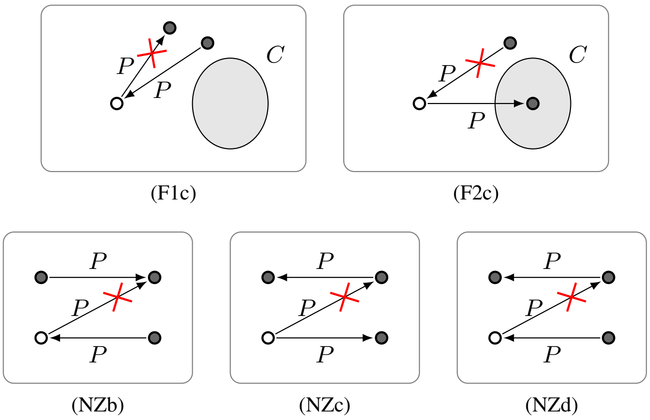

Figure 2. Interpretations forbidden by axioms (F1c), (F2c), and (NZb-d). White objects and crossed out edges are as in Figure 1.

$\exists P^- \sqsubseteq \exists P$ then replace axioms of the form $C \sqsubseteq \exists P \sqcup \exists P^-$ with $C \sqsubseteq \exists P$, and if $\mathcal{T}$ implies $\exists P \sqsubseteq \exists P^-$ then replace axioms of the form $C \sqsubseteq \exists P \sqcup \exists P^-$ with $C \sqsubseteq \exists P^-$. Next, define $\mathsf{Pos}^*_{\mathcal{T}}(P)$ to be the set of all *DL-Lite*$^{\mathcal{H}}_{core}$ inclusions implied by $\mathcal{T}$ and positive w.r.t. P, or inclusions in $\mathcal{T}$ of the form $C \sqsubseteq \exists P \sqcup \exists P^-$, or inclusions in $\mathcal{T}$ of the form $\exists P^- \sqsubseteq \exists P$, $\exists P \sqsubseteq \exists P^-$ if $\mathcal{T} \not\models P^- \sqsubseteq P$, and define $\mathsf{Neg}^*_{\mathcal{T}}(P)$ to be the set of inclusions in $\mathcal{T}$ negative w.r.t. P, or inclusion $P^- \sqsubseteq P$ if $\mathcal{T} \models P^- \sqsubseteq P$.

In the following theorem we compute circumscription of an atomic role in a *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox.

Theorem 6. *Let $\mathcal{T}$ be a DL-Lite$^{\mathcal{H}}_{bool}$ TBox, P an atomic role, and $\mathcal{T}^{P,\sqcup}$ the transformation of $\mathcal{T}$ defined as above. Further, let $\mathsf{Pos}^*_{\mathcal{T}^{P,\sqcup}}(P)$ be the set of the form*

$$\begin{array}{lll} \{R_i \sqsubseteq P\}_{i=0}^m & \cup & \{C_i^* \sqsubseteq \exists P \sqcup \exists P^-\}_{i=0}^k \quad \cup \\ \{C_i \sqsubseteq \exists P\}_{i=0}^n & \cup & \{C_i' \sqsubseteq \exists P^-\}_{i=0}^l \end{array}$$

Then $\mathsf{Circ}(\mathcal{T}; P)$ can be computed as the union of $\mathcal{T}$ and the following TBox Π:

$$\begin{array}{lll} D_1 & \equiv & (C_1 \sqcup \cdots \sqcup C_n) \sqcap \neg(\exists R_1 \sqcup \cdots \sqcup \exists R_m) \\ D_2 & \equiv & (C_1' \sqcup \cdots \sqcup C_l') \sqcap \neg(\exists R_1^- \sqcup \cdots \sqcup \exists R_m^-) \\ D & \equiv & (C_1^* \sqcup \cdots \sqcup C_k^*) \sqcap \neg(\exists R_1 \sqcup \cdots \sqcup \exists R_m) \sqcap \\ & & \quad \neg(D_1 \sqcup D_2) \sqcap \neg(\exists R_1^- \sqcup \cdots \sqcup \exists R_m^-) \\ P' & \equiv & P \sqcap \neg(R_1 \sqcup \cdots \sqcup R_m) \\ \multicolumn{3}{l}{\mathsf{min}_{core}(P', D_1 \sqcup D, D_2 \sqcup D)} \\ \multicolumn{3}{l}{\mathsf{min}_{bool}(P', D_1 \sqcup D, D_2 \sqcup D) \sqcap D} \end{array}$$

where P' is a fresh atomic role, D_1, D_2, and D are fresh atomic concepts, and $\mathsf{min}_{bool}(P', D_1 \sqcup D, D_2 \sqcup D) \sqcap D$ denotes the set of axioms of the form $D \sqcap C_l \sqsubseteq C_r$ for each axiom $C_l \sqsubseteq C_r$ in $\mathsf{min}_{bool}(P', D_1 \sqcup D, D_2 \sqcup D)$.

4.3 Adding an ABox

To fully address the problem of computing circumscription w.r.t. a single predicate in *DL-Lite*$^{\mathcal{H}}_{bool}$, it remains to add an ABox to the theory.

First, we show how to compute circumscription of a role or a concept in an ABox.

Proposition 7. *Let $\mathcal{A}$ be a DL-Lite$^{\mathcal{H}}_{bool}$ ABox. Then circumscription of a predicate X in $\mathcal{A}$ is equivalent to the KB $\langle \mathcal{T}_{\bar{X}}, \mathcal{A} \rangle$, where*

- *if X is an atomic concept A and $\{a_1, \ldots, a_n\} = \{a \mid A(a) \in \mathcal{A}\}$, then $\mathcal{T}_{\bar{A}} = \{A \sqsubseteq \{a_1, \ldots, a_n\}\}$.*
- *if X is an atomic role P, for individuals a and b, k_a denotes the number of P-successors of a in $\mathcal{A}$, k_b denotes the number of P-predecessors of b in $\mathcal{A}$, $\{a_1, \ldots, a_n\} = \{a \mid \mathcal{A} \models \exists P(a)\}$ and $\{b_1, \ldots, b_m\} = \{b \mid \mathcal{A} \models \exists P^-(b)\}$, then $\mathcal{T}_{\bar{P}}$ is the following TBox:*

$$\begin{array}{c} \{\{a\} \sqsubseteq\, \leq k_a\, P \mid \mathcal{A} \models \exists P(a)\} \cup \\ \{\{b\} \sqsubseteq\, \leq k_b\, P^- \mid \mathcal{A} \models \exists P^-(b)\} \cup \\ \{\exists P \sqsubseteq \{a_1, \cdots, a_n\}, \exists P^- \sqsubseteq \{b_1, \cdots, b_m\}\} \end{array}$$

Intuitively, the TBox $\mathcal{T}_{\bar{X}}$ encodes the closure of the predicate X. It does so by using nominals and number restrictions for the case of a role name.

Finally, we are ready to compute circumscription in a *DL-Lite*$^{\mathcal{H}}_{bool}$ KB.

Theorem 8. *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be a DL-Lite$^{\mathcal{H}}_{bool}$ KB and X a concept or role name. Let $\mathcal{A}'$ be the ABox obtained from $\mathcal{A}$ by renaming each occurrence of X to a fresh predicate X', $\mathcal{A}' = \mathcal{A}[X/X']$, and $\mathcal{T}' = \mathcal{T} \cup \{X' \sqsubseteq X\}$.*

Then $\mathsf{Circ}(\langle \mathcal{T}, \mathcal{A} \rangle; X)$ is equivalent to $\langle \mathsf{Circ}(\mathcal{T}'; X) \cup \mathcal{T}_{\bar{X}'}, \mathcal{A}' \rangle$.

5 Checking Entailment in Circumscribed *DL-Lite*$^{\mathcal{H}}_{bool}$

In the previous section we showed that for a *DL-Lite*$^{\mathcal{H}}_{bool}$ KB $\mathcal{K}$ and a role P, $\mathsf{Circ}(\mathcal{K}; P)$ is not a *DL-Lite*$^{\mathcal{H}}_{bool}$ KB anymore. It requires the language of $\mathcal{ALCHOIQ}$ with union of roles. Reasoning in $\mathcal{ALCHOIQ}$ extended with Boolean constructors on roles can be reduced to reasoning in $\mathcal{SHOIQB}_s$, which is an extension of $\mathcal{SHOIQ}$ with arbitrary Boolean constructors on simple roles and has been shown to be NEXPTIME-complete in [29].

On the other hand, if we only want to check concept or role subsumption in a circumscribed *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox $\mathcal{T}$, then the check can be done by encoding the problem in $\mathcal{ALCQIb}_{reg}$, which has been shown to be EXPTIME-complete (see [9]). However, in most of the cases, the complexity of checking whether $\mathsf{Circ}(\mathcal{T}; P) \models X_1 \sqsubseteq X_2$ for *DL-Lite*$^{\mathcal{H}}_{bool}$ concepts or roles X_1, X_2 is in NP, i.e., is does not exceed the complexity of *DL-Lite*$^{\mathcal{H}}_{bool}$:

a) if $P \notin \Sigma(X_1, X_2)$, $\mathsf{Circ}(\mathcal{T}; P) \models X_1 \sqsubseteq X_2$ iff $\mathcal{T} \models X_1 \sqsubseteq X_2$,
b) if $P \in \Sigma(X_2)$, $\mathsf{Circ}(\mathcal{T}; P) \models X_1 \sqsubseteq X_2$ iff $\mathcal{T} \models X_1 \sqsubseteq X_2$,
c) if $P \in \Sigma(X_1)$

1) if $\mathcal{T}$ does not contain inclusions of the form $C_1 \sqsubseteq \exists P$, $C_2 \sqsubseteq \exists P^-$, and $C \sqsubseteq \exists P \sqcup \exists P^-$, then $\mathsf{Circ}(\mathcal{T}; P)$ is a *DL-Lite*$^{\mathcal{H}}_{bool}$ with union of roles KB and the entailment can be checked using, e.g., the algorithm for $\mathcal{ALCQIb}_{reg}$ (see [9]),
2) if $\mathcal{T}$ contains inclusions of the form $C_1 \sqsubseteq \exists P$ but not $C_2 \sqsubseteq \exists P^-$ and $C \sqsubseteq \exists P \sqcup \exists P^-$, then
   - $\mathsf{Circ}(\mathcal{T}; P) \models X_1 \sqsubseteq X_2$ iff $\mathcal{T} \models X_1 \sqsubseteq X_2$ if $X_1 = \exists P^-$ or $X_1 = P$, and
   - $\mathsf{Circ}(\mathcal{T}; P) \models \exists P \sqsubseteq X_2$ iff $\mathcal{T} \models \exists P \sqsubseteq X_2$ or $\mathcal{T} \models D \sqsubseteq X_2$, where $D = \bigsqcup_{i=1}^n D_i$, $\mathcal{T} \models D_i \sqsubseteq \exists P$ and n is the maximal such number.
3) if $\mathcal{T}$ contains inclusions of the form $C_2 \sqsubseteq \exists P^-$ but not $C_1 \sqsubseteq \exists P$ and $C \sqsubseteq \exists P \sqcup \exists P^-$, then this is symmetric to the previous case.

4) if $\mathcal{T}$ contains both inclusions of the form $C_1 \sqsubseteq \exists P$ and $C_2 \sqsubseteq \exists P^-$, or $C \sqsubseteq \exists P \sqcup \exists P^-$, then $\mathsf{Circ}(\mathcal{T}; P) \models X_1 \sqsubseteq X_2$ iff $\mathcal{T} \models X_1 \sqsubseteq X_2$.

For an atomic concept A, $\mathsf{Circ}(\mathcal{K}; A)$ is a *DL-Lite*$^{\mathcal{H}}_{bool}$ KB and the entailment check can be done in NP.

In most of the cases the complexity of checking entailment does not exceed that of *DL-Lite*$^{\mathcal{H}}_{bool}$ (i.e., in NP). As for the case c)-1), the complexity of checking entailment in $\mathcal{ALCQIb}_{reg}$ is EXPTIME. The exact complexity of *DL-Lite*$^{\mathcal{H}}_{bool}$ with union of roles is unknown and lies between NP and EXPTIME.

6 Conclusions

We have studied circumscribed *DL-Lite* and addressed the problem of computing circumscription in *DL-Lite*$^{\mathcal{H}}_{bool}$ KBs. We computed circumscription of a single predicate (a concept or a role) in a *DL-Lite*$^{\mathcal{H}}_{bool}$ KB, which turned out to be first-order expressible. We showed that circumscription of a concept in a *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox is a *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox, whereas circumscription of a role a *DL-Lite*$^{\mathcal{H}}_{bool}$ TBox is an $\mathcal{ALCHIQ}$ plus union of roles TBox. Moreover adding an ABox to the circumscribed theory requires nominals in the language. We also showed that checking entailment of concept or role inclusions in a circumscribed KB can be done in EXPTIME.

To fully address the problem of circumscribing *DL-Lite*$^{\mathcal{H}}_{bool}$, we need to consider multiple minimized predicates and varying predicates. It is quite straightforward to compute prioritized circumscription of a set of concepts with strict priority as follows: first, circumscribe the concept with the highest priority; then, circumscribe the concept with the second priority in the result of the first circumscription; and continue by analogy. Conversely, parallel circumscription and varied predicates require more investigation.

Another interesting point is to study the exact complexity of checking entailment in *DL-Lite*$^{\mathcal{H}}_{bool}$ with Boolean constructors on roles. In the existing literature on complex role constructors only expressive DLs starting from $\mathcal{ALC}$ are considered. Therefore, analysis of the exact complexity of a low complexity logic such as *DL-Lite*$^{\mathcal{H}}_{bool}$ combined with Boolean constructors on roles could result in a better bound than EXPTIME.

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