This is a closed book exam: the only resources allowed are blank paper, pens, and your head. Explain your reasoning. Write clearly, in the sense of logic, language and legibility. The clarity of your explanations affects your grade. Write your name and ID on every solution sheet. Good luck!

Problem 1.1 [6 points] You must give a brief explanation of your answer to receive full credit.
(a) Decide whether the following statement is TRUE or FALSE: The complement of a recursive-enumerable language is recursively-enumerable.
(b) Let $M_2$ be a 2-tape (deterministic) TM, and let $M_1$ be the result of converting $M_2$ into a (1-tape deterministic) TM. Sketch using two to three sentences the main idea underlying the conversion. How are the running times of $M_1$ and $M_2$ related to each other?
(c) Decide whether the following statement is TRUE or FALSE: For all languages $L_1, L_2$, and $L_3$, if there exist a reduction from $L_1$ to $L_3$ and a reduction from $L_2$ to $L_3$, then there exists a reduction from $L_1$ to $L_2$.

Problem 1.2 [6 points] Construct a TM $M$ that accepts the language $L = \{n\#w \mid n$ is a number represented in binary with the least significant digit on the right, and $w \in \{a, b, c\}^*$ with $|w|_a + |w|_b = n\}$, where $|w|_x$ denotes the number of occurrences of $x$ in $w$.
E.g.: $10\#accbc \in L$, $0\# \not\in L$, $10\#ccac \not\in L$.

Show the sequence of IDs of $M$ on the input strings ”10#acbc” and ”10#cb”.

Problem 1.3 [6 points] The extraction $L_1 \ominus L_2$ of two languages $L_1$ and $L_2$ is defined as:
$L_1 \ominus L_2 = \{vw \mid vw_2w \in L_1$, for some $w_2 \in L_2\}$

Show that the class of recursively enumerable languages is closed under the extraction operation, i.e., that if $L_1$ and $L_2$ are recursively enumerable, then so is $L_1 \ominus L_2$.

[Hint: Show how to construct, from two (deterministic) TMs $M_1$ accepting $L_1$ and $M_2$ accepting $L_2$, a (possibly multi-tape) non-deterministic TM $N$ accepting $L_1 \ominus L_2$. You need not detail completely the construction of $N$, but you should provide sufficient information to make clear that you would be able to detail the construction, if you were given enough time to do so.]

Problem 1.4 [6 points]
(a) Let $f$ and $g$ be primitive recursive functions. Show that the following predicate $p$ is primitive recursive:
$p(x) = \begin{cases} 1 & \text{if } f(i) > g(j), \text{ for all } 1 \leq i \leq x \text{ and } 1 \leq j \leq x \\ 0 & \text{otherwise} \end{cases}$

(b) Show that the following function $f$ is primitive recursive:
$f(x) = \begin{cases} 2 & \text{if } x = 0 \text{ or } x = 1 \\ 3 \cdot f(x-1) \cdot f(x-2) & \text{if } x \geq 2 \end{cases}$

Problem 1.5 [6 points]
(a) Let $f$ be a total number-theoretic function with $n + 1$ variables. Provide the definition of the $(n + 1)$-variable function $g_n f$ such that $g_n f(x, i)$ encodes the values of $f(x, i)$ for $1 \leq i \leq y$.

(b) Let $g$ and $h$ be total number-theoretic functions, respectively with $n$ and $n + 2$ variables. Define the $(n + 1)$-variable function $f$ obtained from $g$ and $h$ by course-of-values recursion.
1.1 e) FALSE. Consider, e.g., $L_2$
   b) $M_1$ has 4 triple, 2 for the 2 tapes, 2 with a marker for the 2 head positions. On each move of $M_2$, one scan back and forth of $M_0$
   $M_1$ has quadratic running time in the running time of $M_2$
   c) FALSE, e.g., $L_1$, a RE language
      $L_2$, a RE language
      $L_3$, a non RE language
   \[
   \begin{align*}
   L_1 &< L_3 \\
   L_2 &< L_3 \\
   \text{but } L_1 &\not< L_2
   \end{align*}
   \]

1.2.

1.3. $N$ is a 3-steps NTM working as follows, when given an input string $w$ on tape 1:
   1) Guess a prefix $v$ of $w$ and copy it to tape 3
   2) Guess an arbitrary string $w_2$ on tape 2
   3) Copy $w_2$ to tape 3 immediately after $v$
   4) Run $M_2$ on $w_2$ on tape 2
      If $M_2$ accepts, then proceed.
      If $M_2$ rejects or loops, then this non-deterministic run of $N$ will also reject or loop.
   5) Copy the remaining part $w$ of $w$ from tape 1 to tape 3, immediately after $w_2$. Tape 3 now contains $v w_2 w$.
   6) Run $M_1$ on $v w_2 w$, and accept if $M_1$ accepts.
      Otherwise, this non-deterministic run of $N$ will reject or loop.
1.4 a) \( \eta(x) = \prod_{i=0}^{x-1} \eta_i(f(x), g(x)) \)

Since \( f, g, \eta \) are PRFs,
- The composition of PRFs is a PRF.
- The bounded product of a PRF is a PRF.

We get that also \( \eta \) is a PRF.

b) We define an auxiliary function \( h(x) = \eta_{m_f}(f(x), f(x+1)) \)

\[ h(0) = \eta_{m_f}(f(0), f(1)) = \eta_{m_f}(2, 2) = 2^3 \cdot 3^3 = 8 \cdot 27 = 216 \]
\[ h(x+1) = \eta_{m_f}(f(x+1), f(x+2)) = \]
\[ = \eta_{m_f}(f(x+1), 3 \cdot f(x+1), f(x)) = \]
\[ = \eta_{m_f}\left( \text{dec}(1, h(x)), 3 \cdot \text{dec}(1, h(x)) = \text{dec}(0, h(x)) \right) \]

Since \( \eta_{m_f} \) and \( \text{dec} \) are PRFs, this is in a definition.

By PR.
\( f(x) = \text{dec}(0, h(x)) \)

Hence \( f \) is a PRF.

1.5 a) \( \phi_y(x, y) = \prod_{i=0}^{y} \phi_i(x, y) \cdot 1 \)

b) \[
\begin{cases}
    f(x, 0) = \phi(x) \\
    f(x, y+1) = h(x, y, \eta_{m_f}(x, y))
\end{cases}
\]
Exercise 1: Consider a TM $M_0 = (Q_0, \Sigma, \Gamma_0, \delta_0, q_0, \emptyset, F_0)$.

Show that $L(M)$ is also accepted by a TM $M_1$ that never moves left of its initial position (i.e., it uses a TM with a semi-infinite tape).

Idea. $M_1$ is a two track TM: $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, \emptyset, F_1)$

Let us call $q_0$ the initial tape position of $M_0$

The states of $M_1$ are all the states of $M_0$, with an additional component $P$ on $\mathbb{N}$, indicating whether $M_1$ is currently working on the track representing the positive or negative position of the tape of $M_0$: $Q_1 = (Q_0 \times \{P, N\}) \cup \{q_1, q_1'\}$

- $\Gamma_1$ is the set of pairs of symbols of $\Gamma_0$, plus symbols with $\star$ on $\mathbb{T}_0$

$$\Gamma_1 = \Gamma_0 \times (\Gamma_0 \cup \{\star\})$$

The $\star$ on $\mathbb{T}_1$ is used to detect when $M_1$ reaches the leftmost tape position.

Initially, $\delta_1$ writes $\star$ on $\mathbb{T}_1$ of the leftmost position

$$\delta(q_1, [\star], L) = (q_1, [\star], L) \quad \forall x \in \Sigma$$

- For the transitions of $M_1$, we need to distinguish 4 cases:

1) $M_0$ is to the right of $q_0 \Rightarrow M_1$ works on track $\mathbb{T}_1$

2) $M_0$ is on $q_0 \Rightarrow M_1$ is on $[\star]$

3) $M_0$ is to the left of $q_0 \Rightarrow M_1$ works on track $\mathbb{T}_0$
Let \( \delta_0(q, x) = (q', y, d) \) be a transition of \( M_0 \).

Then we have

1. \( \delta_1([q, P], [\overline{x}]) = ([q', P], [\overline{y}], d) \) for every \( z \in \Gamma_0 \) (i.e. \( z \neq x \))

2. \( \delta_1([q, N], [\overline{x}] ) = ([q', N], [\overline{y}], \overline{d}) \) for every \( z \in \Gamma_0 \)
   where \( \overline{d} = L \) if \( d = R \)
   \( \overline{d} = R \) if \( d = L \)

3. if \( M_0 \) moves right, i.e. \( d = R \)
   \( \delta_1([q, -], [\overline{x}]) = ([q', P], [\overline{y}], R) \)

   if \( M_0 \) moves left, i.e. \( d = L \)
   \( \delta_1([q, -], [\overline{x}]) = ([q', N], [\overline{y}], R) \)

- Final states of \( M_1 \): \( F_1 = F_0 \times \{P, N\} \)
Exercise 2: Construct a TM that computes the length of its input string, represented as a binary number (with the least significant digit on the right). Assume \( \Sigma = \{A, B\} \).

Idea: we write a counter to the left of the input separated by a $.

We repeatedly move to the right of the input, delete the last symbol, come back and increment the counter.
Exercise 3: For a TM $M$ with input alphabet $Σ$, let $<M, w>$ denote the encoding $E(M)$ of $M$ followed by input $w$.

Consider the language $L = \{ <M, w> | M$ when started on an input string $w$, eventually does three consecutive transitions in which it moves the head in the same direction $\}$.

a) Show that $L$ is recursively enumerable.

b) Show that $L$ is not recursive.

e) We reduce $L$ to $L_n$.

The reduction $R$ is a TM that takes as input $<M, w>$ and produces as output $R(<M, w>) = <M', w>$ such that $<M, w> \in L \iff <M', w> \in L_n$.

We describe how $R$ has to transform $E(M)$ to obtain $E(M')$:

- $R$ has to add to the states of $M$ a second component that counts how many consecutive transitions $M$ has made in the same direction:

  - the values of the counter component are $-3, -2, -1, 1, 2, 3$
  - the transitions of $M$ are modified to update the counter:

    - if $δ(q, x) = (q', y, d)$ is a transition of $M$, then $M'$ contains transitions $\delta'(q, c, x) = (q, c', y, d)$ for each $c ∈ \{-3, -2, -1, 1, 2\}$, with $c'$ given by:

      $\begin{array}{c|cccccc}
        & -3 & -2 & -1 & 0 & 1 & 2 \\
        \hline
        L & -3 & -2 & -1 & 1 & 1 & 1 \\
        R & 1 & 1 & 1 & 1 & 2 & 3 \\
      \end{array}$

- the final states of $M'$ are exactly those where the counter is $3$, i.e. $F' = Q × \{3\}$.
6) We reduce the halting problem $L_H$ to $L$.

The reduction $R$ is a TM that takes as input $<M, w>$ and produces as output $R(<M, w>) = <M', w>$ such that $<M, w> \in L_H$ iff $<M', w> \in L$.

We describe how $R$ has to transform $E(M)$ to obtain $E(M')$:

- The final states of $M$ are made non-final in $M'$.
- From a final or blocking state of $M$ we add to $M'$ a transition to a new state from which $M'$ makes 3 transitions to the right.
- We have to make sure that $M'$ never does 3 consecutive transitions in the same direction (except the ones above).

Hence:

If $M$ does an $R$-move, then
$M'$ does an $R$-$L$-$R$ move.

If $M$ does an $L$-move, then
$M'$ does an $L$-$R$-$L$ move.

- The tape symbol is changed only in the first of the three moves, while the other two leave the tape unchanged.
- For the dummy moves, additional states are needed, and these need to be distinct for each state of $M$.

Formally:

If $\delta(q, x) = (q', y, R)$, then
$\delta'(q, x) = (q'_R, y, R)$
$\delta'(q'_R, -) = (q''_R, -, L)$
$\delta'(q''_R, -) = (q', - R)$

If $\delta(q, x) = (q', y, L)$, then
$\delta'(q, x) = (q'_L, y, L)$
$\delta'(q'_L, -) = (q''_L, - R)$
$\delta'(q''_L, -) = (q', - L)$
Exercise 4: Let \( g(x) \) be a PRF.

1) Show that the following predicate is a PRF:

\[
(f(x)) = \begin{cases} 
1 & \text{if } g(i) < g(x) \text{ for all } 0 \leq i \leq y \\
0 & \text{otherwise}
\end{cases}
\]

\[
f(x,y) = \sum_{i=0}^{y} g(i)
\]

2) Let \( f \) be defined by

\[
f(k) = \begin{cases} 
2 & \text{if } k = 0 \\
3 & \text{if } k = 1 \\
(k-3) + f(k-1) & \text{if } k \geq 3
\end{cases}
\]

Give the values \( f(4), f(5), f(6) \).

\[
f(3) = f(0) + f(2) = 1 + 3 = 4
\]
\[
f(4) = f(1) + f(3) = 2 + 4 = 6
\]
\[
f(5) = f(2) + f(4) = 3 + 6 = 9
\]
\[
f(6) = f(3) + f(5) = 4 + 9 = 13
\]

Show that \( f \) is a PRF.

We have that \( f(y+1) = f(y-2) + f(y) \).

We introduce an auxiliary function \( h \) with

\[
h(y) = [f(y), f(y+1), f(y+2)] = \text{qm}_3(f(y), f(y+1), f(y+2))
\]

\[
h(0) = \text{qm}_3(f(0), f(1), f(2)) = \text{qm}_3(1, 2, 3) = 2^2 \cdot 3^3 \cdot 5^4
\]

\[
h(y+1) = [f(y+1), f(y+2), f(y+3)] =
\]

\[
= [f(y+1), f(y+2), f(y) + f(y+2)] =
\]

\[
= [\text{dec}(1, h(y)), \text{dec}(2, h(y)), \text{dec}(0, h(y)) + \text{dec}(2, h(y))]
\]

\[
= \text{qm}_3(\ldots)
\]

Hence \( h \) is PR. Then \( f(y) = \text{dec}(0, h(y)) \) is also PR.