Problem 1.1 [6 points] You must give a brief explanation of your answer to receive full credit.

(a) Decide whether the following statement is TRUE or FALSE: The complement of a recursive-enumerable language is recursively-enumerable.

(b) Let $M_2$ be a 2-tape (deterministic) TM, and let $M_1$ be the result of converting $M_2$ into a (1-tape deterministic) TM. Sketch using two to three sentences the main idea underlying the conversion. How are the running times of $M_1$ and $M_2$ related to each other?

(c) Decide whether the following statement is TRUE or FALSE: For all languages $L_1$, $L_2$, and $L_3$, if there exist a reduction from $L_1$ to $L_3$ and a reduction from $L_2$ to $L_3$, then there exists a reduction from $L_1$ to $L_2$.

Problem 1.2 [6 points] Construct a TM $M$ that accepts the language $L = \{n\#w \mid n \text{ is a number represented in binary with the least significant digit on the right}, \text{and } w \in \{a, b, c\}^* \text{ with } |w|_a + |w|_b = n\}$, where $|w|_x$ denotes the number of occurrences of $x$ in $w$.

E.g.: $10\#accbc \in L$, $0\# \in L$, $10\#accbc \notin L$, $10\#ccac \notin L$.

Show the sequence of IDs of $M$ on the input strings “10#accbc” and “10#ccac”.

Problem 1.3 [6 points] The extraction $L_1 \ominus L_2$ of two languages $L_1$ and $L_2$ is defined as:

$$L_1 \ominus L_2 = \{vw \mid vw_2w \in L_1, \text{ for some } w_2 \in L_2\}$$

Show that the class of recursively enumerable languages is closed under the extraction operation, i.e., that if $L_1$ and $L_2$ are recursively enumerable, then so is $L_1 \ominus L_2$.

[Hint: Show how to construct, from two (deterministic) TMs $M_1$ accepting $L_1$ and $M_2$ accepting $L_2$, a (possibly multi-tape) non-deterministic TM $N$ accepting $L_1 \ominus L_2$. You need not detail completely the construction of $N$, but you should provide sufficient information to make clear that you would be able to detail the construction, if you were given enough time to do so.]

Problem 1.4 [6 points]

(a) Let $f$ and $g$ be primitive recursive functions. Show that the following predicate $p$ is primitive recursive:

$$p(x) = \begin{cases} 1 & \text{if } f(i) > g(j), \text{ for all } 1 \leq i \leq x \text{ and } 1 \leq j \leq x \\ 0 & \text{otherwise} \end{cases}$$

(b) Show that the following function $f$ is primitive recursive:

$$f(x) = \begin{cases} 2 & \text{if } x = 0 \text{ or } x = 1 \\ 3 \cdot f(x-1) \cdot f(x-2) & \text{if } x \geq 2 \end{cases}$$

Problem 1.5 [6 points]

(a) Let $f$ be a total number-theoretic function with $n + 1$ variables. Provide the definition of the $(n + 1)$-variable function $g_n$ such that $g_n(x, i)$ encodes the values of $f(x, i)$ for $1 \leq i \leq y$.

(b) Let $g$ and $h$ be total number-theoretic functions, respectively with $n$ and $n + 2$ variables. Define the $(n + 1)$-variable function $f$ obtained from $g$ and $h$ by course-of-values recursion.
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1.1 e) FALSE. Consider e.g. L_n

b) M_1 has 4 tapes, 2 for the 2 tapes, 2 with a marker for the 2 head positions. For each move of M_2, one can back and forth of M_1.
M_1 has quadratic running time in the running time of M_2.

c) FALSE: e.g. L_1 a RE language
L_2 a REC language
L_3 a non-RE language
\{ L_1 \subset L_3 \}
\{ L_2 \subset L_3 \}
buts L_1 \neq L_2

1.2.

1.3. N is a 3-tape N TM working as follows, when given an input string x on tape 1:

1) guess a prefix w of x and copy it to tape 3
2) guess an arbitrary string w_2 on tape 2
3) copy w_2 to tape 3 immediately after w
4) run M_2 on w_2 on tape 2

If M_2 accepts, then proceed.
If M_2 rejects on loops, then this non-deterministic run of N will also reject or loop.

5) copy the remaining part w of x from tape 1 to tape 3, immediately after w_2. Tape 3 now contains vv_2 w.
6) run M_1 on vv_2 w, and accept if M_1 accepts.
Otherwise, this non-deterministic run of N will reject or loop.
1.4 a) \( \varphi(x) = \prod_{i=1}^{n} f^{*i}(f(i), f(j)) \)

Since \( f, g, q \) are PRFs

the composition of PRFs is a PRF

the bounded product of a PRF is a PRF

we get that also \( \varphi \) is a PRF

b) We define an emulation function \( h(x) = q_{m, q}(f(x), f(x+1)) \)

\[
\begin{align*}
h(0) &= q_{m, q}(f(0), f(1)) = q_{2}(2, 2) = 2^3 \cdot 3^3 = 8 \cdot 27 = 216 \\
h(x+1) &= q_{m, q}(f(x+1), f(x+2)) = q_{m, q}(f(x+1), 3 \cdot f(x+1), f(x)) = q_{m, q}(\text{dec}(1, h(x)), 3 \cdot \text{dec}(1, h(x)) = \text{dec}(0, h(x)))
\end{align*}
\]

Since \( q_{m, q} \) and \( \text{dec} \) are PRFs, this is a definition of \( h \) by PR.

\( f(x) = \text{dec}(0, h(x)) \)

Hence \( f \) is a PRF

1.5 a) \( \psi_f(R, y) = \prod_{i=0}^{n} q^{m(i)}(R, i) \) + 1

b) \[
\begin{align*}
f(R, 0) &= f(R) \\
f(R, 1) &= h(R, y, q_{m, q}(R, y))
\end{align*}
\]
Exercise 1: Consider a TM $M_0 = (Q_0, \Sigma, \Gamma_0, \delta_0, q_0, \delta_0', F_0)$.

Show that $L(M)$ is also accepted by a TM $M_1$ that never moves left of its initial position (i.e., by a TM with a semi-infinite tape).

Idea: $M_0$ is a two track TM: $M_0 = (Q_0, \Sigma, \Gamma_1, \delta_1, q_0, \delta_1', F_0)$

Let us call $p_0$ the initial tape position of $M_0$

- The states of $M_1$ are all the states of $M_0$, with an additional component $P$ or $N$, indicating whether $M_0$ is currently working on the track representing the positive or negative portion of the tape of $M_0$: $Q_1 = Q_0 \times \{P, N\}$

- $\Gamma_1$ is the set of pairs of symbols of $\Gamma_0$, plus symbols with $*$ on $T_0$:
  \[ \Gamma_1 = \Gamma_0 \times (\Gamma_0 \cup \{\ast\}) \]

- The $\ast$ on $T_1$ is used to detect when $M_1$ reaches the leftmost tape position.

- Initially, $P_1$ writes $\ast$ on $T_1$ of the leftmost position (for this it actually needs two additional states).

- For the transitions of $M_1$, we need to distinguish 4 cases:
  1) $M_0$ is to the right of $p_0 \Rightarrow M_1$ works on track $T_1$.
  2) $M_0$ is to the left of $p_0 \Rightarrow M_1$ works on track $T_0$.
  3) $M_0$ is on $p_0 \Rightarrow M_1$ is on $[\ast]$.
Let \( \delta_c(q, x) = (q', y, d) \) be a transition of \( M_0 \).

Then we have

1) \( \delta_n([q, P], [\Sigma]) = ([q', P], [\Sigma], d) \) for every \( z \in F \) (i.e. \( z \neq \epsilon \))

2) \( \delta_n([q, N], [\Sigma]) = ([q', N], [\Sigma], d) \) for every \( z \in F \)

where \( \overline{d} = L \) if \( d = R \)

3) if \( M_0 \) moves right, i.e. \( d = R \)

\( \delta_n([q, -], [\epsilon]) = ([q', P], [\epsilon], R) \)

if \( M_0 \) moves left, i.e. \( d = L \)

\( \delta_n([q, -], [\epsilon]) = ([q', N], [\epsilon], R) \)

Final states of \( M_1 \): \( F_1 = F_0 \times \{ P, N \} \)
Exercise 2: Construct a TM that computes the length of its input string, represented as a binary number (with the least significant digit on the right). Assume \( \Sigma = \{0, 1\} \).

Idea: We write a counter to the left of the input separated by a $\$$. We repeatedly move to the right of the input, delete the last symbol, come back and increment the counter.
Exercise 3: For a TM $M$ with input alphabet $\Sigma$, let $\langle M, w \rangle$ denote the encoding $E(M)$ of $M$ followed by input $w$.

Consider the language $L = \{ \langle M, w \rangle \mid M \text{ when started on an input string } w, \text{ eventually does three consecutive transitions in which it moves the head in the same direction} \}$.

a) Show that $L$ is recursively enumerable.

b) Show that $L$ is not recursive.

We reduce $L$ to $L_\mu$.

The reduction $R$ is a TM that takes as input $\langle M, w \rangle$ and produces as output $R(\langle M, w \rangle) = \langle M', w \rangle$ such that $\langle M, w \rangle \in L \iff \langle M', w \rangle \in L_\mu$.

We describe how $R$ has to transform $E(M)$ to obtain $E(M')$:

- $R$ has to add to the states of $M$ a second component that counts how many consecutive transitions $M$ has made in the same direction:

  The values of the counter component are $-3, -2, -1, 0, 1, 2, 3$

- The transitions of $M$ are modified to update the counter:

  If $M$ moves right, then in $M'$:
  
  \[
  \begin{align*}
  C = -2 &\rightarrow C = -1 \\
  C = -1 &\rightarrow C = 0 \\
  C = 0 &\rightarrow C = 1 \\
  C = 1 &\rightarrow C = 2 \\
  C = 2 &\rightarrow C = 3
  \end{align*}
  \]

  If $M$ moves left, then in $M'$:
  
  \[
  \begin{align*}
  C = -2 &\rightarrow C = -3 \\
  C = -1 &\rightarrow C = -2 \\
  C = 0 &\rightarrow C = -1 \\
  C = 1 &\rightarrow C = 0
  \end{align*}
  \]

- The states with the counter $3$ or $-3$ are the only final states.
b) We reduce the halting problem \( \mathcal{L}_H \) to \( \mathcal{L} \).

The reduction \( \mathcal{R} \) is a TM that takes as input \( \langle M, w \rangle \) and produces as output \( \mathcal{R}(\langle M, w \rangle) = \langle M', w \rangle \) such that \( \langle M, w \rangle \in \mathcal{L}_H \) iff \( \langle M', w \rangle \in \mathcal{L} \).

We describe how \( \mathcal{R} \) has to transform \( \mathcal{E}(M) \) to obtain \( \mathcal{E}(M') \):

- the final states of \( M \) are made non-final in \( M' \);
- from a final or blocking state of \( M \) we add to \( M' \) a transition to a new state from which \( M' \) makes 3 transitions to the right;
- we have to make sure that \( M' \) never does 3 consecutive transitions in the same direction (except the ones above).

Hence:

if \( M \) does an R-move, then
   \( M' \) does an L-R-L move

if \( M \) does an L-move, then
   \( M' \) does an R-L-R move

- the tape symbol is changed only in the first of the three moves, while the other two leave the tape unchanged.
- for the dummy moves, additional states are needed, and these need to be distinct for each state of \( M \).
Exercise 4: Let \( f(x) \) be a PRF.

a) Show that the following predicate is a PRF:

\[
\begin{cases} 
 1 & \text{if } g(i) < g(x) \text{ for all } 0 \leq i \leq y \\
 0 & \text{otherwise}
\end{cases}
\]

Let \( f(x, y) = \sum_{i=0}^{y} \cdot \text{lgt}(g(i), g(x)) \)

b) Let \( f \) be defined by

\[
\begin{align*}
  f(k) &= \begin{cases} 
    1 & \text{if } k = 0 \\
    3 & \text{if } k = 1 \\
    2 & \text{if } k = 2 \\
    (k-3) + f(k-1) & \text{if } k \geq 3
  \end{cases}
\end{align*}
\]

Give the values \( f(4) \), \( f(5) \), \( f(6) \).

\[
\begin{align*}
  f(3) &= f(0) + f(2) = 1 + 3 = 4 \\
  f(3) &= f(1) + f(3) = 2 + 4 = 6 \\
  f(5) &= f(2) + f(4) = 3 + 6 = 9 \\
  f(6) &= f(3) + f(5) = 4 + 9 = 13
\end{align*}
\]

Show that \( f \) is a PRF.

We have that \( f(y+1) = f(y-2) + f(y) \).

We introduce an auxiliary function \( h \) with

\[
h(y) = [f(y), f(y+1), f(y+2), f(y+3)] = \text{gm}_2([f(y), f(y+1), f(y+2)])
\]

\[
h(0) = \text{gm}_2([f(0), f(1), f(2)]) = \text{gm}_2(1, 2, 3) = 2^2, 3, 5, 4
\]

\[
h(y+1) = [f(y+1), f(y+2), f(y+3)] =
\]

\[
= [f(y+1), f(y+2), f(y) + f(y+2)] =
\]

\[
= [\text{dec}(1, h(y)), \text{dec}(2, h(y)), \text{dec}(0, h(y)) + \text{dec}(2, h(y))]
\]

\[
= \text{gm}_2(\ldots)
\]

Hence \( h \) is PR. Then \( f(y) = \text{dec}(0, h(y)) \) is also PR.