Problem 1.1 [6 points] You must give a brief explanation of your answer to receive full credit.
(a) Decide whether the following statement is TRUE or FALSE: The complement of a recursive-enumerable language is recursively-enumerable.
(b) Let $M_2$ be a 2-tape (deterministic) TM, and let $M_1$ be the result of converting $M_2$ into a (1-tape deterministic) TM. Sketch using two to three sentences the main idea underlying the conversion. How are the running times of $M_1$ and $M_2$ related to each other?
(c) Decide whether the following statement is TRUE or FALSE: For all languages $L_1$, $L_2$, and $L_3$, if there exist a reduction from $L_1$ to $L_3$ and a reduction from $L_2$ to $L_3$, then there exists a reduction from $L_1$ to $L_2$.

Problem 1.2 [6 points] Construct a TM $M$ that accepts the language $L = \{n\#w \mid n$ is a number represented in binary with the least significant digit on the right, and $w \in \{a,b,c\}^* \}$ with $|w|_a + |w|_b = n\}$,
E.g.: $10\#acbc \in L$, $0\# \in L$, $10\#acbc \notin L$, $10\#ccac \notin L$.
Show the sequence of IDs of $M$ on the input strings "$10\#acbc$" and "$10\#cb$".

Problem 1.3 [6 points] The *extraction* $L_1 \oplus L_2$ of two languages $L_1$ and $L_2$ is defined as:
$L_1 \oplus L_2 = \{vw \mid vw_2w \in L_1$, for some $w_2 \in L_2\}$
Show that the class of recursively enumerable languages is closed under the extraction operation, i.e., that if $L_1$ and $L_2$ are recursively enumerable, then so is $L_1 \oplus L_2$.

Problem 1.4 [6 points]
(a) Let $f$ and $g$ be primitive recursive functions. Show that the following predicate $p$ is primitive recursive:
$p(x) = 1$ if $f(i) > g(j)$, for all $1 \leq i \leq x$ and $1 \leq j \leq x$, otherwise
(b) Show that the following function $f$ is primitive recursive:
$f(x) = \begin{cases} 2 & \text{if } x = 0 \text{ or } x = 1 \\ 3 \cdot f(x-1) \cdot f(x-2) & \text{if } x \geq 2 \end{cases}$

Problem 1.5 [6 points]
(a) Let $f$ be a total number-theoretic function with $n + 1$ variables. Provide the definition of the $(n + 1)$-variable function $g_n f$ such that $g_n f(\vec{x}, y)$ encodes the values of $f(\vec{x}, i)$ for $1 \leq i \leq y$.
(b) Let $g$ and $h$ be total number-theoretic functions, respectively with $n$ and $n + 2$ variables. Define the $(n + 1)$-variable function $f$ obtained from $g$ and $h$ by *course-of-values recursion*.
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9.1 c) FALSE. Consider, e.g. L_n
   b) M_1 has 4 heads, 2 for the 2 tapes, 2 with a marker for the 2 head positions. For each move of M_2, one move back and forth of M_1
   M_1 has quadratic running time in the running time of M_2
   c) FALSE: e.g., L_1, a RE language
      L_2 a REC language
      L_3 a non-REC language
      \{ L_1 < L_3 \}
      \{ L_2 < L_3 \}
      but L_1 \neq L_2

9.2

9.3. N is a 3-tape NTM working as follows, when given an input string x on tape 1:
   1) guess a prefix w of x and copy it to tape 3
   2) guess an arbitrary string w_2 on tape 2
   3) copy w_2 to tape 3 immediately after w
   4) run M_2 on w_2 on tape 2
      If M_2 accepts, then proceed.
      If M_2 rejects or loops, then this non-deterministic run of N will also reject or loop.

   5) copy the remaining part w of x from tape 1 to tape 3, immediately after w_2. Tape 3 now contains \( w w_2 w \).
   6) run M_1 on \( w w_2 w \) and accept if M_1 accepts.
      Otherwise, this non-deterministic run of N will reject or loop.
1.4 e) \( q(x) = \prod_{i=0}^{x} q_1 \left( f(i), g(i) \right) \)

Since \( f, g \) are PRFs,
the composition of PRFs is a PRF
the bounded product of a PRF is a PRF.
we get that also \( q \) is a PRF.

b) We define an equivalence function \( h(x) = q_{m_1}(f(x), f(x+1)) \)

\[
\begin{align*}
    h(0) &= q_{m_1}(f(0), f(1)) = q_{m_1}(2, 2) = 2^3 \cdot 3^3 = 8 \cdot 27 = 216 \\
    h(x+1) &= q_{m_1}(f(x+1), f(x+2)) = \\
        &= q_{m_1}(f(x+1), 3 \cdot f(x+1), f(x)) = \\
        &= q_{m_1}(\text{dec}(1, h(x)), 3 \cdot \text{dec}(1, h(x)) = \text{dec}(0, h(x)))
\end{align*}
\]

Since \( q_{m_1} \) and \( \text{dec} \) are PRFs, this is a definition of \( h \) by PR.

\( f(x) = \text{dec}(0, h(x)) \)

Hence \( f \) is a PRF.

1.5 a) \( f_{m_2}(x, y) = \prod_{i=0}^{x} q_{m}(i) \)

b) \[
\begin{align*}
    f(x, 0) &= q_1(x) \\
    f(x, y+1) &= h(x, y, q_1(x, y))
\end{align*}
\]
Exercise 1: Consider a TM $M_0 = (Q_0, \Sigma, \Gamma_0, \delta_0, q_0, \phi, F_0)$.

Show that $L(M)$ is also accepted by a TM $M_1$ that never moves left of its initial position (i.e., $L$ by a TM with a semi-infinite tape).

Idea: $M_0$ is a two-track TM: $M_0 = (Q_0, \Sigma, \Gamma_0, \delta_0, q_0, \phi, F_0)$

Let us call $q_0$ the initial tape position of $M_0$.

The states of $M_1$ are all the states of $M_0$, with an additional component $P \in \{P, N\}$, indicating whether $M_0$ is currently working on the track representing the positive or negative portion of the tape of $M_0$.

- $Q_1 = Q_0 \times \{P, N\}$
- $\Gamma_1 = \Gamma_0 \times (\Gamma_0 \cup \{\star\})$

The $\star$ on $\Gamma_1$ is used to detect when $M_1$ reaches the leftmost tape position.

Initially, $M_1$ writes $\star$ on $\Gamma_1$ of the leftmost position (for this it actually needs two additional states).

For the transitions of $M_1$, we need to distinguish 4 cases:

1) $M_0$ is to the right of $q_0 \Rightarrow M_1$ works on right track ($T_r$)
2) $M_0$ is to the left of $q_0 \Rightarrow M_1$ works on left track ($T_l$)
3) $M_0$ is on $q_0 \Rightarrow M_1$ is on $[\star]$
Let $\delta_0(q, x) = (q', y, d)$ be a transition of $M_0$.

Then we have:

1) $\delta_0([q, P], [\bar{x}]) = ([q', P], [\bar{y}], d)$ for every $2 \in \Gamma_0$ (i.e., $2 \neq x$)

2) $\delta_0([q, N], [\bar{x}]) = ([q', N], [\bar{y}], d)$ for every $2 \in \Gamma_0$

where $\bar{d} = L$ if $d = R$
    $\bar{d} = R$ if $d = L$

3) If $M_0$ moves right, i.e. $d = R$

   $\delta_0([q, -], [\bar{x}]) = ([q', P], [\bar{y}], R)$

   If $M_0$ moves left, i.e. $d = L$

   $\delta_0([q, -], [\bar{x}]) = ([q', N], [\bar{y}], R)$

- Final states of $M_1$: $F_1 = F_0 \times \{P, N\}$
Exercise 2. Construct a TM that computes the length of its input string, represented as a binary number (with the least significant digit on the right). Assume $\Sigma = \{0, 1\}$.

Idea: we write a counter to the left of the input separated by a $\$$. We repeatedly move to the right of the input, delete the last symbol, come back and increment the counter.
Exercise 3: For a TM $M$ with input alphabet $\Sigma$, let $\langle M, w \rangle$ denote the encoding $E(M)$ of $M$ followed by input $w$.

Consider the language $L = \{ \langle M, w \rangle \mid M$ when started on an input string $w$, eventually does three consecutive transitions in which it moves the head in the same direction $\}$.

a) Show that $L$ is recursively enumerable.
b) Show that $L$ is not recursive.

c) We reduce $L$ to $L_w$.

The reduction $R$ is a TM that takes as input $\langle M, w \rangle$ and produces as output $R(\langle M, w \rangle) = \langle M', w \rangle$ such that $\langle M, w \rangle \in L$ iff $\langle M', w \rangle \in L_w$.

We describe how $R$ has to transform $E(M)$ to obtain $E(M')$:

- $R$ has to add to the states of $M$ a second component that counts how many consecutive transitions $M$ has made in the same direction:

  The values of the counter component are $-3, -2, -1, 1, 2, 3$.

- The transitions of $M$ are modified to update the counter:

  If $M$ moves right:
  
  if $M$ moves right:
  
  then in $M'$:
  
  - $C = -2 \rightarrow C = -1$
  - $C = -1 \rightarrow C = 0$
  - $C = 0 \rightarrow C = 1$
  - $C = 1 \rightarrow C = 2$
  - $C = 2 \rightarrow C = 3$

  If $M$ moves left:
  
  then in $M'$:
  
  - $C = -2 \rightarrow C = -3$
  - $C = -3 \rightarrow C = -2$
  - $C = -2 \rightarrow C = -1$
  - $C = -1 \rightarrow C = 0$
  - $C = 0 \rightarrow C = 1$

- The states with the counter $3$ or $-3$ are the only final states.
b) We reduce the halting problem \( L_H \) to \( L \).

The reduction \( R \) is a TM that takes as input \( \langle M, w \rangle \) and produces as output \( R(\langle M, w \rangle) = \langle M', w \rangle \) such that \( \langle M, w \rangle \in L_H \iff \langle M', w \rangle \in L \).

We describe how \( R \) has to transform \( E(M) \) to obtain \( E(M') \):

- the final states of \( M \) are made non-final in \( M' \);
- from a final or blocking state of \( M \) we add to \( M' \) a transition to a new state from which \( M' \) makes 3 transitions to the right;
- we have to make sure that \( M' \) never does 3 consecutive transitions in the same direction (except the ones above).

Hence:

if \( M \) does an R-move, then
\( M' \) does an R-L-R move.

if \( M \) does an L-move, then
\( M' \) does an L-R-L move.

- the tape symbol is changed only in the first of the three moves, while the other two leave the tape unchanged.
- for the dummy moves, additional states are needed, and these need to be distinct for each state of \( M \).
Exercise 4: Let \( f(x) \) be a PRF.

(a) Show that the following predicate is a PRF:
\[
f(x,y) = \begin{cases} 1 & \text{if } g(i) < g(x) \text{ for all } 0 \leq i \leq y \\ 0 & \text{otherwise} \end{cases}
\]
\[
f(x,y) = \sum_{i=0}^{y} \text{lt}(g(i), g(x))
\]

(b) Let \( f \) be defined by
\[
f(x) = \begin{cases} 2 & \text{if } x = 0 \\ 3 & \text{if } x = 1 \\ 4 & \text{if } x = 2 \\ (f(x-3) + f(x-1)) & \text{if } x \geq 3 \\ \end{cases}
\]

Give the values \( f(4) \), \( f(5) \), \( f(6) \).

\[
\begin{align*}
 f(3) &= f(0) + f(2) = 1 + 3 = 4 \\
f(4) &= f(1) + f(3) = 2 + 4 = 6 \\
f(5) &= f(2) + f(4) = 3 + 6 = 9 \\
f(6) &= f(3) + f(5) = 4 + 9 = 13 \\
\end{align*}
\]

Show that \( f \) is a PRF.

We have that \( f(y+1) = f(y-2) + f(y) \).

We introduce an auxiliary function \( h \) with
\[
h(y) = [f(y), f(y+1), f(y+2)] = q_{m_2}(f(y), f(y+1), f(y+2)),
\]
\[
h(0) = q_{m_2}(f(0), f(1), f(2)) = q_{m_2}(1, 2, 3) = 2^2 \cdot 3^3 \cdot 5^4
\]
\[
h(y+1) = [f(y+1), f(y+2), f(y+3)] =
\[
= [f(y+1), f(y+2), f(y) + f(y+2)] =
\[
= [\text{dec}(1, h(y)), \text{dec}(2, h(y)), \text{dec}(0, h(y)) + \text{dec}(2, h(y))]
\]
\[
= q_{m_2}(...)
\]

Hence \( h \) is PR. Then \( f(y) = \text{dec}(0, h(y)) \) is also PR.