5. Reasoning in Description Logics

**Exercise 5.1** Let $T$ be a TBox consisting of concept inclusions of the form $A_1 \sqsubseteq A_2$ and concept disjointness assertion of the form $A_1 \sqsubseteq \neg A_2$, for atomic concepts $A_1$ and $A_2$.

Describe an algorithm for checking concept satisfiability with respect to $T$, i.e., whether for some concept $A$ it holds that $A$ is satisfiable with respect to $T$.

**Exercise 5.2** Consider TBoxes $T$ consisting of axioms of the form $B_1 \sqsubseteq B_2$, where $B_1, B_2 ::= A \mid \exists R \mid \exists R^-$, $A$ denotes an atomic concept, and $R$ an atomic role.

1. Describe an algorithm for checking subsumption with respect to a given $T$, i.e., whether for two concepts $B_1$ and $B_2$ it holds that $T \models B_1 \sqsubseteq B_2$.
2. Let $A = \{A_0(a)\}$ and $T$ a(n arbitrary) TBox of the above form. Can we determine whether $\langle T, A \rangle$ is satisfiable?

**Exercise 5.3** Show that concept satisfiability in $ALC$ is NP-hard.

Hint: show the claim by reduction from SAT.

**Exercise 5.4** Let $q_n$, for $n \geq 2$, be a Boolean conjunctive query with $n$ existential variables of the form $\exists x_1, \ldots, x_n. P(x_1, x_2) \land \cdots \land P(x_{n-1}, x_n)$. Given $n \geq 2$:

1. construct an $ALC$ KB $K_n$ such that $K_n \models q_n$.
2. construct an $ALC$ KB $K'_n$ of size polynomial in $n$ such that $K'_n \models q_n$ and $K'_n \not\models q_{2^n+1}$.

Hint: $K'_n$ “implements” a binary counter by means of $n$ atomic concepts representing the bits of the counter, and such that the models of $K'_n$ contain a $P$-chain of objects of length $2^n$.  