This is a closed book exam: the only resources allowed are blank paper, pens, and your head. Explain your reasoning. Write clearly, in the sense of logic, language and legibility. The clarity of your explanations affects your grade. Write your name and ID on every solution sheet. Good luck!

**Problem 1.1** [6 points] You must give a brief explanation of your answer to receive full credit.

(a) Decide whether the following statement is TRUE or FALSE: The complement of a recursive-enumerable language is recursively-enumerable.

(b) Let $M_2$ be a 2-tape (deterministic) TM, and let $M_1$ be the result of converting $M_2$ into a (1-tape deterministic) TM. Sketch using two to three sentences the main idea underlying the conversion. How are the running times of $M_1$ and $M_2$ related to each other?

(c) Decide whether the following statement is TRUE or FALSE: For all languages $L_1$, $L_2$, and $L_3$, if there exist a reduction from $L_1$ to $L_3$ and a reduction from $L_2$ to $L_3$, then there exists a reduction from $L_1$ to $L_2$.

**Problem 1.2** [6 points] Construct a TM $M$ that accepts the language $L = \{n\#w \mid n$ is a number represented in binary with the least significant digit on the right, and $w \in \{a, b, c\}^*$ with $|w|_a + |w|_b = n\}$, where $|w|_x$ denotes the number of occurrences of $x$ in $w$.

E.g.: $10\#acbc \in L$, $0\# \notin L$, $10\#acccbc \notin L$, $10\#ccac \notin L$.

Show the sequence of IDs of $M$ on the input strings "10#acbc" and "10#cb".

**Problem 1.3** [6 points] The extraction $L_1 \ominus L_2$ of two languages $L_1$ and $L_2$ is defined as:

$$ L_1 \ominus L_2 = \{vw \mid vw_2w \in L_1, \text{ for some } w_2 \in L_2 \} $$

Show that the class of recursively enumerable languages is closed under the extraction operation, i.e., that if $L_1$ and $L_2$ are recursively enumerable, then so is $L_1 \ominus L_2$.

**[Hint:]** Show how to construct, from two (deterministic) TMs $M_1$ accepting $L_1$ and $M_2$ accepting $L_2$, a (possibly multi-tape) non-deterministic TM $N$ accepting $L_1 \ominus L_2$. You need not detail completely the construction of $N$, but you should provide sufficient information to make clear that you would be able to detail the construction, if you were given enough time to do so.]

**Problem 1.4** [6 points]

(a) Let $f$ and $g$ be primitive recursive functions. Show that the following predicate $p$ is primitive recursive:

$$ p(x) = \begin{cases} 1 & \text{if } f(i) > g(j), \text{ for all } 1 \leq i \leq x \text{ and } 1 \leq j \leq x \\ 0 & \text{otherwise} \end{cases} $$

(b) Show that the following function $f$ is primitive recursive:

$$ f(x) = \begin{cases} 2 & \text{if } x = 0 \text{ or } x = 1 \\ 3 \cdot f(x-1) \cdot f(x-2) & \text{if } x \geq 2 \end{cases} $$

**Problem 1.5** [6 points]

(a) Let $f$ be a total number-theoretic function with $n + 1$ variables. Provide the definition of the $(n + 1)$-variable function $g_n$ such that $g_n(\vec{x}, i)$ encodes the values of $f(\vec{x}, i)$ for $1 \leq i \leq y$.

(b) Let $g$ and $h$ be total number-theoretic functions, respectively with $n$ and $n + 2$ variables. Define the $(n + 1)$-variable function $f$ obtained from $g$ and $h$ by course-of-values recursion.
Solutions to the TOC exam of 30/1/2003, Part 1

9.1c) FALSE. Consider, e.g., L_2

b) M_1 has 4 heads, 2 for the 2 tapes, 1 with a marker for
the 2 head positions. For each move of M_2, one seen back
and forth of M_0.
M_3 has quadratic running time in the running time of M_2.

c) FALSE: e.g., L_1 a RE language
L_2 a REC language
L_3 a non-RE language,
\{ L_1 \subseteq L_3 \}
L_2 \subseteq L_3
but L_1 \neq L_2

9.2

9.3. N is a 3-tape N TM working as follows, when given an input
string \( x \) on tape 1:

1) Guess a prefix \( w_1 \) of \( x \) and copy it to tape 3.
2) Guess an arbitrary string \( w_2 \) on tape 2.
3) Copy \( w_2 \) to tape 3 immediately after \( w_1 \).
4) Run \( M \) on \( w_2 \) on tape 2.
   - If \( M \) accepts, then proceed.
   - If \( M \) rejects or loops, then this non-deterministic run
     of \( N \) will also reject or loop.
5) Copy the remaining part \( w_3 \) of \( x \) from tape 1 to tape 3,
   immediately after \( w_2 \). Tape 3 now contains \( w_1 w_2 w_3 \).
6) Run \( M_1 \) on \( w_1 w_2 w_3 \), and accept if \( M_1 \) accepts.
   Otherwise, this non-deterministic run of \( N \) will
   reject or loop.
1.4 a) \( \eta(x) = \prod_{i=0}^{n} x \quad q_i \quad q_i(f(i), g(i)) \)

Since \( f, g \) are PRFs

the composition of PRFs is a PRF

the bounded product of a PRF is a PRF

we get that also \( \eta \) is a PRF

b) We define an equality function \( h(x) = \eta \eta_1(f(x), f(x+1)) \)

\[
\begin{align*}
    h(0) &= \eta \eta_1(f(0), f(1)) = \eta \eta_1(2, 2) = 2^3 \cdot 3^3 = 8 \cdot 27 = 216 \\
    h(x+1) &= \eta \eta_1(f(x+1), f(x+2)) = \\
    &= \eta \eta_1(f(x+1), 3 \cdot f(x+1), f(x)) = \\
    &= \eta \eta_1(\text{dec}(1, h(x)), 3 \cdot \text{dec}(1, h(x)) = \text{dec}(0, h(x))
\end{align*}
\]

Since \( \eta \eta_1 \) and \( \text{dec} \) are PRFs, this is a definition of \( h \) by PR.

\( f(x) = \text{dec}(0, h(x)) \)

Hence \( f \) is a PRF

1.5 a) \( \phi_\sigma(R, y) = \prod_{i=0}^{\sigma} \phi_i(R, y + 1) \)

b) \( f(\emptyset, 0) = \phi(\emptyset) \)

\( f(R, y + 1) = \eta(R, y, \phi_\sigma(R, y)) \)
Exercise 1: Consider a TM $M_0 = (Q_0, \Sigma, \Gamma, \delta, q_0, \phi, F_0)$.

Show that $L(M)$ is also accepted by a TM $M_1$ that never moves left of its initial position (i.e., by a TM with a semi-infinite tape).

Idea: $M_0$ is a two-track TM: $M_0 = (Q_0, \Sigma, \Gamma_1, \delta_1, q_0, \phi, F_0)$

Let us call $q_0$ the initial tape position of $M_0$.

The states of $M_1$ are all the states of $M_0$, with an additional component $P \in \{0, 1\}$, indicating whether $M_0$ is currently working on the tracks representing the positive or negative portion of the tape of $M_0$:

$$Q_1 = Q_0 \times \{P, N\}$$

$\Gamma_1$ is the set of pairs of symbols of $\Gamma$, plus symbols with * on $\Gamma_0$:

$$\Gamma_1 = \Gamma_0 \times (\Gamma_0 \cup \{*\})$$

The * on $\Gamma_1$ is used to detect when $M_1$ reaches the leftmost tape position.

Initially, $G_1$ writes * on $T_1$ of the leftmost position (for this it actually needs two additional states).

For the transitions of $M_1$, we need to distinguish 4 cases:

1) $M_0$ is to the right of $q_0 \Rightarrow M_1$ works on track $T_1$.
2) $M_0$ is to the left of $q_0 \Rightarrow M_1$ works on track $T_0$.
3) $M_0$ is on $q_0 \Rightarrow M_1$ is on $T_0$. 

...
Let \( \delta_0(q, x) = (q', y, d) \) be a transition of \( M_0 \).

Then we have

1) \( \delta_1([q, P], [\overline{z}]) = ([q', P], [\overline{y}], \overline{d}) \) for every \( z \in \Gamma_0 \) (i.e. \( z \neq x \))

2) \( \delta_1([q, N], [\overline{x}]) = ([q', N], [\overline{y}], \overline{d}) \) for every \( z \in \Gamma_0 \)

where \( \overline{d} = L \) if \( d = R \)
\( \overline{d} = R \) if \( d = L \)

3) if \( M_0 \) moves right, i.e. \( d = R \)
\( \delta_1([q, P], [\overline{x}]) = ([q', P], [\overline{y}], R) \)

if \( M_0 \) moves left, i.e. \( d = L \)
\( \delta_1([q, N], [\overline{x}]) = ([q', N], [\overline{y}], R) \)

Initial states of \( M_1 \): \( F_1 = F_0 \times \{ P, N \} \)
Exercise 2. Construct a TM that computes the length of its input string, represented as a binary number (with the least significant digit on the right). Assume \( \Sigma = \{0, 1\} \).

Idea: we write a counter to the left of the input separated by a $.

We repeatedly move to the right of the input, delete the last symbol, come back and increment the counter.
Exercise 3: For a TM \( M \) with input alphabet \( \Sigma \), let \( \langle M, w \rangle \) denote the encoding \( E(M) \) of \( M \) followed by input \( w \).

Consider the language \( L = \{ \langle M, w \rangle \mid \text{M when started on an input string } w, \text{eventually does three consecutive transitions in which it moves the head in the same direction} \} \).

a) Show that \( L \) is recursively enumerable.
b) Show that \( L \) is not recursive.

d) We reduce \( L \) to \( L_{\text{accept}} \).

The reduction \( R \) is a TM that takes as input \( \langle M, w \rangle \) and produces as output \( R(\langle M, w \rangle) = \langle M', w \rangle \) such that \( \langle M, w \rangle \in L \iff \langle M', w \rangle \in L_{\text{accept}} \).

We describe how \( R \) has to transform \( E(M) \) to obtain \( E(M') \):

- \( R \) has to add to the states of \( M \) a second component that counts how many consecutive transitions \( M \) has made in the same direction:

  the values of the counter component are: \(-3, -2, -1, 0, 1, 2, 3\)

- the transitions of \( M \) are modified to update the counter

  if \( M \) moves right:
  
  \[
  \begin{align*}
  c &\rightarrow c - 2 \\
  c &\rightarrow c - 1 \\
  c &\rightarrow c \\
  c &\rightarrow c + 1 \\
  c &\rightarrow c + 2 \\
  c &\rightarrow c + 3
  \end{align*}
  \]

  then in \( M' \):

  \[
  \begin{align*}
  c &\rightarrow c + 1 \\
  c &\rightarrow c + 1 \\
  c &\rightarrow c + 2 \\
  c &\rightarrow c + 3
  \end{align*}
  \]

  if \( M \) moves left:

  \[
  \begin{align*}
  c &\rightarrow c + 2 \\
  c &\rightarrow c + 1 \\
  c &\rightarrow c \\
  c &\rightarrow c - 1 \\
  c &\rightarrow c - 2 \\
  c &\rightarrow c - 3
  \end{align*}
  \]

  then in \( M' \):

  \[
  \begin{align*}
  c &\rightarrow c - 1 \\
  c &\rightarrow c - 1 \\
  c &\rightarrow c - 2 \\
  c &\rightarrow c - 3 \\
  c &\rightarrow c - 2 \\
  c &\rightarrow c - 2
  \end{align*}
  \]

- the states with the counter \( 3 \) or \(-3 \) are the only final states.
b) We reduce the halting problem $L_H$ to $L$.

The reduction $R$ is a TM that takes as input $<M, w>$ and produces as output $R(<M, w>) = <M', w>$ such that $<M, w> \in L_H$ iff $<M', w> \in L$.

We describe how $R$ has to transform $E(M)$ to obtain $E(M')$:

- the final states of $M$ are made non-final in $M'$
- from a final or blocking state of $M$ we add to $M'$ a transition to a new state from which $M'$ makes 3 transitions to the right
- we have to make sure that $M'$ never does 3 consecutive transitions in the same direction (except the ones above).

Hence:

- if $M$ does an $R$-move, then $M'$ does an $R-L-R$ more
- if $M$ does an $L$-move, then $M'$ does an $L-R-L$ more

- the tape symbol is changed only in the first of the three moves, while the other two leave the tape unchanged
- for the dummy moves, additional states are needed, and there need to be distinct for each state of $M$. 
Exercise 4: Let \( f(x) \) be a PRF.

a) Show that the following predicate is a PRF:

\[
\begin{cases} 
1 & \text{if } \frac{g(i)}{g(x)} < 1 \text{ for all } 0 \leq i \leq y \\
0 & \text{otherwise} 
\end{cases}
\]

\[
f(x,y) = \prod_{i=0}^{y} \left( \frac{g(i)}{g(x)} \right)
\]

b) Let \( f \) be defined by

\[
f(x) = \begin{cases} 
2 & \text{if } x = 0 \\
3 & \text{if } x = 1 \\
1 & \text{if } x = 2 \\
(k-3) + f(k-1) & \text{if } x \geq 3 
\end{cases}
\]

Give the values \( f(4) \), \( f(5) \), \( f(6) \).

\[
\begin{align*}
\{3\} &= \{0\} + \{2\} = 1 + 3 = 4 \\
\{4\} &= \{1\} + \{3\} = 2 + 4 = 6 \\
\{5\} &= \{2\} + \{4\} = 3 + 6 = 9 \\
\{6\} &= \{3\} + \{5\} = 4 + 3 = 13
\end{align*}
\]

Show that \( f \) is a PRF.

We have that \( f(y+1) = \{y\} + \{y\} \).

We introduce an auxiliary function \( h \) with

\[
h(y) = [f(y), \{y+1\}, f(y+2)] = \text{gm}_{3}(f(y), \{y\} + \{y\} + \{y\})
\]

\[
h(0) = \text{gm}_{3}(\{0\}, \{1\}, \{2\}) = \text{gm}_{3}(1,2,3) = 2^2 \cdot 3^3 \cdot 5^4
\]

\[
h(y+1) = [f(y+1), \{y+2\}, f(y+3)] =
\]

\[
= [f(y+1), \{y+2\}, f(y) + f(y+2)] =
\]

\[
= [\text{dec}(1, h(y)), \text{dec}(2, h(y)), \text{dec}(0, h(y)) + \text{dec}(2, h(y))] =
\]

\[
= \text{gm}_{3}(\ldots)
\]

Hence \( h \) is PR. Then \( f(y) = \text{dec}(0, h(y)) \) is also PR.