Exercise 1

Show that multiplication is a primitive recursive function.

Solution:

\[
\text{mult} \ (x, 0) = g(x) = 0 \\
\text{mult} \ (x, y + 1) = R(x, y, \text{mult} \ (x, y)) = \text{mult} \ (x, y) + x
\]

where \( g = 2 \) and \( R = \text{add} \circ (p_2^{(2)}, p_2^{(3)}) \)

Exercise 2

Let \( g(x, y) \) be a primitive recursive function. Then the following functions obtained from \( g \) are also PR.

a) \( f(x_1, y_1, z_1, \ldots, z_n) = g(x, y) \)

b) \( f(x, y) = g(y, x) \)

c) \( f(x) = g(x, x) \)

Solution:

a) \( f = g \circ (p_2^{(2)}, p_2^{(3)}) \)

b) \( f = g \circ (p_2^{(3)}, p_2^{(3)}) \)

c) \( f = g \circ (p_2^{(3)}, p_2^{(3)}) \)
Exercise 3

Let \( p(x, z) \) be a primitive recursive predicate. Show that the following functions are primitive recursive.

a) \( f_1(x, y_0, y) = \) the first value \( z \) in \([y_0, y]\) for which \( p(x, z) \) is true
b) \( f_2(x, y) = \) the second value \( z \) in \([0, y]\) for which \( p(x, z) \) is true
c) \( f_3(x, y) = \) the largest value \( z \) in \([0, y]\) for which \( p(x, z) \) is true

If there is no value \( z \) in the range such that \( p(x, z) \) is true, then \( f_3 \) is \( y + 1 \).

Solution:

a) \( f_1(x, y_0, y) = \mu z \in [y_0, y] [ p(x, z) \land \ \geq \ (z, y_0) ] \)

The PRF \( \geq \) ("greater or equal to") is used to enforce the lower bound; multiplication \( \cdot \) works as "boolean and".

b) \( f_2(x, y) = \mu z \in [0, y] [ p(x, z) \land \ gt \ (z, \mu z' \in [0, y] [ p(x, z') ] ) ] \)

The PRF \( \gt \) ("greater than") makes sure we skip the first value.

c) Let \( f'(x, y) = y \cdot \mu z \in [p(x, y) = z] \)

reverses the order of examination (i.e., we go from \( y \) down to \( 0 \))

Then:

\[ f_3(x, y) = \text{eq}(y + 1, \mu z \in [p(x, z)] \cdot (y + 1) + \text{neq}(y + 1, \mu z \in [p(x, z)] \cdot f'(x, y)) \]

It checks whether there is a \( z \) such that \( z \leq y \) and \( p(x, z) \) is true, and outputs \( f'(x, y) \) if it is the case and \( y + 1 \) otherwise.
Exercise 4

Consider integer division \( \text{div}(x, y) \): it's not defined for \( 0 \), hence not total and hence not PR. Let
\[
\text{quo}(x, y) = \begin{cases} 
0 & \text{if } y = 0 \\
\text{div}(x, y) & \text{otherwise}
\end{cases}
\]
a) Define \( \text{quo}(x, y) \) using bounded minimization.
b) Show that remainder, divides, number of divisors, and prime are primitive recursive.

Solution:

a) \( \text{quo}(x, y) = \text{sgn}(y) \cdot \mu z \leq x \left[ \text{gt}((z+1) \cdot y, x) \right] \)

b) Remainder:
\[
\text{rem}(x, y) = x \div (y \cdot \text{quo}(x, y))
\]
Divides:
\[
\text{divides}(x, y) = \begin{cases} 
1 & \text{if } x > 0, y > 0, \text{ and } y \text{ is a divisor of } x \\
0 & \text{otherwise}
\end{cases}
\]
\[
\text{divides}(x, y) = \text{eq}(\text{rem}(x, y), 0) \cdot \text{sgn}(y)
\]
Number of divisors:
\[
\text{ndivisors}(x, y) = \sum_{i=0}^{x} \text{divides}(x, y)
\]
Prime:
\[
\text{prime}(x) = \begin{cases} 
1 & \text{if } x \text{ is prime} \\
0 & \text{otherwise}
\end{cases}
\]
\[
\text{prime}(x) = \text{eq}(\text{ndivisors}(x), 2)
\]
Exercise 5

Show that the function \( p_n(i) \) computing the \( i \)-th prime is \( PR \) by exploiting the fact that \( p_n(x+1) \leq p_n(x)! + 1 \).

Solution:

\[
\begin{cases}
p_n(0) = 2 \\
p_n(x+1) = \mu z \leq (p_n(x)! + 1) \left[ \text{prime}(z) \cdot g(z, p_n(x)) \right]
\end{cases}
\]

Exercise 6

Show that the Ackermann function

\[
\begin{cases}
A(0, y) = y + 1 \\
A(x+1, 0) = A(x, 1) \\
A(x+1, y+1) = A(x, A(x+1, y))
\end{cases}
\]

is defined for every pair \( x, y \in \mathbb{N} \).

Solution:

By induction on \( x \) (main induction)

Base case: \( A(0, y) = y + 1 \)

Inductive step: By induction on \( y \) (secondary induction)

\( A(x+1, y) \)

Base case: \( A(x+1, 0) = A(x, 1) \) and the main induction hypothesis applies

Inductive step: By the secondary induction hypothesis \( A(x+1, y) \) is defined; then for \( A(x+1, y+1) = A(x, A(x+1, y)) \) the main induction hypothesis applies
Exercise 7

Define a primitive recursive function $f : \mathbb{N} \rightarrow \mathbb{N}$ that counts the number of occurrences of the digit 5 in a natural number.

Solution:

We need some auxiliary primitive recursive functions:

- exponential $m^n : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$

  \[
  \exp(m,n) = \begin{cases} 
  \exp(m,0) = 1 \\
  \exp(m,n+1) = \exp(m,n) \cdot m 
  \end{cases}
  \]

- length (number of digits) : $\mathbb{N} \rightarrow \mathbb{N}$

  \[\text{length}(n) = (\mu z \leq n [ \text{gt}(10^{z+1}, n) ] ) + 1\]

  examples: $\text{length}(0) = \text{length}(1) = \ldots = \text{length}(9) = 1$, $\text{length}(10) = 2$, ...

$f : \mathbb{N} \rightarrow \mathbb{N}$ is then defined as follows:

\[f(n) = \sum_{i=1}^{\text{length}(n)} \text{eq}(5, \text{rem}[(\text{quot}(n, 10^{i-1})), 10])\]

Example: $f(523) = \text{eq}(5, \text{rem}[(\text{quot}(523, 4)), 10])$

\[
= \frac{523}{2} + \text{eq}(5, \text{rem}[(\text{quot}(523, 10)), 10])
\]

\[
= \frac{523}{2} + \frac{5}{5} + \text{eq}(5, \text{rem}[(\text{quot}(523, 100)), 10])
\]

\[
= 0 + 1 + 0 = 1
\]
Exercise 8

Define a primitive recursive function \( f : \mathbb{N} \rightarrow \mathbb{N} \) that reverses the digits of a natural number, i.e. \( f(253) = 352 \), \( f(5524) = 4255 \).

Solution:

\[
f(n) = \sum_{i=1}^{\text{length}(n)} (\text{rem}(n, 10^{\text{length}(n)-i+1}) \times 10^{\text{length}(n)-i}) \times 10^{i-1}
\]

Example: \( f(5524) = \text{rem}(5524,10000) \cdot 1000 + \text{rem}(5524,1000) \cdot 100 + \text{rem}(5524,100) \cdot 10 + \text{rem}(5524,10) \cdot 1 \)

\[
= 5 + 50 + 200 + 4000 = 4255
\]