Exercise

Using the characterization of regular languages in terms of DFAs, show the following:

If $L_1$ and $L_2$ are regular, then so is $L_1 \cap L_2$.

Do not rely on De Morgan's law: $L_1 \cap L_2 = \overline{L_1} \cup \overline{L_2}$.

Apply the construction of a DFA for $L_1 \cap L_2$ to the following DFAs $A_1$ for $L_1$ and $A_2$ for $L_2$:

$A_1 : \quad \begin{array}{c}
\text{State 0} \\
\text{State 1} \\
\text{State 2}
\end{array} \\
\begin{array}{ccc}
\rightarrow & 9_0 & 9_1 \\
\rightarrow & 9_1 & 9_2
\end{array}$

$A_2 : \quad \begin{array}{c}
\text{State 0} \\
\text{State 1} \\
\text{State 2}
\end{array} \\
\begin{array}{ccc}
\rightarrow & 9_0 & 9_1 \\
\rightarrow & 9_1 & 9_2
\end{array}$

Note: we can assume that $L_1$ and $L_2$ are REs over the same alphabet $\Sigma$. 
Solution:

Let $A_1 = (Q_1, \Sigma, S_1, q_{01}, F_1)$ be a DFA s.t. $L(A_1) = L_1$.

Let $A_2 = (Q_2, \Sigma, S_2, q_{02}, F_2)$ s.t. $L(A_2) = L_2$.

Consider a string $w$ accepted by both $A_1$ and $A_2$.

Let $w = e_1 e_2 \ldots e_n$. Then we have

$$
\begin{align*}
\text{in } A_1 & \quad q_{01} \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \xrightarrow{a_3} \ldots \xrightarrow{a_n} p_n \in F_1 \\
\text{in } A_2 & \quad q_{02} \xrightarrow{a_1} p'_1 \xrightarrow{a_2} p'_2 \xrightarrow{a_3} \ldots \xrightarrow{a_n} p'_n \in F_2
\end{align*}
$$

Hence we can construct a DFA $A_\infty = (Q_\infty, \Sigma, S_\infty, q_{0\infty}, F_\infty)$ that simulates the transitions of both $A_1$ and $A_2$:

- Each state of $A_\infty$ is a pair of states $(q_1, q_2)$, where $q_1 \in Q_1$ and $q_2 \in Q_2$.

  Hence $Q_\infty = Q_1 \times Q_2$.

- The initial state $q_{0\infty}$ is the pair of initial states of $Q_1$ and $Q_2$. Hence $q_{0\infty} = (q_{01}, q_{02})$.

- The set of final states is such that both $A_1$ and $A_2$ accept if $A_\infty$ accepts. Hence $F_\infty = F_1 \times F_2$.

- The transition function $\delta_\infty$ simulates the transitions of both $A_1$ and $A_2$: if $A_\infty$ is in state $(q_1, q_2)$, then on input $a$ it goes to a state $(q'_1, q'_2)$, where

$$
q'_1 = \delta_1(q_1, a) \quad \text{and} \quad q'_2 = \delta_2(q_2, a).
$$

Hence: for all $a \in \Sigma$, $q_1 \in Q_1$, $q_2 \in Q_2$:

$$
\delta_\infty((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a)).
$$
One can show that $A\_n$ constructed in this way accepts $L(A\_1) \cap L(A\_2)$.
$A\_n$ is called the product automaton.

By applying this construction to the automata $A\_1$ and $A\_2$, we obtain

![Diagram]

We have used $q\textsubscript{ij}$ to denote $(q\textsubscript{i1}, q\textsubscript{j1})$.