Exercise: Let $G = (V, E)$ be an undirected graph.

A vertex cover $C$ of $G$ is a subset of the vertices $V$ such that every edge of $G$ touches at least one of the vertices of $C$.

The vertex cover problem:

Input: - graph $G = (V, E)$
- integer $k$

Output: yes iff $G$ has a vertex cover of size $\leq k$

Vertex cover is NP-complete:

Proof:

in NP: easy

- guess a subset $C$ of $V$ of size $\leq k$
- check in poly-time that it is a vertex cover

NP-hard by reduction from 3-SAT

We define a poly-time reduction $R$ that:

- takes as input a 3-CNF formula $F$
- constructs a graph $G = (V, E)$ and an integer $k$
- such that:

$F$ is satisfiable $\iff G$ admits a vertex cover with $k$ nodes

Let $F = C_1 \land \cdots \land C_m$ be a 3-CNF formula over variables

$\{x_1, \ldots, x_n\}$

We construct $G = (V, E)$ as constituted by various components,
- For each variable \( X_i \), we have a truth-setting component \( T_i = (V_i, E_i) \) with \( V_i = \{ X_i, \overline{X}_i \} \)
  \[ E_i = \{ \{ X_i, \overline{X}_i \} \} \]
  
  note: at least one of \( X_i, \overline{X}_i \) will be in every vertex cover to cover \( \{ X_i, \overline{X}_i \} \)

- For each clause \( C_j \) of \( F \), we have a satisfaction testing component \( S_j = (V'_j, E'_j) \)

\[
\begin{align*}
\circ_3j & \quad \circ_2j & \quad \circ_1j \\
\end{align*}
\]

note: at least two of \( V'_j \) will be in every vertex cover to cover \( E'_j \)

- We have a communication component, which is the only part that depends on which literals are in which clauses.

Let \( C_j = c_{1j} + c_{2j} + c_{3j} \)

Then we have \( E''_j = \{ \{ \circ_1j, c_{1j} \}, \{ c_{2j}, \circ_2j \}, \{ c_{3j}, \circ_3j \} \} \)

We then set \( K = m + 2m \equiv \text{number of variables} + \text{number of clauses} \)
Example: \( F = (\overline{x_1} + \overline{x_3} + \overline{x_4}) \cdot (\overline{x_1} + x_2 + \overline{x_4}) \)

\[ k_2 = m + 2m = 4 + 2 \cdot 2 = 8 \]

We show that \( F \) is satisfiable \( \Rightarrow \) \( G \) has a vertex cover of size \( \leq k \)

\( \leq \) Let \( V' \subseteq V \) be a vertex cover for \( G \) with \( |V'| \leq k \).

We need that \( V' \) contains one vertex for each variable at least and two vertices for each clause.

This is already \( k_2 = m + 2m \)

\( \Rightarrow \) at least in actually exactly.

We use \( V' \) to obtain the truth assignment \( \text{if} \)

we set \( x_i = \text{true} \) if \( x_i \in V' \)

\( x_i = \text{false} \) if \( \overline{x_i} \in V' \)

To show that \( \text{if} \) is a truth assignment that satisfies \( F \),

we must that all clauses of the connection components are covered by \( V' \).

Consider a clause \( C_i = \overline{x_j} \vee \overline{x_k} \vee x_l \).

- two of the ones in \( E_i \) are covered by the choice of

\( x_j \) among \( x_i, \overline{x_i}, a_j, \overline{a_j}, a_k, \overline{a_k} \) in \( V' \).

We say, let there be \( a_j, \overline{a_j}, a_k, \overline{a_k} \)
the third one is then covered by the literal \( l_{3j} \) (connected to \( e_{3j} \)) which has to be in \( V' \).

Since, by definition of \( \phi \), \( l_{3j} = \text{true} \), \( C_j \) is satisfied.

\( \Rightarrow \) Let \( \phi \) be a truth assignment that satisfies \( F \).

We define a subset \( V' \subset V \) as follows:

- \( x_i \in V' \) iff \( \phi(x_i) = \text{true} \)
- \( \overline{x_i} \in V' \) iff \( \phi(x_i) = \text{false} \)

Since \( \phi \) satisfies \( F \), for each communication component \( E''_j = \{e_{1j}, l_{1j}\}, \{e_{2j}, l_{2j}\}, \{e_{3j}, l_{3j}\} \)

one of the three edges \( \{e_{ij}, l_{ij}\} \) is covered in \( V' \) by \( l_{ij} \).

W.l.o.g., let \( i = 1 \). Then \( \{e_{2j}, l_{2j}\}, \{e_{3j}, l_{3j}\} \) can be covered by having \( e_{3j} \in V' \) and \( e_{3j} \in V' \).

We get that \( V' \) contains \( n + 2m \) vertices.
Exercise (10.3.2): The problem $4TA-SAT$ is defined as follows:

Given a propositional formula $E$, does $E$ have at least 4 satisfying truth assignments?

Show that $4TA-SAT$ is NP-complete.

Proof:

1) $4TA-SAT$ is in NP

We devise a non-deterministic poly-time algorithm:

1. Guess 4 truth-assignments $T_1, T_2, T_3, T_4$
2. Check that $T_1, T_2, T_3, T_4$ all satisfy $E$

Note that both steps require time polynomial in the size of $E$

2) $4TA-SAT$ is NP-hard

We show this by reducing SAT to $4TA-SAT$.

Let $E$ be a propositional formula, and let $x_1, \ldots, x_n$ be all variables in $E$.

We construct a new formula $E'$ as:

$$ EE \in SAT \iff E' \in 4TA-SAT $$

Let $y_1, y_2$ be two new variables. Then

$$ E' = E \lor ((x_1 \land x_2 \land \ldots \land x_n) \land (y_1 \land \overline{y_2}) \lor (y_1 \land \overline{y_2}) \lor (\overline{y_1} \land y_2)) $$
Consider the truth assignments for \( k_1, ..., k_n, y_1, y_2 \)

\[
\begin{array}{c|c|c|c|c|c|c|c}
   & k_1 & k_2 & k_n & y_1 & y_2 & \text{Case 1} & \text{Case 2} \\
   \hline
   1) & T & T & T & T & T & E & E' \\
   2) & T & T & T & T & F & F & F \\
   3) & T & T & T & F & T & F & F \\
   4) & T & T & T & F & F & F & F \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   2^{n+2} & F & F & F & F & F & E & T \\
\end{array}
\]

Alternative solution:

\[ E' = E \land (y_1 \lor y_2 \lor y_3) \]

- If \( E \) is unsatisfiable, then \( E' \) is unsatisfiable, and hence \( E' \not\in \text{SAT} \)

- If \( E \) is satisfiable, then \( E' \) has at least 7 satisfying truth assignments; these are obtained by combining
  - a TA for \( k_1, ..., k_n \) satisfying \( E \) with
  - the 7 TAs for \( y_1, y_2, y_3 \) satisfying \( y_1 \lor y_2 \lor y_3 \)
Exercise 11.11 b)

Consider the problem \textit{FALSE-SAT}:

Given a boolean expression \( E \) that is false when all its variables are made false, is there some other truth assignment that makes \( E \) false besides all-false?

Decide whether the problem is in \textit{NP} or \textit{coNP}.

Describe its complement.

If the problem or its complement is \textit{NP-complete}, prove it.

Proof:

The problem is \textit{NP-complete}.

- In \textit{NP}:
  - Given a boolean expression \( E \), we need to check:
    1) that \( E \) is false when all variables are assigned false
    2) that there is some other truth assignment making \( E \) false

    (1) can be done in poly-time by a DTM
    (2) can be done in poly-time by a NTM

    - guess a truth assignment \( T \) different from all-false, and
      answer yes if under \( T \), \( E \) evaluates to false

- \textit{NP-hard}:
  - by a reduction from \textit{SAT}

  Let \( E \) be a boolean expression with variables \( x_1, \ldots, x_n \).
  We construct an expression \( E' \) s.t. \( E \in \text{SAT} \iff E' \in \text{FALSE-SAT} \)

  1) test if \( E \) is true when all variables are false (polynomial)

    - if so, \( E \in \text{SAT} \), and we convert it to a fixed expression
      that is in \text{FALSE-SAT}, e.g., \( \land \).

2) Otherwise, let $E'$ be $\neg E \wedge (x_1 \vee x_2 \vee \cdots \vee x_n)$. Clearly, the reduction is poly-time.

We have that $E'$ is false when all of $x_1, \ldots, x_n$ are false. Notice that in case (2), $E$ is false when all variables are false. Hence, if $E \in \text{SAT}$, then it is satisfied by a truth assignment $T$ different from all-false. Thus, $\neg E$ is made false by $T$, and $E' \in \text{FALSE-SAT}$.

Conversely, if $E' \in \text{FALSE-SAT}$, then since $x_1, x_2, \ldots, x_n$ is false only for the all-false truth assignment, there must be some other truth-assignment $T$ that makes $\neg E$ false. Then $T$ makes $E$ true, and $E \in \text{SAT}$.