Problem 1.1 [6 points] You must give a brief explanation of your answer to receive full credit.

(a) Decide whether the following statement is TRUE or FALSE: The complement of a recursive-enumerable language is recursively-enumerable.

(b) Let $M_2$ be a 2-tape (deterministic) TM, and let $M_1$ be the result of converting $M_2$ into a (1-tape deterministic) TM. Sketch using two to three sentences the main idea underlying the conversion. How are the running times of $M_1$ and $M_2$ related to each other?

(c) Decide whether the following statement is TRUE or FALSE: For all languages $L_1$, $L_2$, and $L_3$, if there exist a reduction from $L_1$ to $L_3$ and a reduction from $L_2$ to $L_3$, then there exists a reduction from $L_1$ to $L_2$.

Problem 1.2 [6 points] Construct a TM $M$ that accepts the language $L_1 = \{n\#w \mid n$ is a number represented in binary with the least significant digit on the right, and $w \in \{a, b, c\}^*$ with $|w|_a + |w|_b = n\}$, where $|w|_x$ denotes the number of occurrences of $x$ in $w$.

E.g.: 10#accbc ∈ $L$, 0# ∈ $L$, 10#accbcb /∈ $L$, 10#ccac /∈ $L$. Show the sequence of IDs of $M$ on the input strings “10#accbc” and “10#cc”.

Problem 1.3 [6 points] The extraction $L_1 \ominus L_2$ of two languages $L_1$ and $L_2$ is defined as:

$L_1 \ominus L_2 = \{vw \mid vw_2w \in L_1,$ for some $w_2 \in L_2\}$

Show that the class of recursively enumerable languages is closed under the extraction operation, i.e., that if $L_1$ and $L_2$ are recursively enumerable, then so is $L_1 \ominus L_2$.

[Hint: Show how to construct, from two (deterministic) TMs $M_1$ accepting $L_1$ and $M_2$ accepting $L_2$, a (possibly multi-tape) non-deterministic TM $N$ accepting $L_1 \ominus L_2$. You need not detail completely the construction of $N$, but you should provide sufficient information to make clear that you would be able to detail the construction, if you were given enough time to do so.]

Problem 1.4 [6 points]

(a) Let $f$ and $g$ be primitive recursive functions. Show that the following predicate $p$ is primitive recursive:

$$p(x) = \begin{cases} 
1 & \text{if } f(i) > g(j), \text{ for all } 1 \leq i \leq x \text{ and } 1 \leq j \leq x \\
0 & \text{otherwise}
\end{cases}$$

(b) Show that the following function $f$ is primitive recursive:

$$f(x) = \begin{cases} 
2 & \text{if } x = 0 \text{ or } x = 1 \\
3 \cdot f(x-1) \cdot f(x-2) & \text{if } x \geq 2
\end{cases}$$

Problem 1.5 [6 points]

(a) Let $f$ be a total number-theoretic function with $n + 1$ variables. Provide the definition of the $(n + 1)$-variable function $g_n f$ such that $g_n f(x, i)$ encodes the values of $f(x, i)$ for $1 \leq i \leq y$.

(b) Let $g$ and $h$ be total number-theoretic functions, respectively with $n$ and $n + 2$ variables. Define the $(n + 1)$-variable function $f$ obtained from $g$ and $h$ by course-of-values recursion.
1.1. c) FALSE. Counter, e.g. $L_1$

M₁ has 4 tracks, 2 for the 2 tapes, 2 with a marker for the 2 head positions. On each move of M₂, one move back and forth of M₁. M₁ has quadratic running time in the running time of M₂.

1.2. Diagram:

1.3. N is a 3-tape NTM working as follows, when given an input string $X$ on tape 1:

1) Guess a prefix $v$ of $X$ and copy it to tape 3
2) Guess an arbitrary string $w₂$ on tape 2
3) Copy $w₂$ to tape 3 immediately after $v$
4) Run $M₂$ on $w₂$ on tape 2

If $M₂$ accepts, then proceed.
If $M₂$ rejects or loops, then this non-deterministic run of N will also reject or loop.

5) Copy the remaining part $w$ of $X$ from tape 1 to tape 3, immediately after $w₂$. Tape 3 now contains $v w₂ w$.
6) Run $M₁$ on $v w₂ w$, and accept if $M₁$ accepts.
Otherwise, this non-deterministic run of N will reject or loop.
1.4 e) \( \eta(x) = \prod_{i=0}^{x} \prod_{j=0}^{x} \eta(f(i), f(j)) \)

Since \( f, g, \eta \) are PRFs,

the composition of PRFs is a PRF.

the bounded product of a PRF is a PRF.

we get that also \( \eta \) is a PRF.

b) We define an auxiliary function \( h(x) = \eta_2(f(x), f(x+1)) \)

\[
\begin{align*}
    h(0) & = \eta_2(f(0), f(1)) = \eta_2(2, 2) = 2^3 \cdot 3^3 = 8 \cdot 27 = 216 \\
    h(x+1) &= \eta_2(f(x+1), f(x+2)) = \\
    & = \eta_2(f(x+1), 3 \cdot f(x+1) \cdot f(x)) = \\
    & = \eta_2(\text{dec}(1, h(x)), 3 \cdot \text{dec}(1, h(x)) \cdot \text{dec}(0, h(x)))
\end{align*}
\]

Since \( \eta_2 \) and \( \text{dec} \) are PRFs, this is in a definition.

of \( h \) by PR.

\( f(x) = \text{dec}(0, h(x)) \)

Hence \( f \) is a PRF.

1.5 a) \( \eta_3(x, y) = \prod_{i=0}^{x} \eta_2(i, i+1) \)

b) \[
\begin{align*}
    f(x, 0) & = f(x) \\
    f(x, y+1) & = h(x, y, \eta_2(x, y))
\end{align*}
\]
Exercise 1: Consider a TM $M_0 = (Q_0, \Sigma, \Gamma_0, \delta_0, q_0, \phi, F_0)$.

Show that $L(M)$ is also accepted by a TM $M_1$ that may move left of its initial position (i.e., $e$ by a TM with a semi-infinite tape).

Idea: $M_0$ is a two track TM: $M_0 = (Q_0, \Sigma, \Gamma_0, \delta_0, q_0, \phi, F_0)$

Let us call $q_0$ the initial tape position of $M_0$

\[ \vdash \quad |q_0|a_1b_1c_1d_1e_1f_1 \]

\[ M_0 \quad \vdash \quad M_1 \]

The states of $M_1$ are all the states of $M_0$, with an additional component $P \in N$, indicating whether $M_0$ is currently working on the track representing the positive or negative portion of the tape of $M_0$.

- $Q_1 = Q_0 \times \{P, N\}$

- $\Gamma_1$ is the set of pairs of symbols of $\Gamma_0$, plus symbols with $*$ on $T_1$

\[ \Gamma_1 = \Gamma_0 \times (\Gamma_0 \cup \{\star\}) \]

The $\star$ on $T_1$ is used to detect when $M_1$ reaches the leftmost tape position.

Initially, $Q_1$ writes $\star$ on $T_1$ of the leftmost position (for this it actually needs two additional states).

For the transitions of $M_1$, we need to distinguish 4 cases:

1. $M_0$ is to the right of $q_0 \Rightarrow M_1$ works on track $T_1$.
2. $M_0$ works on track $T_1$.
3. $M_0$ is on $q_0 \Rightarrow M_1$ is on $[\star]$.
Let $\delta_0(q, x) = (q', y, d)$ be a transition of $M_0$.

Then we have

1) $\delta_1([q, P], [x]) = ([q', P], [y], d)$ for every $x \in \Gamma_0$ (i.e., $x \neq \ast$)

2) $\delta_1([q, N], [x]) = ([q', N], [y], \bar{d})$ for every $x \in \Gamma_0$
   
   where $\bar{d} = L$ if $d = R$
   
   $\bar{d} = R$ if $d = L$

3) if $M_0$ moves right, i.e., $d = R$

   $\delta_1([q, -], [x]) = ([q', P], [y], R)$

   if $M_0$ moves left, i.e., $d = L$

   $\delta_1([q, -], [x]) = ([q', N], [y], R)$

- Final states of $M_1$: $F_1 = F_0 \times \{P, N\}$
Exercise 2. Construct a TM that computes the length of its input string, represented as a binary number (with the least significant digit on the right). Assume $\Sigma = \{0, 1\}$.

Idea: we write a counter to the left of the input separated by a $\$$.

We repeatedly move to the right of the input, delete the last symbol, come back and increment the counter.
Exercise 3. Let \( M \) be a Turing machine with input alphabet \( \Sigma \), let \( \langle M, w \rangle \) denote the encoding \( E(M) \) of \( M \) followed by input \( w \).

Consider the language \( L = \{ \langle M, w \rangle \mid M \text{ when started on an input string } w, \text{ eventually does three consecutive transitions in which it moves the head in the same direction} \} \).

(a) Show that \( L \) is recursively enumerable.
(b) Show that \( L \) is not recursive.

2. We reduce \( L \) to \( L_m \).

The reduction \( R \) is a Turing machine that takes as input \( \langle M, w \rangle \)
and produces as output \( R(\langle M, w \rangle) = \langle M', w \rangle \)
such that \( \langle M, w \rangle \in L \iff \langle M', w \rangle \in L_m \).

We describe how \( R \) has to transform \( E(M) \) to obtain \( E(M') \):

- \( R \) has to add to the states of \( M \) a second component that counts how many consecutive transitions \( M \) has made in the same direction.

  The values of the counter component are: -3, -2, -1, 1, 2, 3

- The transitions of \( M \) are modified to update the counter.

  if \( M \) moves right:
  \begin{align*}
  C = -2 & \quad \Rightarrow \quad C = 1 \\
  C = -1 & \quad \Rightarrow \quad C = 1 \\
  C = 1 & \quad \Rightarrow \quad C = 2 \\
  C = 2 & \quad \Rightarrow \quad C = 3
  \end{align*}

  if \( M \) moves left:
  \begin{align*}
  C = -2 & \quad \Rightarrow \quad C = -3 \\
  C = -1 & \quad \Rightarrow \quad C = -2 \\
  C = 1 & \quad \Rightarrow \quad C = -1 \\
  C = 2 & \quad \Rightarrow \quad C = -1
  \end{align*}

- The states with the counter 3 or -3 are the only final states.
b) We reduce the halting problem \( \text{L}_H \) to \( \text{L} \).

The reduction \( R \) is a TM that takes as input \( \langle M, w \rangle \)
and produces as output \( R(\langle M, w \rangle) = \langle M', w \rangle \)
such that \( \langle M, w \rangle \in \text{L}_H \iff \langle M', w \rangle \in \text{L} \).

We describe how \( R \) has to transform \( \Sigma(M) \) to obtain \( \Sigma(M') \):

- The final states of \( M \) are made non-final in \( M' \).
- From a final or blocking state of \( M \) we add to \( M' \) a transition to a new state from which \( M' \) makes 3 transitions to the right.
- We have to make sure that \( M' \) never does 3 consecutive transitions in the same direction (except the ones above).

Hence:

- if \( M \) does an \( R \)-move, then
  \( M' \) does an \( R - L - R \) move
- if \( M \) does an \( L \)-move, then
  \( M' \) does an \( L - R - L \) move

- the tape symbol is changed only in the first of the three moves, while the other two leave the tape unchanged.
- for the dummy moves, additional states are needed, and these need to be distinct for each state of \( M \).
Exercise 4: Let \( f(x) \) be a PRF.

a) Show that the following predicate is a PRF:

\[
(f(x), y) = \begin{cases} 
1 & \text{if } g(i) < g(k) \text{ for all } 0 \leq i \leq y \\
0 & \text{otherwise}
\end{cases}
\]

\[
(f(x), y) = \bigwedge_{i=0}^{y} \neg (g(i) \leq g(x))
\]

b) Let \( f \) be defined by

\[
f(x) = \begin{cases} 
2 & \text{if } x = 0 \\
3 & \text{if } x = 1 \\
(k-3) + f(k-1) & \text{if } x \geq 3
\end{cases}
\]

Give the values \( f(4) \), \( f(5) \), \( f(6) \).

\[
f(4) = f(1) + f(3) = 2 + 4 = 6
\]

\[
f(5) = f(2) + f(4) = 3 + 6 = 9
\]

\[
f(6) = f(3) + f(5) = 4 + 3 = 13
\]

Show that \( f \) is a PRF.

We have that \( f(y+1) = f(y) + f(2) \).

We introduce an auxiliary function \( h \) with

\[
h(y) = [f(y), f(y+1), f(y+2)] = gm_2(f(y), f(y+1), f(y+2))
\]

\[
h(0) = gm_2(f(0), f(4), f(2)) = gm_2(4, 2, 3) = 2^2 \cdot 3^3 \cdot 5^4
\]

\[
h(y+1) = [f(y+1), f(y+2), f(y+3)] =
\]

\[
= [f(y+1), f(y+2), f(y) + f(y+2)] =
\]

\[
= [\text{dec}(1, h(y)), \text{dec}(2, h(y)), \text{dec}(0, h(y)) + \text{dec}(2, h(y))]
\]

\[
= gm_2(\ldots)
\]

Hence \( h \) is PR. Then \( f(y) = \text{dec}(0, h(y)) \) is also PR.