**Exercise 1**

Show that multiplication is a primitive recursive function.

**Solution:**

\[
\begin{align*}
\text{mult} (x, 0) &= q(x) = 0 \\
\text{mult} (x, y+1) &= R(x, y, \text{mult} (x, y)) = \text{mult} (x, y) \cdot x
\end{align*}
\]

where \( q = \text{z} \) and \( R = \text{add} o (p^{(3)}, p^{(3)}) \)

**Exercise 2**

Let \( g(x,y) \) be a primitive recursive function. Then the following functions obtained from \( g \) are also PR.

a) \( f(x_1, y_1, z_1, \ldots, z_n) = g(x, y) \)

b) \( f(x, y) = g(y, x) \)

c) \( f(x) = g(x, x) \)

**Solution:**

a) \( f = g \circ (p^{(m+2)}, p^{(m+2)}) \)

b) \( f = g \circ (p^{(3)}, p^{(3)}) \)

c) \( f = g \circ (p^{(4)}, p^{(4)}) \)
Exercise 3

Let \( p(x,z) \) be a primitive recursive predicate. Show that the following functions are primitive recursive.

a) \( f_1(x,y_0,y_1) = \) the first value \( z \) in \([y_0,y_1]\) for which \( p(x,z) \) is true

b) \( f_2(x,y) = \) the second value \( z \) in \([0,y]\) for which \( p(x,z) \) is true

c) \( f_3(x,y) = \) the largest value \( z \) in \([0,y]\) for which \( p(x,z) \) is true

If there is no value \( z \) in the range such that \( p(x,z) \) is true, then \( f_i \) is \( y+1 \).

Solution:

a) \( f_1(x,y_0,y_1) = \mu z \leq y [ p(x,z) \cdot ge(z,y_0) ] \)

The PRF \( ge \) ("greater or equal to") is used to enforce the lower bound; multiplication \( \cdot \) works as "boolean and".

b) \( f_2(x,y) = \mu z \leq y [ p(x,z) \cdot gt(z, \mu z' \leq y [ p(x,z') ] ) ] \)

The PRF \( gt \) ("greater than") makes sure we skip the first value.

c) Let \( f'(x,y) = y \cdot \mu z \leq y [ p(x,y-z) ] \)

\[ \text{reverses the order of examination (i.e., we go from } y \text{ down to } 0) \]

Then:

\[ f_3(x,y) = eq(y+1, \mu z \leq y [ p(x,z) ] ) \cdot (y+1) \]

\[ + \text{eq}(y+1, \mu z \leq y [ p(x,z) ] ) \cdot f'(x,y) \]

It checks whether there is a \( z \) such that \( z \leq y \) and \( p(x,z) = \text{true} \), and outputs \( f'(x,y) \) if it is the case and \( y+1 \) otherwise.
Exercise 4

Consider integer division \( \text{div}(x, y) \): it's not defined for 0, hence not total and hence not PR. Let

\[
\text{quo}(x, y) = \begin{cases} 
0 & \text{if } y = 0 \\
\text{div}(x, y) & \text{otherwise}
\end{cases}
\]

(a) Define \( \text{quo}(x, y) \) using bounded minimization.
(b) Show that remainder, divides, number of divisors, and prime are primitive recursive.

Solution:

(a) \( \text{quo}(x, y) = \text{sgn}(y) \cdot \mu z \leq x \ [q \mid ((z+1) \cdot y, x)] \)

(b) Remainder:

\[
\text{rem}(x, y) = x \div (y \cdot \text{quo}(x, y))
\]

Divides:

\[
\text{divides}(x, y) = \begin{cases} 
1 & \text{if } x > 0, y > 0, \text{ and } y \text{ is a divisor of } x \\
0 & \text{otherwise}
\end{cases}
\]

\[
\text{divides}(x, y) = \text{sgn}(\text{rem}(x, y), 0) \cdot \text{sgn}(x)
\]

Number of divisors:

\[
\text{ndivisors}(x, y) = \sum_{i=0}^{x} \text{divides}(x, y)
\]

Prime:

\[
\text{prime}(x) = \begin{cases} 
1 & \text{if } x \text{ is prime} \\
0 & \text{otherwise}
\end{cases}
\]

\[
\text{prime}(x) = \text{sgn}(\text{ndivisors}(x), 2)
\]
Exercise 5

Show that the function \( p_m(i) \) computing the \( i \)-th prime is \( PR \) by exploiting the fact that \( p_m(x+1) \leq p_m(x)! + 1 \).

Solution:

\[
\begin{align*}
  p_m(0) &= 2 \\
p_m(x+1) &= \mu z \leq (p_m(x)! + 1) [ \text{prime}(z) \cdot \text{gt}(z, p_m(x))] 
\end{align*}
\]

Exercise 6

Show that the Ackermann function

\[
\begin{align*}
  A(0, y) &= y + 1 \\
  A(x+1, 0) &= A(x, 1) \\
  A(x+1, y+1) &= A(x, A(x+1, y))
\end{align*}
\]

is defined for every main \( x, y \in \mathbb{N} \).

Solution:

By induction on \( x \) (main induction)

Base case: \( A(0, y) = y + 1 \)

Inductive step: By induction on \( y \) (secondary induction)

\( A(x+1, y) \)

Base case: \( A(x+1, 0) = A(x, 1) \) and the main induction hypothesis applies

Inductive step: By the secondary induction hypothesis \( A(x+1, y) \) is defined; then for \( A(x+1, y+1) = A(x, A(x+1, y)) \) the main induction hypothesis applies
Exercise 7

Define a primitive recursive function \( f : \mathbb{N} \to \mathbb{N} \) that counts the number of occurrences of the digit 5 in a natural number.

Solution:

We need some auxiliary primitive recursive functions:

- exponential: \( m^n : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \)
  \[
  \exp(m, n) = \begin{cases} 
  \exp(m, 0) = 1 \\
  \exp(m, n+1) = \exp(m, n) \cdot m
  \end{cases}
  \]

- length (number of digits): \( \mathbb{N} \to \mathbb{N} \)
  \[
  \text{length}(n) = (\mu z \leq n [ \text{eq}(10^z, n)] + 1
  \]
  Examples: \( \text{length}(0) = \text{length}(1) = \ldots = \text{length}(9) = 1 \), \( \text{length}(10) = 2 \), ...

\( f : \mathbb{N} \to \mathbb{N} \) is then defined as follows:

\[
 f(n) = \sum_{i=1}^{\text{length}(n)} \text{eq}(5, \text{rem}(\text{que}(n, 10^{i-1}), 10))
\]

Example: \( f(253) = \text{eq}(5, \text{rem}(\text{que}(253, 1), 10)) \)

\[
= \text{eq}(5, \text{rem}(253, 2)) + \text{eq}(5, \text{rem}(\text{que}(253, 10), 10))
\]

\[
= 0 + 1 = 1
\]

\[
= 0 + 1 + 0 = 1
\]
Exercise 8

Define a primitive recursive function \( f : \mathbb{N} \to \mathbb{N} \) that reverses the digits of a natural number, i.e. \( f(253) = 352 \), \( f(5524) = 4255 \).

Solution:

\[
f(n) = \sum_{i=1}^{\text{length}(n)} \left( \text{rem} \left(n, 10^{\text{length}(n)-i+1} \right), 10^{\text{length}(n)-i} \right) \cdot 10^{i-1}
\]

Example: \( f(5524) = \text{rem}(5524, 10000) \cdot 1000 + \text{rem}(5524, 1000) \cdot 100 + \text{rem}(5524, 100) \cdot 10 + \text{rem}(5524, 10) \cdot 1 \cdot 10000 \)

\[
= 5 + 50 + 200 + 4000 = 4255
\]