Basic notations and definitions

Sets

Set:
- explicit notation: e.g. \( V = \{0, 1, 2, \ldots m\} \)
- informally, we also use... e.g. \( \mathbb{N} = \{0, 1, 2, \ldots \} \)
- using a set former, i.e.: \( \{ x \mid E(x) \} \)
  where \( E(x) \) is a boolean expression depending on \( x \)
  e.g.: \( \{ x \mid x \in \mathbb{N} \land x \geq 10 \land x \leq 50 \} \)

Subset: \( A \subseteq B \) denotes that \( A \) is a subset of \( B \) (or \( A \) is contained in \( B \))

\( \forall x : \text{if } x \in A \text{ then } x \in B \)

NB: \( A \subseteq B \) means \( A \subseteq B \) and \( A \neq B \)

\( \mathcal{B} \subseteq \mathcal{A} \)

We may have sets whose elements are themselves sets

e.g.: \( A = \{0, 3\}, \{0, 2, 3\} \)
\( B = \{0, 3\}, \{0, 2\}, \{1, 2, 3\} \)

If \( A \subseteq B \), this does not simply everything not the containment between \( x \in A \) and \( y \in B \), e.g.: \( x \subseteq y \)

Gonerset: of a set \( A \): denoted \( 2^A \)

\( 2^A = \{ x \mid x \subseteq A \} \)

NB: \( x \in 2^A \iff x \subseteq A \)
Set operations:
- Intersection: \( A \cap B = \{ x \mid x \in A \land x \in B \} \)
- Union: \( A \cup B = \{ x \mid x \in A \lor x \in B \} \)
- Difference: \( A \setminus B = \{ x \mid x \in A \land x \notin B \} \)

When we refer to an implicit universe \( U \), we may denote with \( \overline{A} \) the complement of \( A \) (with \( U \))
\[ \overline{A} = U \setminus A \quad \text{(e.g. } U = \mathbb{N} \text{ or } U = \Sigma^*) \]

Cartesian product of sets \( A_1, A_2, \ldots, A_n \)
\[ A_1 \times A_2 \times \cdots \times A_n = \{ (x_1, \ldots, x_n) \mid x_1 \in A_1, \ldots, x_n \in A_n \} \]
... set of n-tuples of elements respectively of \( A_1, \ldots, A_n \)

Relations
- Binary relation between two sets \( A \) and \( B \)
\[ R \subseteq A \times B \]
\[ \text{e.g. } \leq \subseteq \mathbb{N} \times \mathbb{N} \text{ is defined as } \]
\[ \leq = \{ (x, y) \mid x \in \mathbb{N}, y \in \mathbb{N}, \text{then } x + k = y \} \]
- We may use n-ary notation: \( (x, y) \in R \iff x R y \)
- \( (x, y) \in \leq \iff x \leq y \)
- A relation \( R \subseteq S \times S \) for some set \( S \), is called a precedence relation.
  - Reflexive: \( \forall s \in S : s R s \)
  - Symmetric: \( \forall a, b \in S : \text{if } a R b \text{ then } b R a \)
  - Transitive: \( \forall a, b, c \in S : \text{if } a R b \text{ and } b R c \text{ then } a R c \)
  - Antisymmetric: \( \forall a, b : \text{if } a R b \text{ and } b R a \text{ then } a = b \)
- Types of precedence relations:
  - equivalence: reflexive, symmetric, and transitive
  - preorder: reflexive and transitive
  - partial order: antisymmetric preorder
  - total order on $S$: for all $x, y \in S$, either $x \leq y$ or $y \leq x$

When $\prec \in S \times S$ is a partial order (on $S$), we say also that $(S, \prec)$ is a partially ordered set.

- minimal element $\preceq \in S$: $\forall y \in S : y \not\prec x$
- maximal $\quad \supseteq \quad x \not\prec y$

- Transitive closure of $R \subseteq S \times S$, denoted $R^+$

$$R^+ = \bigcup_{n \in \mathbb{N}} R^n$$

with

$$R^0 = R$$

$$R^{n+1} = \{ (e, c) | \exists b : (e, b) \in R^n \land (b, c) \in R \}$$

Functions:

Consider an $n$-ary relation $R \subseteq A_1 \times \cdots \times A_n$ and $k \leq n$.

Then $R$ is a $k$-argument function if

for each $k$-tuple $(x_1, \ldots, x_k) \in A_1 \times \cdots \times A_k$

there is a unique $(n-k)$-tuple $(x_{k+1}, \ldots, x_n) \in A_{k+1} \times \cdots \times A_n$

such that $(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \in R$.

We denote this as $R : A_1 \times \cdots \times A_k \to A_{k+1} \times \cdots \times A_n$.
\( A_0 \times \cdots \times A_n \) \text{ domain of } R

\( A_0 \times \cdots \times A_n \) \text{ co-domain of } R

We may use \( R \) to denote an \( n \)-tuple of elements, i.e.

\( R = \langle x_1, \ldots, x_n \rangle \) \text{ (where } n \text{ depends on the context)}

For simplicity, we consider now just functions \( f : A \to B \)

\( f : A \to B \) is also a relation \( f \subseteq A \times B \).

The converse does in general not hold.

But we can associate to each \( R \subseteq A \times B \) a function

\[ f_R : A \to 2^B \text{ with } f_R(x) = \{ y \mid x R y \} \]

\( f : A \to B \) is

- bijective if \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \)
- injective if \( \forall y \in B, \exists x \in A : f(x) = y \)
- surjective if \( \forall y \in B, \exists x \in A : f(x) = y \)

For \( D \subseteq A \), \( \{D\} \) denotes the image of \( D \) via \( f \), i.e.

\[ \{D\} = \{ y \mid \exists x \in D, f(x) = y \} \]

\( f^{-1} \) denotes the inverse of \( f \).

\( f^{-1} \) may not be a function,

But we can always define for \( D \subseteq B \) the inverse image of \( D \)

\[ f^{-1}(D) = \{ x \mid x \in A \land \{x\} \in D \} \]
Partial functions:

\[ f : A \rightarrow B \] is total if it is defined for every \( x \in A \).

i.e. if \( \forall x \in A : \exists y \in B : f(x) = y \) (i.e., \( x \in \text{dom } f \))

If \( f \) is not defined for some \( x \in A \) it is called partial.

We denote partial functions with Greek letters.

We use \( A \rightarrow B \) to denote the set of total functions from \( A \) to \( B \).

We use \( \downarrow x \) when \( f \) is defined on \( x \).

\[ \cdots \quad f(x) \uparrow \quad \cdots \quad \text{is not defined} \quad \cdots \]

Domain of \( f \):
\[ \text{dom } (f) = \{ x \mid f(x) \downarrow \} \]

Range of \( f \):
\[ \text{range } (f) = \{ x \mid \exists y, f(y) = x \uparrow \} \]

(Where \( \uparrow \) denotes the undefined value.)

Cardinality of sets:

\( |S| \) denotes the cardinality of a set \( S \).

- When \( S \) is finite, then \( |S| \) is the number of its elements.
- When \( S \) is infinite, defining \( |S| \) is more complicated.

Definitions:
- \( A \) and \( B \) are equinumerous if there is a bijection \( f : A \rightarrow B \), written \( A \cong B \).
- Then \( |S| \) denotes the collection of sets \( Y \) such that \( Y \cong S \).
- \( |A| \leq |B| \) if there is an injection \( f : A \rightarrow B \).

Case \( : \) if \( A \subset B \) then \( |A| \leq |B| \).

\( A < B \) if \( A \subset B \) but \( A \neq B \).
Basic definitions about languages:

- **Alphabet**: finite, nonempty set of symbols: \( \Sigma \)
  
  - e.g.: \( \Sigma = \{ 0, 1 \} \)  
  - \( \Sigma = \{ a, b, \ldots, z \} \)  
  - \( \Sigma \) = set of Unicode characters

- **String**: finite sequence of symbols from \( \Sigma \)
  
  \[ w = a_1 a_2 \ldots a_n, \text{ with } a_i \in \Sigma \text{ for } i \in \{1, \ldots, n\} \]
  
  - e.g.: \( 01101 \)
  - ciano ciao

  - **Empty string**: denoted \( \varepsilon \); string with no symbols

  - **Length of a string**: number of (positions for) symbols in the string
    
    \[ |w| \]
    
    - of \( w = a_1 \ldots a_n \), then \( |w| = n \)
    
    - e.g.: \( |\varepsilon| = 0 \)  
    - \( |\varepsilon| = 0 \) is the only string of length 0
    - \( |\varepsilon| = 1 \)  
    - ciao ciao  | = 8

    Notice: strictly speaking, the number of symbols in ciao ciao is 4

- **Powers of an alphabet**:
  
  \[ \Sigma^k = \Sigma \times \Sigma \times \cdots \times \Sigma \]

  - \( k \) times  
  - length \( k \)

  - e.g.: \( \Sigma^0 = \{ \varepsilon \} \)
    
    \[ \{0,1\}^4 = \{0,1\} \]
    
    - what is the difference between this and \( \{0,1\}^2 \)?
Closure of an alphabet $\Sigma$: $\Sigma^*$ is the set of all finite strings over $\Sigma$

i.e. $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$

also $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \ldots$  hence $\Sigma^* = \Sigma^0 \cup \Sigma^+

Note: all strings in $\Sigma^*$ are finite
$\Sigma^*$ is an infinite set

i.e. $\Sigma = \{0, 1\}$
$\Sigma^* = \{0, 1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$

Concatenation of two strings:
$x = a, a_2, \ldots, a_m \in \Sigma^*$
$y = b_1, b_2, \ldots, b_n \in \Sigma^*$

$\Rightarrow xy = a, a_2, \ldots, a_m b_1, b_2, \ldots, b_n$ (we may omit the $\varepsilon$)

Note: $\varepsilon, x = x \cdot \varepsilon = x$, i.e. $\varepsilon$ is the identity for conc.
$|xy| = |x| + |y|

Language $L$ over $\Sigma$: is any subset of $\Sigma^*$ (i.e. $L \subseteq \Sigma^*$)

Note: $L$ contains only finite strings, but it may be infinite

Examples:

$\Sigma = \{0, 1, \ldots, 2\}$
$L = \text{set of all English words}$

$\{0, 1\}$
$L = \{\varepsilon, 01, 0011, 000111, \ldots\}$ all strings with equal # of 0 and 1, with all 0's preceding the 1's

$\emptyset$ the empty language (i.e. $\{\varepsilon\}$)
$L = \text{valid ASCII characters}$
$L = \text{all legal C programs}$
$L = \text{all legal Java programs}$