Exercise 1: Consider a TM $M_0 = (Q_0, \Sigma, \Gamma, \delta, q_0, \delta', F_0)$.

Show that $L(M)$ is also accepted by a TM $M_1$ that moves left of its initial position (i.e., $e$ by a TM with a semi-infinite tape).

Idea: $M_0$ is a two-track TM: $M_0 = (Q_0, \Sigma, \Gamma, \delta, q_0, \delta', F_0)$

Let us call $q_0$ the initial tape position of $M_0$.

\[ \begin{array}{cccc} & a & b & c & d & e & f & g \\ \downarrow & & & \uparrow & & & & \end{array} \]

\[ \hspace{1cm} M_0 \hspace{4cm} M_1 \]

\[ \begin{array}{cccc} \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \hspace{0.5cm} \rightarrow \hspace{0.5cm} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \]

The states of $M_1$ are all the states of $M_0$, with an additional component $P$ or $N$, indicating whether $M_0$ is currently working on the track representing the positive or negative portion of the tape of $M_0$:

$Q_1 = Q_0 \times \{P, N\}$

$\Gamma_1$ is the set of pairs of symbols of $\Gamma$, plus symbols with * on $T_0$:

$\Gamma_1 = \Gamma_0 \times (\Gamma_0 \cup \{*\})$

The * on $T_0$ is used to detect when $M_1$ reaches the leftmost tape position.

Initially, $\Gamma_1$ writes * on $T_0$ of the leftmost position.

For the transitions of $M_1$, we need to distinguish 4 cases:

1) $M_0$ is to the right of $q_0 \rightarrow M_1$ works on track $T_0$

2) $M_0$ is to the left of $q_0 \rightarrow M_1$ works on track $T_0$

3) $M_0$ is on $q_0 \rightarrow M_1$ is on $[q_0]$
Let \( \delta_0(q, x) = (q', y, d) \) be a transition of \( M_0 \).

Then we have

1) \( \delta_1([q, P], [x]) = ([q', P], [y], d) \) for every \( z \in \Gamma \)

(i.e. \( z \neq x) \))

2) \( \delta_1([q, N], [x]) = ([q', N], [y], d) \) for every \( z \in \Gamma \)

where \( \bar{d} = L \) if \( d = R \)

\( \bar{d} = R \) if \( d = L \)

3) if \( M_0 \) moves right, i.e. \( d = R \)

\( \delta_1([q, -], [x]) = ([q', P], [y], R) \)

if \( M_0 \) moves left, i.e. \( d = L \)

\( \delta_1([q, -], [x]) = ([q', N], [y], R) \)

Final states of \( M_1 \) : \( F_1 = F_0 \times \{P, N\} \)
Exercise 2: Construct a TM that computes the length of its input string, represented as a binary number (with the least significant digit on the right). Assume $\Sigma = \{0, 1\}$.

Idea: We write a counter to the left of the input separated by a $\$$. We repeatedly move to the right of the input, delete the last symbol, come back and increment the counter.
Exercise 3: For a TM \( M \) with input alphabet \( \Sigma \), let \( \langle M, w \rangle \) denote the encoding \( E(M) \) of \( M \) followed by input \( w \).

Consider the language \( L = \{ \langle M, w \rangle \mid M \text{ when started on an input string } w, \text{ eventually does three consecutive transitions in which it moves the head in the same direction} \} \).

a) Show that \( L \) is recursively enumerable.
b) Show that \( L \) is not recursive.

e) We reduce \( L \) to \( L_m \).

The reduction \( R \) is a TM that takes as input \( \langle M, w \rangle \) and produces as output \( R(\langle M, w \rangle) = \langle M', w \rangle \) such that \( \langle M, w \rangle \in L \iff \langle M', w \rangle \in L_m \).

We describe how \( R \) has to transform \( E(M) \) to obtain \( E(M') \):

- \( R \) has to add to the states of \( M \) a second component that counts how many consecutive transitions \( M \) has made in the same direction:

  - The values of the counter component are \( -3, -2, -1, 0, 1, 2, 3 \).

- The transitions of \( M \) are modified to update the counter:

  - If \( M \) moves right:
    \[
    \begin{align*}
    \text{if } c = -2 & \Rightarrow c = 1 \\
    \text{then in } M' : & \begin{cases} 
    c = -1 & \Rightarrow c = 1 \\
    c = 1 & \Rightarrow c = 2 \\
    c = 2 & \Rightarrow c = 3
    \end{cases}
    \end{align*}
    \]

  - If \( M \) moves left:
    \[
    \begin{align*}
    \text{if } c = -2 & \Rightarrow c = -3 \\
    \text{then in } M' : & \begin{cases} 
    c = -1 & \Rightarrow c = -2 \\
    c = 1 & \Rightarrow c = -1 \\
    c = 2 & \Rightarrow c = -1
    \end{cases}
    \end{align*}
    \]

- The states with the counter \( 3 \) or \( -3 \) are the only final states.
b) We reduce the halting problem $L_M$ to $L$.

The reduction $R$ is a TM that takes as input $<M, w>$ and produces as output $R(<M, w>) = <M', w>$ such that $<M, w> \in L_M$ iff $<M', w> \in L$.

We describe how $R$ has to transform $E(M)$ to obtain $E(M')$:

- The final states of $M$ are made non-final in $M'$.
- From a final or blocking state of $M$ we add to $M'$ a transition to a new state from which $M'$ makes 3 transitions to the right.
- We have to make sure that $M'$ never does 3 consecutive transitions in the same direction (except the ones above).

Hence:

- if $M$ does an R-move, then $M'$ does an R-L-R move.
- if $M$ does an L-move, then $M'$ does an L-R-L move.

- the tape symbol is changed only in the first of the three moves, while the other two leave the tape unchanged.

- for the dummy move, additional states are needed, and these need to be distinct for each state of $M$. 

Exercise 4: Let $g(x)$ be a PRF.

a) Show that the following predicate is a PRF:
\[ f(x, y) = \begin{cases} 1 & \text{if } g(i) < g(x) \text{ for all } 0 \leq i \leq y \\ 0 & \text{otherwise} \end{cases} \]

\[ f(x, y) = \sum_{i=0}^{y} \text{lt}(g(i), g(x)) \]

b) Let $f$ be defined by
\[ f(x) = \begin{cases} 2^n & \text{if } x = 0 \\ 3 & \text{if } x = 1 \\ f(x) = f(x-3) + f(x-1) & \text{if } x \geq 3 \end{cases} \]

Give the values $f(4)$, $f(5)$, $f(6)$.

- $f(3) = f(0) + f(2) = 1 + 3 = 4$
- $f(4) = f(1) + f(3) = 2 + 4 = 6$
- $f(5) = f(2) + f(4) = 3 + 6 = 9$
- $f(6) = f(3) + f(5) = 4 + 9 = 13$

Show that $f$ is a PRF.

We have that $f(y+1) = f(y-2) + f(y)$.

We introduce an auxiliary function $h$ with
\[ h(y) = [f(y), f(y+1), f(y+2)] = qm_3(f(y), f(y-1), f(y-2)) \]
\[ h(0) = qm_3([f(0), f(1), f(2)]) = qm_3(1, 2, 3) = 2^2 \cdot 3^2 \cdot 5^4 \]
\[ h(y+1) = [f(y+1), f(y+2), f(y+3)] = \]
\[ = [f(y+1), f(y+2), f(y) + f(y+2)] = \]
\[ = [\text{dec}(1, h(y)), \text{dec}(2, h(y)), \text{dec}(0, h(y)) + \text{dec}(2, h(y))] = qm_3(\cdots) \]

Hence $h$ is PR. Then $f(y) = \text{dec}(0, h(y))$ is also PR.