

# PRIMITIVE RECURSIVE FUNCTIONS

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E6.1

## Exercise 1

Show that multiplication is a primitive recursive function.

Solution:

$$\begin{cases} \text{mult}(x, 0) = g(x) = 0 \end{cases}$$

$$\begin{cases} \text{mult}(x, y+1) = h(x, y, \text{mult}(x, y)) = \text{mult}(x, y) + x \end{cases}$$

where  $g = z$  and  $h = \text{add} \circ (P_3^{(3)}, P_1^{(3)})$

## Exercise 2

Let  $g(x, y)$  be a primitive recursive function. Then the following functions  $\square$  obtained from  $g$  are also PR.

a)  $f(x, y, z_1, \dots, z_n) = g(x, y)$

b)  $f(x, y) = g(y, x)$

c)  $f(x) = g(x, x)$

Solution:

a)  $f = g \circ (P_1^{(n+2)}, P_2^{(n+2)})$

b)  $f = g \circ (P_2^{(2)}, P_1^{(2)})$

c)  $f = g \circ (P_1^{(1)}, P_1^{(1)})$

Exercise 3

Let  $p(x, z)$  be a primitive recursive predicate. Show that the following functions are primitive recursive.

- a)  $f_1(x, y_0, y) =$  the first value  $z$  in  $[y_0, y]$  for which  $p(x, z)$  is true
- b)  $f_2(x, y) =$  the second value  $z$  in  $[0, y]$  for which  $p(x, z)$  is true
- c)  $f_3(x, y) =$  the largest value  $z$  in  $[0, y]$  for which  $p(x, z)$  is true

If there is no value  $z$  in the range such that  $p(x, z)$  is true, then  $f_i$  is  $y + 1$ .

Solution:

a)  $f_1(x, y_0, y) = \mu z \leq y [p(x, z) \cdot \mu z \leq y_0]$

The PRF  $\mu z \leq y$  ("greater<sup>than</sup> or equal to") is used to enforce the lower bound; multiplication works as "boolean and".

b)  $f_2(x, y) = \mu z \leq y [p(x, z) \cdot \mu z' \leq y [p(x, z')]]$

The PRF  $\mu z' \leq y [p(x, z')]$  ("greater than") makes sure we skip the first value.

c) Let  $f'(x, y) = \mu z \leq y [p(x, y - z)]$

reverses the order of examination (i.e. we go from  $y$  down to  $0$ )

Then:

$$f_3(x, y) = \mu z \leq y + 1 [p(x, z) \cdot \mu z \leq y [p(x, z)]] \cdot (y + 1) \\ + \text{neg}(y + 1, \mu z \leq y [p(x, z)]) \cdot f'(x, y)$$

It checks whether there is a  $z$  such that  $z \leq y$  and  $p(x, z) = \text{true}$ , and outputs  $f'(x, y)$  if it is the case and  $y + 1$  otherwise.

Exercise 4

Consider integer division  $\text{div}(x, y)$ : it's not defined for 0, hence not total and hence not PR. Let

$$\text{quo}(x, y) = \begin{cases} 0 & \text{if } y=0 \\ \text{div}(x, y) & \text{otherwise} \end{cases}$$

- Define  $\text{quo}(x, y)$  using bounded minimization.
- Show that remainder, divides, number of divisors, and prime are primitive recursive.

Solution:

$$a) \text{ quo}(x, y) = \text{sg}(y) \cdot \mu z \leq x [\text{gt}((z+1) \cdot y, x)]$$

b) Remainder:

$$\text{rem}(x, y) = x - (y \cdot \text{quo}(x, y))$$

Divides:

$$\text{divides}(x, y) = \begin{cases} 1 & \text{if } x > 0, y > 0, \text{ and } y \text{ is a divisor of } x \\ 0 & \text{otherwise} \end{cases}$$

$$\text{divides}(x, y) = \text{eq}(\text{rem}(x, y), 0) \cdot \text{sg}(x)$$

Number of divisors:

$$\text{ndivisors}(x, y) = \sum_{i=0}^x \text{divides}(x, y)$$

Prime:

$$\text{prime}(x) = \begin{cases} 1 & \text{if } x \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{prime}(x) = \text{eq}(\text{ndivisors}(x), 2)$$

Exercise 5

Show that the function  $pn(i)$  computing the  $i$ -th prime is PR by exploiting the fact that  $pn(x+1) \leq pn(x)! + 1$ .

Solution:

$$\begin{cases} pn(0) = 2 \\ pn(x+1) = \mu z \leq (pn(x)! + 1) [ \text{prime}(z) \cdot gt(z, pn(x)) ] \end{cases}$$

Exercise 6

Show that the Ackermann function

$$\begin{cases} A(0, y) = y + 1 \\ A(x+1, 0) = A(x, 1) \\ A(x+1, y+1) = A(x, A(x+1, y)) \end{cases}$$

is defined for every pair  $x, y \in \mathbb{N}$ .

Solution:

By induction on  $x$  (main induction).

Base case:  $A(0, y) = y + 1$

Inductive step: By induction on  $y$  (secondary induction)

$A(x+1, y)$

Base case:  $A(x+1, 0) = A(x, 1)$  and the

$A(x+1, 0)$

main induction hypothesis applies

Inductive step: By the secondary induction

$A(x+1, y+1)$

hypothesis  $A(x+1, y)$  is defined;

thus for  $A(x+1, y+1) = A(x, A(x+1, y))$  the main induction hypothesis applies

Exercise 7

Define a primitive recursive function  $f: \mathbb{N} \rightarrow \mathbb{N}$  that counts the number of occurrences of the digit 5 in a natural number.

Solution:

We need some auxiliary primitive recursive functions

- exponential  $m^n: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$

$$\text{exp}(m, n) = \begin{cases} \text{exp}(m, 0) = 1 \\ \text{exp}(m, n+1) = \text{exp}(m, n) \cdot m \end{cases}$$

- length (number of digits) :  $\mathbb{N} \rightarrow \mathbb{N}$

$$\text{length}(n) = (\mu z \leq n [\text{gt}(10^{z+1}, n)]) + 1$$

examples :  $\text{length}(0) = \text{length}(1) = \dots = \text{length}(9) = 1, \text{length}(10) = 2, \dots$

$f: \mathbb{N} \rightarrow \mathbb{N}$  is then defined as follows

$$f(n) = \sum_{i=1}^{\text{length}(n)} \text{eq}(5, \text{rem}(\text{quo}(n, 10^{i-1}), 10))$$

Example :  $f(253) =$

$\text{eq}(5, \text{rem}(\text{quo}(253, 1), 10))$	$\underbrace{253}_{3}$	0
	+	
$+ \text{eq}(5, \text{rem}(\text{quo}(253, 10), 10))$	$\underbrace{25}_{5}$	1
	+	
$+ \text{eq}(5, \text{rem}(\text{quo}(253, 100), 10))$	$\underbrace{2}_{2}$	0
	+	
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	1	

$= 0 + 1 + 0 = 1$

### Exercise 8

Define a primitive recursive function  $f: \mathbb{N} \rightarrow \mathbb{N}$  that reverses the digits of a natural number, i.e.  $f(253) = 352$ ,  $f(5524) = 4255$ .

Solution:

$$f(n) = \sum_{i=1}^{\text{length}(n)} (\text{quo}(\text{rem}(n, 10^{\text{length}(n)-i+1}), 10^{\text{length}(n)-i}) \cdot 10^{i-1})$$

Example:  $f(5524) =$

$$\begin{aligned} & \underbrace{\text{quo}(\text{rem}(5524, 10000), 1000)}_5 \cdot 1 \\ & + \underbrace{\text{quo}(\text{rem}(5524, 1000), 100)}_5 \cdot 10 \\ & + \underbrace{\text{quo}(\text{rem}(5524, 100), 10)}_2 \cdot 100 \\ & + \underbrace{\text{quo}(\text{rem}(5524, 10), 1)}_4 \cdot 1000 \\ & = 5 + 50 + 200 + 4000 = 4255 \end{aligned}$$