Exercise 1

Decide which of the following statements is true and which is false. Give a brief explanation of your answer.

a) For all languages $L_1$ and $L_2$, it holds that $(L_1^* L_2^*)^* = (L_1^* L_2^*)^*$

b) If $L_1$ and $L_2$ are both non-regular then $L_1 \cup L_2$ could be regular.

c) For all languages $L_1$ and $L_2$, if $L_1 \subseteq L_2$ then $L_1^* \subseteq L_2^*$.

Solution:

a) False. Consider the languages $L_1 = \{a\}$ and $L_2 = \{b\}$. Then $b \in (L_1^* L_2^*)^*$ but $b \notin (L_1^* L_2^*)^*$.

b) True. Assume that $L_2 = \overline{L_1}$, i.e. $L_2 = \overline{L_1}$. If $L_2$ is non-regular then no is in $L_2$ because, if $L_2$ would be regular then, by the closure properties of regular languages, $L_1 = \overline{L_2}$ would be regular too, thus leading to a contradiction. Since $L_1 \cup L_2 = L_1 \cup \overline{L_1} = \Sigma^*$ we have that the union of two non-regular languages can be regular.

c) True. Given that for all $w \in L_1$ we have that $w \in L_2$, the argument goes as follows. If $w' \in L_1^*$ then $w' = w_1 w_2 \ldots w_n$ for some $n \in \mathbb{N}$ and $w_i \in L_1$ ($1 \leq i \leq n$). But then each $w_i$ is also in $L_2$ ($w_i \in L_2$ for $1 \leq i \leq n$) and therefore $w' \in L_2^*$. 

Exercise 2

Show that the language
\[ L = \{ 0^n 1^n 0^{n+m} \mid m, n \geq 0 \} \]
is not regular.

Solution:

Assume that \( L \) is regular. Then, by the pumping lemma, we have that:

There exists \( n \) such that for all \( w \in L \) such that \( |w| \geq n \), there are three strings \( x, y, z \) such that:
\[ w = xyz, \quad |xy| \leq n, \quad |y| > 0, \quad xy^kz \in L. \]

Now, given some \( n \), let \( w = 0^n 1^n 0^{2n} \).

\[ w = \underbrace{0\cdots 0}_{n} \underbrace{1\cdots 1}_{n} \underbrace{0\cdots 0}_{2n} \]

Since \( |w| = 4n \) we have that \( |w| > n \).

In order to apply the pumping lemma, we need to find strings \( x \) and \( y \) such that \( |xy| \leq n \). The only possible choices are: \( x = 0^a \) and \( y = 0^b \) where \( b > 1 \).

But then we have that \( xy = 0^a 1^n 0^b \) and thus that \( n + n - b = 2n \). Therefore, for \( k = 0 \), \( xy^kz \notin L \).

Since we assumed that \( L \) is regular, this is a contradiction. Hence \( L \) cannot be regular.
Exercise 3

Show that the language
\[ L = \{ w \in \{0,1\}^* \mid w \text{ is a palindrome} \} \]
is not regular.

[A string \( w \) is a palindrome if \( w = w^R \) where \( (-)^R \) denotes string reversal.]

Solution:

Again, we use the pumping lemma.

Given some \( n \), let \( w = 0^n10^n \).

If we consider \( x, y, z \) such that
   a) \( w = xyz \)
   b) \( |xy| \leq n \)
   c) \( |y| \geq 1 \)

then \( y \) can only be a non-empty string of 0's.

Thus, for each \( k > 1 \), the string \( xyz^kz \) has more 0's on the left-hand side than on the right-hand side. We conclude that, for \( k > 1 \), \( xyz^kz \notin L \).

Therefore we have that \( L \) is not regular.
Exercise 4 (4.3.3 from textbook)

Give an algorithm to tell whether a regular language $L$ is universal (i.e. $L = \Sigma^*$?).

Solution:

If $L$ is universal then $\overline{L} = \Sigma^* - L = \emptyset$. Therefore we only need to check whether $\overline{L}$ is empty.

Exercise 5 (4.3.4 from textbook)

Give an algorithm to tell whether two regular languages have at least one string in common.

Solution:

We can check whether the intersection $L$ of the two languages that we denote with $L_1$ and $L_2$ is non-empty.

$$L = L_1 \cap L_2 = \overline{L_1} \cup \overline{L_2}$$

An automaton accepting $L$ can be easily constructed from automata accepting $L_1$ and $L_2$. Note that all automata need to be deterministic, otherwise complement of a language might not be accepted.