E 12.1

Exercise: Reduction from 3-SAT to CSAT

(see textbook 10.3.4)

Given a CNF formula $E = C_1 \cdot C_2 \cdots C_k$

with each $C_i = \sum_{j=1}^{k_i} l_{ij}$,

we construct a 3-CNF formula F as follows.

For each clause C_i of E

1) if $C_i = (l)$ (i.e., a single literal)

introduce two new variable u, v, and replace C_i by 4 clauses

$(l + u + v) \cdot$

$(l + u + \bar{v}) \cdot$

$(l + \bar{u} + v) \cdot$

$(l + \bar{u} + \bar{v})$

Since u, v appear in all 4 combinations, the 4 clauses can be satisfied only if l is true

2) if $C_i = (l_1 + l_2)$

introduce a new variable z, and replace C_i by 2 clause

$(l_1 + l_2 + z) \cdot$

$(l_1 + l_2 + \bar{z})$

as in 1

3) if $C_i = (l_1 + l_2 + l_3)$, just leave it

4) if $C_i = (l_1 + l_2 + \cdots + l_m)$ with $m \geq 4$

introduce $y_1, y_2, \ldots, y_{m-3}$ and replace C_i by

$(l_1 + l_2 + y_1) \cdot (l_3 + \bar{y}_1 + y_2) \cdot (l_4 + \bar{y}_2 + y_3) \cdots$

$\cdot (l_{m-2} + \bar{y}_{m-4} + y_{m-3}) \cdot (l_{m-1} + l_m + \bar{y}_{m-3})$

E 12.2

An assignment T satisfying E makes at least one literal of C_i true. Let it be l_j.
Then, by making $y_1, \dots, y_{j-2}$ true and

$y_{j-1}, \dots, y_{m-3}$ false,

we satisfy all clauses replacing C_i.
Thus we can extend T to satisfy F.
Conversely, if T makes all l_j of C_i false, then not all new clauses can be satisfied.
Why? each y_j can make at most 1 clause true, but there are $m-2$ clauses and

$m-3$ y_j's.

The 3-CNF formula F is linear in E and can be constructed in linear time

We get: CSAT $<_{poly}$ 3-SAT

$\Rightarrow$ from CSAT NP-hard, we get 3-SAT NP-hard
We also know 3-SAT $\in$ P (since SAT $\in$ P)
$\Rightarrow$ 3-SAT is NP-complete

Exercise: Let $G=(V,E)$ be an undirected graph (E.12.3)

A vertex cover C of G is a subset of the nodes V s.t. every edge of G touches at least one of the nodes of C.

The vertex cover problem:

input: - graph $G=(V,E)$
- integer k

output: yes iff G has a vertex cover of size $\leq k$

Vertex-cover is NP-complete:

Proof:

in NP: easy
- guess a subset C of V of size $\leq k$
- check in poly-time that it is a vertex-cover

NP-hard by reduction from 3-SAT

We define a poly-time reduction R that:
- takes as input a 3-CNF formula F
- constructs a graph $G=(V,E)$ and an integer k

such that:

F is satisfiable $\Longleftrightarrow$ G admits a vertex cover with k nodes

Let $F = C_1 \cdot \dots \cdot C_m$ be a 3-CNF formula over variables $\{x_1, \dots, x_n\}$

We construct $G=(V,E)$ as constituted by various components

E12.4

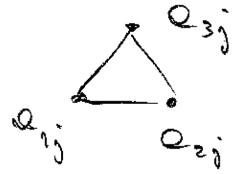
- For each variable x_i, we have a truth-setting component $T_i = (V_i, E_i)$ with $V_i = \{x_i, \bar{x}_i\}$

  $E_i = \{\{u_i, \bar{u}_i\}\}$

  undirected edge

  note: at least one of $u_i, \bar{u}_i$ will be in every vertex cover to cover $\{u_i, \bar{u}_i\}$

- For each clause C_j in F we have a satisfaction testing component $S_j = (V_j', E_j')$



  note: at least two of V_j' will be in every vertex cover to cover E_j'

- We have a communication component, which is the only part that depends on which literals are in which clauses

  let $C_j = l_{1j} + l_{2j} + l_{3j}$

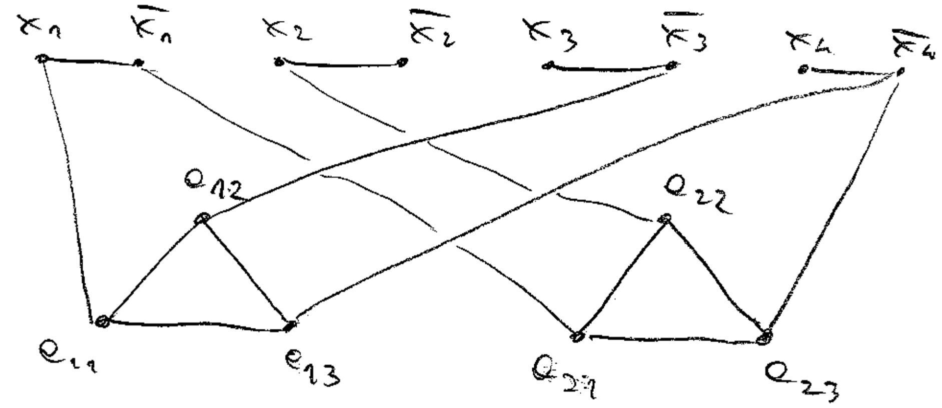
  Then we have $E_j'' = \{\{a_{1j}, l_{1j}\}, \{a_{2j}, l_{2j}\}, \{a_{3j}, l_{3j}\}\}$

We then set $K = n + 2m$

variables

clauses

Example : $F = (x_1 + \overline{x}_3 + \overline{x}_4) \cdot (\overline{x}_1 + x_2 + \overline{x}_4)$ E 12.5



$k = n + 2m = 4 + 2 \cdot 2 = 8$

We show that F is satisfiable $\Leftrightarrow$ G has a vertex cover of size $\leq k$

"$\Leftarrow$" Let $V' \subseteq V$ be a vertex cover for G with $|V'| \leq k$.

We said that V' contains at least
- one vertex for each variable
- 2 vertices for each clause

This is already $k = n + 2m$

$\Rightarrow$ at least is actually exactly

We use V' to obtain the truth assignment

we set x_i = true if $x_i \in V$

x_i = false if $x_i \notin V$

We can show that the truth assignment satisfies F

"$\Rightarrow$" Left as an exercise