<u>Exercise 2.2.2</u>: Prove that $\forall q \in Q$, $\forall x, y \in \Sigma^*$

$$\hat{\delta}(q, x \cdot y) = \hat{\delta}(\hat{\delta}(q, x), y)$$

Solution: by induction on $|y|$

Base case: $y = \varepsilon$

$$\hat{\delta}(q, x \cdot \varepsilon) = \hat{\delta}(q, x) = \hat{\delta}(\hat{\delta}(q, x), \varepsilon) = \hat{\delta}(\hat{\delta}(q, x), y)$$

since for any state q'
$\hat{\delta}(q', \varepsilon) = q'$

Inductive case: $y = z \cdot a$

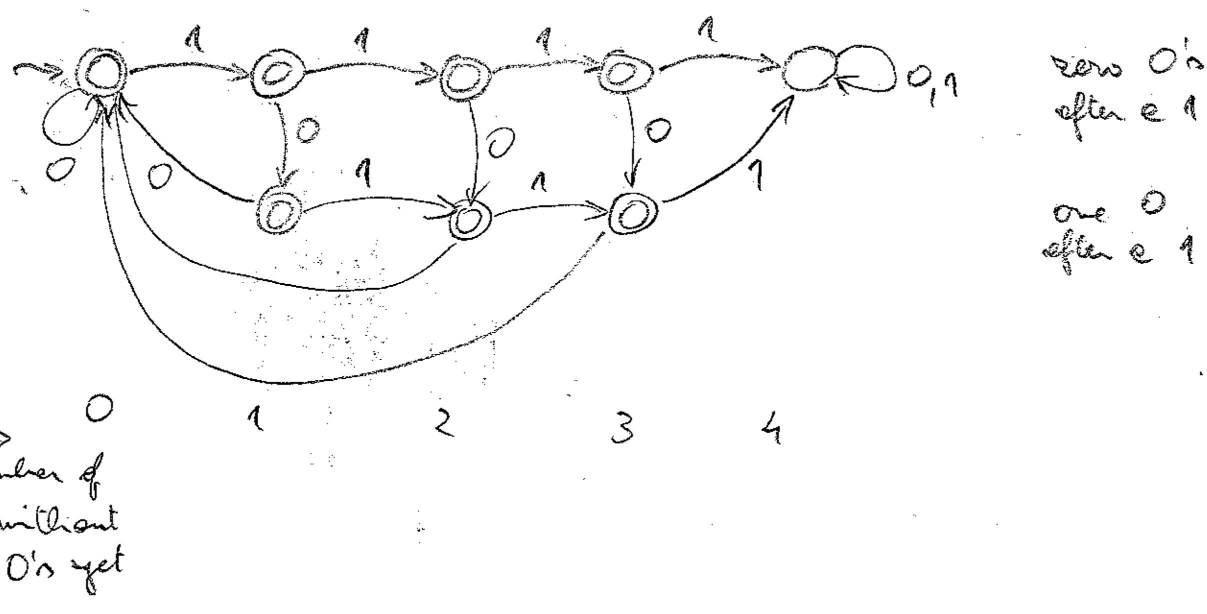
$$\hat{\delta}(q, x \cdot y) = \hat{\delta}(q, x \cdot z \cdot a) \underset{[\text{Def of } \hat{\delta}]}{=} \delta(\hat{\delta}(q, x \cdot z), a) \underset{[\text{Inductive hypothesis}]}{=}$$

$$= \delta(\hat{\delta}(\hat{\delta}(q, x), z), a) = \ [\text{Def. of } \hat{\delta}]$$

$$= \hat{\delta}(\hat{\delta}(q, x), z \cdot a) = \hat{\delta}(\hat{\delta}(q, x), y)$$

<u>Exercise 2.2.5</u>:

a) DFA that accepts $\{w \in \{0,1\}^* \mid$ each consecutive block of 5 symbols contains at least two 0's$\}$



Exercise 2.2.8: (E1.2)

Let A be a DFA s.t. for some $a \in \Sigma$ and all $q \in Q$ we have $\delta(q, a) = q$.

a) Show that for all $n > 0$, and all $q \in Q$, $\hat{\delta}(q, a^n) = q$

b) Show that either $\{a\}^* \subseteq \mathcal{L}(A)$ or $\{a\}^* \cap \mathcal{L}(A) = \emptyset$

Proof:

a) By induction on n

- $n = 1$: $\hat{\delta}(q, a^1) = \delta(\hat{\delta}(q, \varepsilon), a) = \delta(q, a) = q$
- suppose that for all $i < n$, $\hat{\delta}(q, a^i) = q$
we show that also $\hat{\delta}(q, a^n) = q$

$$
\begin{aligned}
\hat{\delta}(q, a^n) &= && [\text{def. of } \hat{\delta}] \\
&= \delta(\hat{\delta}(q, a^{n-1}), a) = && [\text{ind. hyp.}] \\
&= \delta(q, a) = && [\text{assumption on } \delta] \\
&= q
\end{aligned}
$$

b) By part (a), we have that $\hat{\delta}(q_0, a^n) = q_0$, $\forall n > 0$

If $q_0 \in F$, then $\hat{\delta}(q_0, \varepsilon) = q_0 \in F$. Hence $\varepsilon \in \mathcal{L}(A)$

Moreover, for $\forall n \geq 0$, we have $\hat{\delta}(q_0, a^n) = q_0 \in F$.

It follows that for all $n \geq 0$, $a^n \in \mathcal{L}(A)$, i.e. $\{a\}^* \subseteq \mathcal{L}(A)$

If $q_0 \notin F$, then $\hat{\delta}(q_0, \varepsilon) = q_0 \notin F$. Hence $\varepsilon \notin \mathcal{L}(A)$

Moreover, for $\forall n > 0$ we have $\hat{\delta}(q_0, a_n) = q_0 \notin F$

It follows that for all $n \geq 0$, $a^n \notin \mathcal{L}(A)$, i.e.

$\{a\}^* \cap \mathcal{L}(A) = \emptyset$

q.e.d.

E1.3

Exercise 2.2.9

Let $A = (Q, \Sigma, \delta, q_0, \{q_f\})$ be a DFA s.t. for all $a \in \Sigma$ we have $\delta(q_0, a) = \delta(q_f, a)$

a) Show that for all $w \neq \varepsilon$, we have $\hat{\delta}(q_0, w) = \hat{\delta}(q_f, w)$

b) Show that for all $x \in \mathcal{L}(A)$ with $x \neq \varepsilon$, we have

$$x^k \in \mathcal{L}(A), \text{ for all } k > 0.$$

Proof:

a) By induction on $|w|$

- $|w| = 1$, i.e. $w = a$ for some $a \in \Sigma$

$$\hat{\delta}(q_0, a) = \delta(q_0, a) = \delta(q_f, a) = \hat{\delta}(q_f, a).$$

- $|w| = n$ with $n > 1$
  assume the claim holds for all x with $|x| < n$

  Let $w = x \cdot a$ with $|x| = n-1$

$$\hat{\delta}(q_0, w) = \hat{\delta}(q_0, x \cdot a) = \delta(\hat{\delta}(q_0, x), a) = \quad [\text{by I.H.}]$$
$$= \delta(\hat{\delta}(q_f, x), a) = \hat{\delta}(q_f, x \cdot a) = \hat{\delta}(q_f, w)$$

b) By induction on k

- $k = 1$: statement is given by assumption $x = x^1 \in \mathcal{L}(A)$
- $k > 1$: assume that $x^h \in \mathcal{L}(A)$, for all $h < k$

$$\hat{\delta}(q_0, x^k) = \hat{\delta}(q_0, x^{k-1} \cdot x) = \quad [\text{by Ex. 2.2.2}]$$
$$= \hat{\delta}(\hat{\delta}(q_0, x^{k-1}), x) = \quad [\text{by I.H. and since } q_f \text{ is the only final state}]$$
$$= \hat{\delta}(q_f, x) = \quad [\text{by part (a)}]$$
$$= \hat{\delta}(q_0, x) = \quad [\text{by assumption}]$$
$$= q_f \qquad \text{Hence } x^k \in \mathcal{L}(A)$$

Exercise 2.3.5: E1.4

Let $A_D = (Q, \Sigma, \delta_D, q_0, F)$ be a DFA
and $A_N = (Q, \Sigma, \delta_N, q_0, F)$ be a NFA
with $\delta_N(q, a) = \{p\}$ if $\delta_D(q, a) = p$ $\forall q \in Q, a \in \Sigma$

Show that $\hat{\delta}_N(q_0, w) = \{\hat{\delta}_D(q_0, w)\}$ $\forall w \in \Sigma^*$

Proof: by induction on $|w|$

- $w = \varepsilon$

$$\hat{\delta}_N(q_0, \varepsilon) = \{q_0\} = \{\hat{\delta}_D(q_0, \varepsilon)\}$$

- ~~Let~~ $|w| = n+1$ and assume the claim holds for all x with $|x| \leq n$.

Let $w = x \cdot a$, $\hat{\delta}_D(q_0, x) = q$ and $\delta_D(q, a) = p$

$$\hat{\delta}_D(q_0, w) = \hat{\delta}_D(q_0, x \cdot a) = \delta_D(\hat{\delta}_D(q_0, x), a) = \delta_D(q, a) = p$$

By inductive hypothesis, we have that

$$\hat{\delta}_N(q_0, x) = \{\hat{\delta}_D(q_0, x)\} = \{q\}$$

Hence $\hat{\delta}_N(q_0, w) = \hat{\delta}_N(q_0, x \cdot a) = \bigcup_{p' \in \hat{\delta}_N(q_0, x)} \delta_N(p', a) =$

$$= \delta_N(q, a) = \{p\} = \{\hat{\delta}_D(q_0, w)\}$$

[by ind. hyp.]

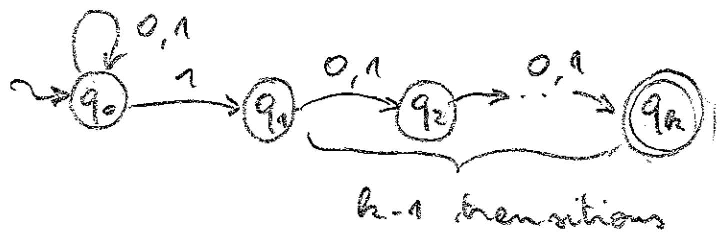
[by def. of δ_N, and since $\delta_D(q, a) = p$]

Exercise E2.2

E1.5

For $k \geq 1$, define an NFA A_N^k s.t.

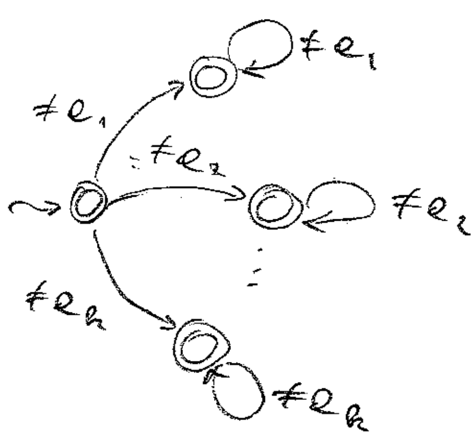
$\mathcal{L}(A_N^k) = \{ w \in \{0,1\}^* \mid \text{the } k\text{-th last symbol of } w \text{ is a } 1 \}$



Exercise E2.3

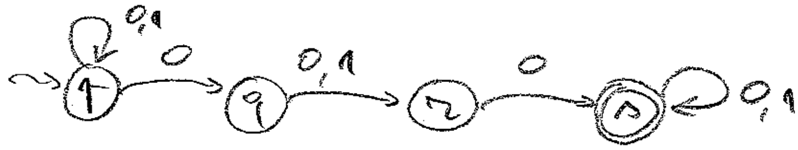
For $\Sigma_k = \{a_1, \dots, a_k\}$, construct an NFA A_N^k s.t.

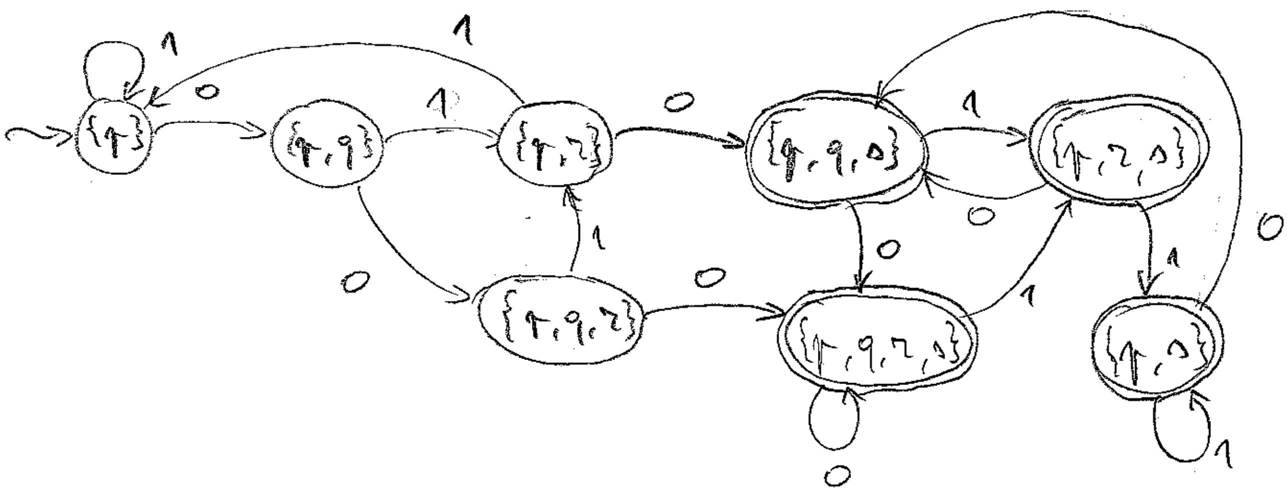
$\mathcal{L}(A_N^k) = \{ w \in \Sigma_k^* \mid w \text{ does not contain at least one of the symbols } a_1, \dots, a_k \}$



Exercise 2.3.1:

Convert the following NFA to a DFA





Exercise 2.3.4:

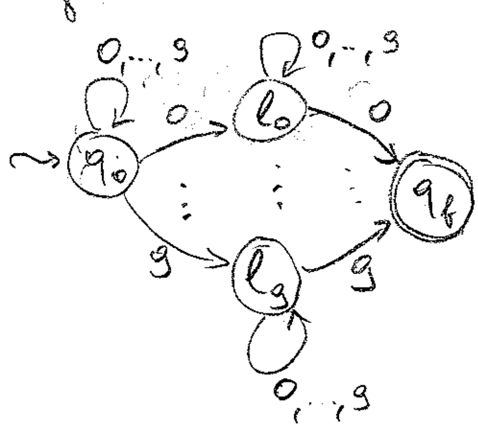
Give NFA's that accept the following languages over {0,..,9}

a) set of strings s.t. the final digit has appeared before

b) --- has not ---

We use states l_i, for i {0,...,9} to guess that the final digit is i

a)



b)